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Abstract

We demostrate how to simulate a one-dimentional fractional Ornstein-Uhlenbeck of the Second kind (fOU_2) using a circulant embedding method. This is an accurate and efficient computational algorithm for generating fOU_2 random vectors.

1 Introduction

Fractional Ornstein-Uhlenbeck of the second kind (fOU_2) comprises a family of Gaussian processes constructed via the Lamperti transformation of fractional Brownian motion with index H $(0 < H < 1)$. The earliest mention in literature of this process is due to Kaarakka and Salminen [12]. A deeper insight in some aspects of fOU_2 are given by Azmoodeh and Morlanes [2] and Amoodeh and Viitari [3]. The main property of fOU_2 is that is short range dependence for all values of H. This feature combined with the long range dependence of fBm with index $1/2 < H < 1$ may have interesting applications in fields such as physics or finance. A genuine application of fOU_2 can reinforce the theory results through demostration. Thus, applying a good simulation algorithm is not only theoretical, but also of practical, importance. The purpose of this paper is to show how to generate fOU_2 with a computer.

A random vector of fOU_2 can be simulated by factorizing its covariance matrix. Among the existing algorithms, Hosking's method [10] and Cholesky decomposition [1] are the standard methods to do this. These methods can simulate accurately one-dimentional fOU_2, but they have a high computational cost when are applied on some fine grid. The reason is that the fOU_2 vector becomes too long and its covariance matrix too large to handle in terms of storage and computational requirements. However, if the covariance matrix is circulant, a better algorithm in terms of the dual

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criteria "accurancy and efficiency" can be applied, the method is called circulant embedding method. This has been introduced in literature by Davis and Harte [6], and generalized by Wood and Chan [17, 4, 5], Dietrich and Newsam [7], and Gneiting et al. [8].

The circulant embedding method exploits the fact that the factorization of a circulant matrix has very low computational cost using the fast Fourier transform; particularly if the number of grid points is a power of two. Although the fOU_2 covariance matrix is not circulant, we can readily embed it into a circulant matrix. The primary motivation of this paper is to use this technique for building and evaluating a one-dimentional fOU_2 random vector in a discrete grid. We found the one-dimensional case to be instructive to show how to apply this algorithm to fOU_2 random vectors in two and three dimentions. It also helps to extend our knowledge on how to simulate non-Gausssian fOU_2 random vectors using a non-linear transformation, see Grigoriu [9].

The rest of the paper is organized as follows. We recall in Section 2 fractional Brownian motion and some of its properties. In Section 3 we introduce the fOU_2 process and some properties of its covariance function. Section 4 deals with the description of the circulant embedding method and the practicalities of the generation of an fOU_2 vector. We also show a practical example. Finally, discussion and conclusions are given in Section 5.

2 Fractional Brownian Motion

We recall the definition and some esential properties of a fractional Brownian motion which is the cornerstone to construct fractional Ornstein-Uhlenbeck of the second kind.

Fractional Brownian motion represents a natural one parameter extension of Brownian motion with parameter $0 < H < 1$, called the Hurst parameter. The parameter is named after the hydrologist Hurst [11] who developed a statistical analysis of the yearly water run-offs of Nile river. Mandelbrot and Van Ness [13] established an integral representation of the process in terms of Brownian motion and named it *Fractional Brownian motion.* More details on fractional Brownian motion, modeling and applications can be also found in Nualart [14].

2.1 Definition and Properties of fBm

A *fractional Brownian motion* (fBm) is defined as a continuous centered Gaussian process $\{B_t^H, t \geq 0\}$ with $B_0^H = 0$ and covariance function

$$\mathrm{Cov}(B_t^H, B_s^H) = \frac{1}{2}\sigma^2(s^{2H} + t^{2H} - (s-t)^{2H}) \tag{1}$$

for all $0 \leq s \leq t$. In particular,

$$\mathrm{Var}(B_t^H) = \sigma^2 t^{2H}. \tag{2}$$

Notice that for $H = \frac{1}{2}$ the variance function is $\sigma^2 min\{t, s\}$ and fBm becomes a Brownian motion. The process B^H is self-similar with Hurst parameter $H > 0$, i.e., for any constant $c > 0$, the rescaled processes $\{c^{-H} B_{ct}^H,\, t \geq 0\}$ have the same probability distribution as B^H. This property is an immediate consequence of the fact that the covariance function (1) is homogeneous of order $2H$, that is,

$$\mathrm{Cov}(B_{ct}^H, B_{ct}^H) = c^{2H}\mathrm{Cov}(B_t^H, B_s^H), \quad c > 0.$$

The increments $(B_t^H - B_s^H)$ are stationary with zero mean Gaussian distribution and variance

$$\mathrm{Var}\left(B_t^H - B_s^H\right) = \sigma^2 |t-s|^{2H}.$$

This is an important property to simulate fBm on a equally spaced grid. Since the increment process is stationary, we can generate it efficiently using the circulant embedding method in Section 4.1. In the next section we introduce this process and give some properties.

2.2 Correlation and Long-Range Dependence of Time Series Data

The generation of fBm on the uniformly spaced grid $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ can be achived by first generating the increment process $\{Y_n = (B_n^H - B_{n-1}^H),\, n \geq 1\}$. The time series Y_n can be characterized as a discrete stationary Gaussian sequence with mean zero and covariance function

$$\mathrm{Cov}(Y_{n+k}, Y_n) = \frac{1}{2}\sigma^2\left((k+1)^{2H} + (k-1)^{2H} - 2k^{2H}\right), \tag{3}$$

called *fractional Gaussian noise* with Hurst parameter H. In particular, for $H = \frac{1}{2}$ the increments are independent and B^H becomes a Brownian motion.

It follows from (3) that the autocorrelation is

$$\begin{aligned}\rho_H(k) &= \frac{1}{2}\left((k+1)^{2H} + (k-1)^{2H} - 2k^{2H}\right)\\ &\approx H(2H-1)k^{2(H-1)}\end{aligned}$$

as k tends to infinity. The parameter H controls the regularity of trajectories. For $1/2 < H < 1$ the autocorrelations are positive $\rho_H(k) > 0$ and decay slowly to zero $\sum_{n=1}^{\infty} \rho_H(n) = \infty$, i.e., they exhibit *long-range dependence.* In the case that $0 < H < 1/2$ the autocorrelations are negative $\rho_H(k) < 0$ and decay to zero with a rate faster than $\frac{1}{n}$ and the increments have the *short-range dependence* property, that is, $\sum_{n=1}^{\infty} \rho_H(n) < \infty$.

3 Fractional Ornstein-Uhlenbeck process of the Second Kind

Fractional Onstein-Uhlenbeck of the second kind represents a family of stationary Gaussian processes. This family was first introduced by Karaakka and Salminen [12]. Inspired by Ornstein-Uhlenbeck diffusions, they defined a new process via the Lamperti transfomation of fractional Brownian motion. This was named *Fractional Brownian Ornstein-Uhlenbeck of the second kind* (fOU_2).

We first recall the well-known Ornstein-Uhlenbeck diffusion [15], also named Vasicek model [16]. The process is the unique strong solution of the Langevin stochastic differential equation

$$\mathrm{d}U_t = \theta(\mu - U_t)\,\mathrm{d}t + \sigma \mathrm{d}B_t, \quad B_0 = 0, \quad \theta > 0$$

where μ and σ are parameters and $B = \{B\}_{t\geq 0}$ is a Brownian motion or a Wiener process. The parameter μ is interpreted as the long-run equilibrium value of the process, σ is the volatility, and θ is the speed of reversion, i.e, the process tends to oscillate around some equilibrium value.

The fractional Ornstein-Uhlenbeck process is a fractional analogue of the Ornstein-Uhlenbeck-process. That is, a continuous process X which is the solution of the stochastic differential equation

$$\mathrm{d}X_t = \theta(X_t - \mu)\mathrm{d}t + \sigma \mathrm{d}B_t^H, \tag{4}$$

where $\theta > 0$, μ and $\sigma > 0$ are parameters and $B^H = \{B_t^H\}_{t\geq 0}$ is a fractional Brownian motion with Hurst parameter $1/2 < H < 1$. This process is also called fractional Onrstein-Uhlenbeck of the first kind by Kaarakka and Salminen [12].

By the Lamperti transformation of fractional Brownian motion B^H we define the process

$$Y_t = \int_0^t e^{-s}\,\mathrm{d}B_{a_s}^H \tag{5}$$

where $a_s = He^{\frac{s}{H}}$. If we change fBm by the process (5) in equation (4) we obtain a new fractional Ornstein-Uhlenbeck stochastic differential equation

$$\mathrm{d}X_t = \theta(\mu - X_t)\,\mathrm{d}t + \mathrm{d}Y_t, \quad X_0 = 0, \quad \theta > 0. \tag{6}$$

The solution may be found by appying fractional Itô lemma to the ansantz $e^{\theta t}X_t$ which leads to

$$X_t = e^{-\theta t}X_0 + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{\theta(s-t)}\,\mathrm{d}Y_s$$

where the stochastic integral is understood as a pathwise Riemann-Stieltjes integral. This process is the so-called fractional Ornstein-Uhlenbeck of the second kind.

3.1 Properties of fOU_2 process

A fractional Ornstein-Uhlenbeck process of the second kind is a continuous stationary Gaussian process with $X_0 = 0$ and covariance

$$\gamma_U(H,\theta,t) = H(2H-1)H^{2H-2}e^{-\theta t}\int_{-\infty}^{t}\int_{-\infty}^{0}\frac{e^{(\theta-1+\frac{1}{H})(u+v)}}{\left|e^{u/H}-e^{v/H}\right|^{2(1-H)}}\,\mathrm{d}u\,\mathrm{d}v \tag{7}$$

where θ is the drift parameter of the equation (6). In particular,

$$\mathrm{Var}(X_t) = \frac{1}{\theta}C(H,\theta)e^{\theta t}\,\mathrm{Beta}(H(\theta-1)+1, 2H-1)$$

where Beta stands for $\mathrm{Beta}(a,b) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$ and $C(H,\theta) = H^{2H(1-\theta)}H\,(2H-1)$.

The covariance function γ_U decays exponentially as a function of the parameters (H,θ). In contrasts to fractional Brownian motion, fOU_2 has the interesting property of short range dependence for all values of the Hurst parameter, i.e.

$$\int_{-\infty}^{\infty}\gamma_U(H,\theta,u)\,\mathrm{d}u = 0$$

for all H and θ. In particular, for $H = \frac{1}{2}$ the covariance function is zero and the process coincides with the Ornstein-Uhlenbeck diffusion, see Kaarakka and Salminen [12].

The rest of properties of fOU_2 are analogous to those of frational Brownian motion by the interpretation of the stochastic integral as a pathwise Riemann-Stieltjes integral.

4 Numerical Simulation of fOU_2

We simulate fractional Ornstein-Uhlenbeck of the second kind by means of the *Circulant Embedding Method* (CEM). The CEM is an exact method to simulate stationary Gaussian processes which is indeed the case of fOU_2. The main step to apply the method is the discretization of its covariance function (7) in equally spaced intervals.

We discretize the interval $[0,T]$ in equally spaced subintervals with $\Delta t = T/N$ for N some interger N and $t_j = j\Delta t$. We also denote $X_{t_j} = X_j$. In order to use the fOU_2 covariance function numerically, we calculate a discrete version of it. We rewrite (7) as a function of the distance of two points $k = t_i - t_j = |i-j|\Delta t$ and use Δt as a parameter. As a result, a discrete version of the fOU_2 covariance function is

$$
\begin{aligned}
R(H,\theta,k) = C(H,\theta)e^{-\theta k}\Bigg(&\frac{1}{\theta}\text{Beta}((\theta-1)H+1, 2H-1)\\
&+\int_H^{He^{\frac{k}{H}}} n^{2\theta H-1}\,\text{InBeta}(\frac{H}{n};(\theta-1)H+1,2H-1)\,\mathrm{d}n\Bigg)
\end{aligned}
\tag{8}
$$

for $k = 0, 1, \ldots, N-1$. The function InBeta stands for the Incomplete Beta function $\mathbf{B}(x;\, a, b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\,\mathrm{d}t$. The calculation of this result can be found in Appendix A.

We next normalize the discretized covariance function. To do this, we redefine fOU_2 as

$$\tilde{X}_j = \beta(H,\theta)^{-\frac{1}{2}} X_j \tag{9}$$

where $\beta(H,\theta)$ stands for $\beta(H,\theta) = C(H,\theta)\theta^{-1}\text{Beta}(H(\theta-1)+1, 2H-1)$ and $C(H,\theta) = H^{2H(1-\theta)}H\,(2H-1)$. The new process has covariance function

$$\tilde{R}(H,\theta,k) = \beta(H,\theta)^{-1}R(H,\theta,k), \quad k = 0, 1, \ldots, N-1.$$

Notice that now $\tilde{R}(0) = 1$.

We are now ready to simulate the normalized fOU_2 process $\tilde{X}$ by means of CEM. The steps of the procedure are explained in the Circulant Embedding Method in a Nutshell Section 6.1 and shown in Algorithm 1. The process fOU_2 is then recovered from the expression

$$X_j = \sqrt{\beta(H,\theta)}\tilde{X}_j$$

with $X_0 = 0$.

4.1 Circulant Embedding Method in a Nutshell

We now give a shortened explanation of the steps involved in simulating the process $\tilde{X}$, defined as the normalized fOU_2 in equation (9). The set of steps for the CEM method is shown in Algorithm 1. The algorithm is based on the fact that the covariance matrix of a stationary discrete Gaussian processes can be embedded in a so called circulant matrix. This latter matrix should be non-negative definite for the algorithm to work, which is the case for the fOU_2.

The advantage of CEM with respect to other methods is that a circulant matrix can be diagonalized explicitly, and the computations are done efficiently with the so-called Fast Fourier Transform (FFT) algorithm. In what follows, we assume that the covariance matrix of $\tilde{X}$ has first been embedded in a circulant matrix. This technical step is explained in the illustrative example Section 6.2.

A circulant matrix $C \in \mathbb{R}^{N\times N}$ with first column c_1 has decomposition $C = WDW^*$ where W is a Fourier matrix and D is a diagonal matrix with diagonal $\lambda = \sqrt{N} W^* c_1$. The columns of W are the eigenvectors of C and D contains the eigenvalues. If all eigenvalues are non-negative then define $R = WD^{1/2}$ and C has the factorization $C = RR^*$.

A complex value vector $\hat{\mathbf{X}} = R\boldsymbol{\xi}$ is next generated where $\boldsymbol{\xi}$ is a complex Gaussian vector of length N and distribution $\boldsymbol{\xi} \sim \mathrm{CN}(\mathbf{0}, 2I_N)$ so $\hat{\mathbf{X}} \sim \mathrm{CN}(\mathbf{0}, 2C)$. From the real and imaginary part of vector $\hat{\mathbf{X}} = \hat{\mathbf{X}}_{\mathbf{1}} + i\hat{\mathbf{X}}_{\mathbf{2}}$, we obtain two sequences of length N with covariance matrix the circulant matrix C, i.e, $\hat{\mathbf{X}}_{\mathbf{1}}$, $\hat{\mathbf{X}}_{\mathbf{2}} \sim \mathrm{N}(0, C)$.

We finally extract two normalized fOU_2 vectors $\tilde{\mathbf{X}}_{\mathbf{1}}$ and $\tilde{\mathbf{X}}_{\mathbf{2}}$ with covariance matrix entries given by $\tilde{R}(H, \theta, k)$ from $\hat{\mathbf{X}}_{\mathbf{1}}$ and $\hat{\mathbf{X}}_{\mathbf{2}}$.

Algorithm 1: Generation of two fOU_2 vectors of length N via Circulant Embedding Method.

Let consider the discretization $0 < t_0 < t_1 < \cdots < t_N = T$ for the desired interval.
Generate the initial value X_0 if it is from a distribution or fix value.
Embed the covariance matrix in a circulant matrix C.
Factorize $C = RR^*$ by the fast Fourier transform.
Generate a complex vector $\boldsymbol{\xi} = \boldsymbol{\xi}_{\mathbf{1}} + i\boldsymbol{\xi}_{\mathbf{2}}$ where $\boldsymbol{\xi}_{\mathbf{1}}, \boldsymbol{\xi}_{\mathbf{2}} \overset{\text{iid}}{\sim} \mathrm{N}(0, I_N)$.
Evaluating $\mathbf{X} = R\boldsymbol{\xi}$ by the fast Fourier transform.
Get the real part $\Re\{\mathbf{X}\}$ and the imaginary part $\Im\{\mathbf{X}\}$.
Output the first N values of $\Re\{\mathbf{X}\}$ and $\Im\{\mathbf{X}\}$.

4.2 Illustrative Example

We illustrate Algorithm 1 with a simulation of a fOU_2 vector with Hurst parameter $H = 0.8$ and $\theta = 3$ on the interval $[0, 1]$. We also describe how to embed its covariance matrix in a circulant matrix.

Let consider the fOU_2 vector $\mathbf{X} = (X_0, X_1, X_2, X_3, X_4)$ with parameters as given above and length $N = 5$. The normalized fOU_2 vector $\tilde{\mathbf{X}}$ has a normalization factor $\beta(0.8, 3) = 0.2870$ and covariance matrix

$$\begin{pmatrix} 1 & 0.25 & 0.15 & 0.11 & 0.09 \\ 0.25 & 1 & 0.25 & 0.15 & 0.11 \\ 0.15 & 0.25 & 1 & 0.25 & 0.15 \\ 0.11 & 0.15 & 0.25 & 1 & 0.25 \\ 0.09 & 0.11 & 0.15 & 0.25 & 1 \end{pmatrix}$$

This is a symetric Toeplitz matrix but not circulant. We can always embed a Toeplitz matrix inside of a larger symmetric circulant matrix by means of the so called *minimal circulant embedding.* We add the first entry of the second column to the last entry of the first column. The penultimate entry of the first column must equal the first entry of the third column and so on.

Here we add three columns and rows to the covariance matrix

$$\left(\begin{array}{ccccc|ccc} 1 & 0.25 & 0.15 & 0.11 & 0.09 & 0.11 & 0.15 & 0.25 \\ 0.25 & 1 & 0.25 & 0.15 & 0.11 & 0.09 & 0.11 & 0.15 \\ 0.15 & 0.25 & 1 & 0.25 & 0.15 & 0.11 & 0.09 & 0.11 \\ 0.11 & 0.15 & 0.25 & 1 & 0.25 & 0.15 & 0.11 & 0.09 \\ 0.09 & 0.11 & 0.15 & 0.25 & 1 & 0.25 & 0.15 & 0.11 \\ \hline 0.11 & 0.09 & 0.11 & 0.15 & 0.25 & 1 & 0.25 & 0.15 \\ 0.15 & 0.11 & 0.09 & 0.11 & 0.15 & 0.25 & 1 & 0.25 \\ 0.25 & 0.15 & 0.11 & 0.09 & 0.11 & 0.15 & 0.25 & 1 \end{array}\right)$$

This is a circulant matrix with first column $c_1 = (1, 0.25, 0.15, 0.11, 0.09, 0.11, 0.15, 0.25)^T$ where ^{T} stands for transpose of a matrix. It has decomposition

$$\underbrace{\frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & -i & -i\omega & -1 & -\omega & i & i\omega \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\omega & i & \omega & -1 & i\omega & -i & -\omega \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega & -i & i\omega & -1 & \omega & i & -i\omega \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\omega & i & -\omega & -1 & -i\omega & -i & \omega \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} 2.12 &&&&&&& \\ & 1.11 &&&&&& \\ && 1.11 &&&&& \\ &&& 0.66 &&&& \\ &&&& 0.79 &&& \\ &&&&& 0.79 && \\ &&&&&& 0.72 & \\ &&&&&&& 0.72 \end{pmatrix}}_{D} W^*$$

where W^* is the conjugate transpose of W. The minimal circulant extension has all eigenvalues non-negative so it is a valid covariance matrix. We generate $\hat{\mathbf{X}} = WD^{1/2}\boldsymbol{\xi}$ where $\boldsymbol{\xi} \sim CN(\mathbf{0}, 2I_8)$, as seen in output Table 1.

Table 1: First five samples for fOU_2 with H=0.8 and $\theta = 3$ on the interval $[0, 4]$.

time	$\hat{\mathbf{X}}$	$\tilde{\mathbf{X}}_1$	$\mathbf{X}$
0	0.45-i0.04	0	0
1	0.37+i0.50	0.45	0.24
2	0.04-i0.49	0.37	0.20
3	0.07-i0.27	0.04	0.02
4	-0.25-i0.20	0.07	0.04
5	-0.24+i0.44		
6	-0.04+i0.47		
7	-0.12-i0.09		

We finally consider the first five elements of the real part of $\hat{\mathbf{X}}$ and multiply them by the squared root of $\beta(0.8, 3)$. As a result, we obtain a fOU_2 sample vector $\mathbf{X}$ of length 5, as shown in output Table 1.

For the sake of generality, we extend the random vector fOU_2 from $N = 5$ to length $N = 1000$ in the interval $[0, 1]$. All the steps in the simulation are the same but the covariance matrix of the normalized fOU_2 vector is embedded in a circulant matrix with dimension 998×998.

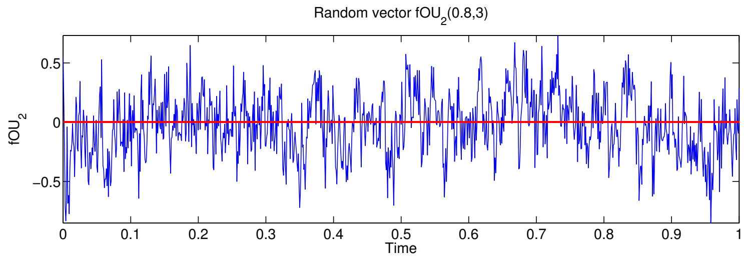

Figure 1: Sample fOU_2 path with Gaussian distribution using Algorithm 2.

5 Conclusion

In this paper, we show how to simulate one-dimentional fOU_2 by a fast and accurante method, the so-called circulant embedding method. The two main steps to do this are first embed its covariance matrix in a circulant one and second use a fast Fourier transform (FFT) algorithm. We exemplified the method with a a random vector fOU_2 with parameters $H = 0.8$ and $\theta = 3$, and length $N = 5$.

The circulant emebedding method also allows us to generate two and three-dimentional fOU_2 vectors by following the same steps of the one-dimentional case exemplified in this paper. The fOU_2 covariance matrix is now embedded into a block circulant matrix with each block being circulant itself. Two and three-dimentional FFT techniques are then applied. Finally, if we wish to simulate a non-Gaussian fOU_2 vector then we could also use the circulant embedding method in combination with a memoryless non-linear transformation, see Grigoriu [9].

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A Discretization of fOU_2 covariance function

We sketch the calculation for the discrete version of fOU_2 covariance function (7). We consider

$$\begin{aligned}
&\int_{-\infty}^{t}\int_{-\infty}^{s} H^{2(H-1)} \frac{e^{(\theta-1+\frac{1}{H})(u+v)}}{\left|e^{u/H}-e^{v/H}\right|^{2(1-H)}}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{0}^{a_t}\int_{0}^{a_s} (mn)^{(\theta-1)H}|m-n|^{2(H-1)}\,\mathrm{d}m\,\mathrm{d}n \\
&= \int_{0}^{a_s}\int_{0}^{m} (mn)^{(\theta-1)H}|m-n|^{2(H-1)}\,\mathrm{d}m\,\mathrm{d}n + \int_{a_s}^{a_t}\int_{0}^{a_s} (mn)^{(\theta-1)H}|m-n|^{2(H-1)}\,\mathrm{d}m\,\mathrm{d}n \\
&\approx \int_{0}^{a_s} m^{2\theta H-1}\int_{0}^{1} \xi^{(\theta-1)H}|1-\xi|^{2(H-1)}\,\mathrm{d}\xi\,\mathrm{d}m + \int_{a_s}^{a_t} m^{2\theta H-1}\int_{0}^{a_s/m} \xi^{(\theta-1)H}|1-\xi|^{2(H-1)}\,\mathrm{d}\xi\,\mathrm{d}m \\
&= \frac{a_s^{2\theta H}}{\theta H}\mathbf{B}((\theta-1)H+1, 2H-1) + \int_{a_s}^{a_t} m^{2\theta H-1}\mathbf{B}(a_s/m; (\theta-1)H+1, 2H-1) \\
&= \frac{1}{\theta}\mathbf{B}((\theta-1)H+1, 2H-1) + \int_{H}^{He^{\frac{k}{H}}} n^{2\theta H-1}\mathbf{B}(\frac{H}{n}; (\theta-1)H+1, 2H-1)\,\mathrm{d}n
\end{aligned}$$

where $\mathbf{B}(\cdot\,;\cdot,\cdot)$ is the incomplete beta function, and $\mathbf{B}(\cdot,\cdot)$ is the beta function.