Testing in multivariate normal models with block circular covariance structures

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Abstract

In this article, likelihood ratio tests concerning multivariate normal models with block circular covariance structures are obtained. Hypotheses about general block structures of the covariance matrix and specific covariance parameters of the block circular covariance structure have been of main interest. In addition, tests about the mean vector have been considered.

Keywords: Beta random variables, Canonical reduction, Covariance parameters, Likelihood ratio test, Restricted model

1. Introduction

The main goal of this paper is to develop likelihood ratio tests (LRT) for multivariate normal models with block circular covariance structure. We consider both testing a block structure (external test) and testing specific (co)variance parameters inside of the block circular structure (internal test). Testing specific (co)variance parameters in case of block structures is a quite challenging problem. It demanded innovative solutions to solve an overparametrization problem, and formulation of a meaningful hypotheses.

For some related works of the external test we refer to Wilks (1946), Votaw (1948), Olkin and Press (1969), Olkin (1973) and Srivastava *et al.* (2008) who

all consider estimation and testing problems for patterned covariance matrices. For example, Votaw (1948) studied a test for block compound symmetry (CS) covariance matrix, extended the testing problem of the CS structure of Wilks (1946) to a "block version" and developed a likelihood ratio test (LRT) criteria for testing 12 hypotheses which are relevant to certain psychometric and medical research problems. Olkin (1973) considered the problem of testing circular Toeplitz (CT) covariance matrix in blocks which is an extension of his previous work Olkin and Press (1969). In comparison to the mentioned works this paper treats a different covariance structure which has only been studied by Liang *et al.* (2014). In this work, we also take into account the mean structure and test both mean and covariance structures simultaneously. Furthermore, we consider testing hypotheses about the mean given a specific covariance matrix.

The organization of the paper is as follows. In Section 2, we present a model with three hypothetical block covariance structures and consider various hypotheses concerning testing mean and covariance matrices. Moreover, the model with a block circular Toeplitz covariance structure will be further studied by introducing two nested random effects. The LRT statistics for testing block structures as well as the corresponding null distributions are given in Section 3. In Section 4, we test hypotheses concerning (co)variance parameters within the block circular Toeplitz covariance structure.

2. Models

Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ be independent samples from $N_p(\mathbf{1}_{n_2} \otimes \boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ is an n_1 variate unknown vector, $p = n_2 n_1$, and let $\boldsymbol{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$. Then we may write

$$\boldsymbol{Y} \sim N_{p,n}((\mathbf{1}_{n_2} \otimes \boldsymbol{\mu})\mathbf{1}_n', \boldsymbol{\Sigma}, \boldsymbol{I}_n), \tag{1}$$

where $N_{p,n}((\mathbf{1}_{n_2} \otimes \boldsymbol{\mu})\mathbf{1}'_n, \boldsymbol{\Sigma}, \boldsymbol{I}_n)$ denotes the $p \times n$ matrix normal distribution with mean matrix $(\mathbf{1}_{n_2} \otimes \boldsymbol{\mu})\mathbf{1}'_n$ and $p \times p$ covariance matrix within elements of columns $\boldsymbol{\Sigma}$ and there are n independent columns. Throughout this paper, $\mathbf{1}_s$ is a column vector of size s with all elements equal to one, $\boldsymbol{J}_s = \mathbf{1}_s \mathbf{1}'_s$, $\boldsymbol{I}_s$ is the $s \times s$ identity matrix and $\otimes$ denotes the Kronecker product. In addition, we denote $\boldsymbol{P}_s = \frac{1}{s}\boldsymbol{J}_s$ and $\boldsymbol{Q}_s = \boldsymbol{I}_s - \boldsymbol{P}_s$ as two mutually orthogonal projectors of size $s \times s$. The notation vec($\bullet$) denotes the vectorization operator and tr($\bullet$) denotes the trace function.

The following three specific structures of $\boldsymbol{\Sigma}$, namely, $\boldsymbol{\Sigma}_I$, $\boldsymbol{\Sigma}_{II}$ and $\boldsymbol{\Sigma}_{III}$, are of interest when constructing the LRT statistics in this article.

(i) $\boldsymbol{\Sigma}_I = \boldsymbol{I}_{n_2} \otimes \boldsymbol{\Sigma}^{(1)} + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{\Sigma}^{(2)}$, where $\boldsymbol{\Sigma}^{(h)} : n_1 \times n_1$ is an unstructured symmetric matrix, $h = 1, 2$;

(ii) $\boldsymbol{\Sigma}_{II} = \boldsymbol{I}_{n_2} \otimes \boldsymbol{\Sigma}^{(1)} + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{\Sigma}^{(2)}$, where $\boldsymbol{\Sigma}^{(h)}$, $h = 1, 2$, is a CT matrix which depends on r parameters, $r = [n_1/2] + 1$, and the symbol $[\bullet]$ stands for the integer part;

(iii) $\boldsymbol{\Sigma}_{III} = \boldsymbol{I}_{n_2} \otimes \boldsymbol{\Sigma}^{(1)} + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{\Sigma}^{(2)}$, where $\boldsymbol{\Sigma}^{(h)}$, $h = 1, 2$, is a CS matrix and can be written as $\boldsymbol{\Sigma}^{(h)} = \sigma_{h1}\boldsymbol{I}_{n_1} + \sigma_{h2}(\boldsymbol{J}_{n_1} - \boldsymbol{I}_{n_1})$.

Throughout this thesis, the notation CT matrix stands for a symmetric circular Toeplitz matrix. Note that the matrix $\boldsymbol{\Sigma}^{(h)} = (\sigma_{ij}^{(h)})$ depends on r, $r = [n_1/2] + 1$, parameters, and for $i, j = 1, \ldots, n_1$, $h = 1, 2$,

$$\sigma_{ij}^{(h)} = \begin{cases} \tau_{|j-i|+1+(h-1)r}, & \text{if } |j - i| \leq r - 1, \\ \tau_{n_1-|j-i|+1+(h-1)r}, & \text{otherwise,} \end{cases}$$

where the τ_q's are unknown parameters, and taking into account that $h = 1, 2$, the index $q = 1, \ldots, 2r$. For spectral properties of CT matrices, see Basilevsky (1983), for example.

The number of unknown parameters in $\boldsymbol{\Sigma}_I$, $\boldsymbol{\Sigma}_{II}$ and $\boldsymbol{\Sigma}_{III}$ are $n_1(n_1 + 1)$, $2r$ and 4, respectively. That is, more restrictions are imposed on $\boldsymbol{\Sigma}$ from $\boldsymbol{\Sigma}_I$ to $\boldsymbol{\Sigma}_{III}$.

Olkin (1973) considered the problems of testing the block versions of hypotheses: sphericity versus intraclass correlation, sphericity versus circular symmetry, intraclass correlation versus circular symmetry and circular symmetry versus general structure by assuming unstructured mean. Olkin's block version of intraclass correlation is the same as $\boldsymbol{\Sigma}_I$. Different from Olkin (1973), this paper will start the null hypothesis of $\boldsymbol{\Sigma}_I$ and test $\boldsymbol{\Sigma}_I$ versus a type of mixed block structure $\boldsymbol{\Sigma}_{II}$ which follows to Barton and Fuhrmann (1993), which has been shown that it can be transformed into Olkin's block circular symmetry model with intraclass blocks inside by using the so-called commutation matrix (Liang *et al.*, 2012). Moreover, it is useful to test further $\boldsymbol{\Sigma}_I$ or $\boldsymbol{\Sigma}_{II}$ versus a more parsimonious structure $\boldsymbol{\Sigma}_{III}$.

It is of interest to test both the mean $\boldsymbol{\mu}$ and block structure of $\boldsymbol{\Sigma}$ simultaneously.

$$H_1^0: \ \boldsymbol{\mu} = \mu \mathbf{1}_{n_1}, \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{II} \text{ versus } H_1^a : \boldsymbol{\mu} \in \mathbb{R}^{n_1}, \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_I,$$

$$H_2^0: \ \boldsymbol{\mu} = \mu \mathbf{1}_{n_1}, \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{III} \text{ versus } H_2^a : \boldsymbol{\mu} \in \mathbb{R}^{n_1}, \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_I.$$

Furthermore, the following hypotheses about the pattern of $\boldsymbol{\Sigma}$ will be tested, i.e.,

$$H_3^0: \ \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{III} \text{ versus } H_3^a : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{II}, \text{ given } \boldsymbol{\mu} = \mu \mathbf{1}_{n_1},$$

$$H_4^0: \ \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{II} \text{ versus } H_4^a : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_I, \text{ given } \boldsymbol{\mu} \in \mathbb{R}^{n_1},$$

$$H_5^0: \ \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{III} \text{ versus } H_5^a : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_I, \text{ given } \boldsymbol{\mu} \in \mathbb{R}^{n_1},$$

or test about the mean, i.e.,

$$H_6^0: \ \boldsymbol{\mu} = \mu \mathbf{1}_{n_1} \text{ versus } H_6^a : \boldsymbol{\mu} \in \mathbb{R}^{n_1}, \text{ given } \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{II},$$

$$H_7^0: \ \boldsymbol{\mu} = \mu \mathbf{1}_{n_1} \text{ versus } H_7^a : \boldsymbol{\mu} \in \mathbb{R}^{n_1}, \text{ given } \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{III}.$$

In this paper we will also pay particular attention to the case when $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{II}$. In Liang *et al.* (2012, 2014), a balanced hierarchical mixed linear model with the covariance structure $\boldsymbol{\Sigma}_{II}$ is considered and the model can be applied to situations when there is a spatial circular layout on one factor and another factor satisfies the

property of exchangeability. Liang *et al.* (2014) gave a real-data example which motivated the pattern of $\boldsymbol{\Sigma}_{II}$. Moreover, $\boldsymbol{\Sigma}_{II}$ characterizes the dependency when both compound symmetry and circular symmetry appear hierarchically (Liang *et al.*, 2011).

Now we will introduce two random effects $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, where $\boldsymbol{\gamma}_2$ is nested within $\boldsymbol{\gamma}_1$, and the response vector $\mathbf{y}_i$ is given by

$$\mathbf{y}_i = \mu\mathbf{1}_p + \boldsymbol{Z}_1\boldsymbol{\gamma}_1 + \boldsymbol{Z}_2\boldsymbol{\gamma}_2 + \boldsymbol{\epsilon}, \quad i = 1, \ldots, n, \tag{2}$$

where $\mathbf{y}_i$ is a $p \times 1$ vector of observations, $p = n_1 n_2$, μ is an unknown constant, $\boldsymbol{\gamma}_1 : n_2 \times 1$, $\boldsymbol{\gamma}_2 : p \times 1$ and $\boldsymbol{\epsilon}$ are independently normally distributed random vectors with zero means and variances-covariance matrices $\boldsymbol{\Sigma}_1$, $\boldsymbol{\Sigma}_2$, and $\sigma^2\boldsymbol{I}_p$, respectively. Here $\boldsymbol{Z}_1 = \boldsymbol{I}_{n_2} \otimes \mathbf{1}_{n_1}$, $\boldsymbol{Z}_2 = \boldsymbol{I}_{n_2} \otimes \boldsymbol{I}_{n_1}$. Hence,

$$\mathbf{y}_i \sim N_p(\mu\mathbf{1}_p, \boldsymbol{V}(\boldsymbol{\theta})), \quad i = 1, \ldots, n,$$
$$\boldsymbol{V}(\boldsymbol{\theta}) = \boldsymbol{Z}_1\boldsymbol{\Sigma}_1\boldsymbol{Z}_1' + \boldsymbol{\Sigma}_2 + \sigma^2\boldsymbol{I}_p, \tag{3}$$

where $\boldsymbol{V}(\boldsymbol{\theta})$ is the covariance matrix with the vector of unknown parameters $\boldsymbol{\theta}$. The covariance matrix $\boldsymbol{V}(\boldsymbol{\theta})$ in (3) may have different structures depending on $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$. Here we will consider $\boldsymbol{V}(\boldsymbol{\theta})$ when the covariance matrix $\boldsymbol{\Sigma}_1 : n_2 \times n_2$ is compound symmetric, i.e.

$$\boldsymbol{\Sigma}_1 = \sigma_1\boldsymbol{I}_{n_2} + \sigma_2(\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}), \tag{4}$$

where σ_1 and σ_2 are unknown parameters. Furthermore, the covariance matrix $\boldsymbol{\Sigma}_2 : p \times p$, in (3), is assumed to have the following block compound symmetric pattern:

$$\boldsymbol{\Sigma}_2 = \boldsymbol{I}_{n_2} \otimes \boldsymbol{\Sigma}^{(1)} + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{\Sigma}^{(2)}, \tag{5}$$

where $\boldsymbol{\Sigma}^{(h)}$ is a CT matrix, $h = 1, 2$. Spectral properties of $\boldsymbol{\Sigma}_1$, $\boldsymbol{Z}_1\boldsymbol{\Sigma}_1\boldsymbol{Z}_1'$ and $\boldsymbol{\Sigma}_2$ have been derived in Liang *et al.* (2012), which will be used later in Section 4 when discussing test for the parameters $\boldsymbol{\theta}$ of $\boldsymbol{V}(\boldsymbol{\theta})$ in (3).

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3. Testing block covariance structures

In this section we are going to derive LRT statistics for testing the null hypotheses H_i^0 versus the alternative hypotheses H_i^a, $i = 1, \ldots, 7$ defined in Section 2.

3.1. Testing simultaneous hypotheses

The LRT statistics corresponding to test H_i^0 versus H_i^a, $i = 1, 2$, are defined as

$$\Lambda_1 = \frac{\max L(\mu, \boldsymbol{\Sigma}_{II})}{\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_I)}, \quad \Lambda_2 = \frac{\max L(\mu, \boldsymbol{\Sigma}_{III})}{\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_I)}.$$

We start with defining an orthogonal matrix (known as Helmert matrix, see Lancaster, 1965) $\boldsymbol{K} : n_2 \times n_2$ such that

$$\boldsymbol{K} = (n_2^{-1/2} \mathbf{1}_{n_2} \vdots \boldsymbol{K}_1), \tag{6}$$

where $\boldsymbol{K}_1' \mathbf{1}_{n_2} = \mathbf{0}$ and $\boldsymbol{K}_1' \boldsymbol{K}_1 = \boldsymbol{I}_{n_2-1}$. Moreover, an orthogonal matrix which contains the known orthonormal eigenvectors, $\boldsymbol{v}_j$, $j = 1, \ldots, n_1$, of any $n_1 \times n_1$ CT matrix is denoted $\boldsymbol{V}$, i.e. $\boldsymbol{V} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n_1})$. We also define the following matrices:

$$\boldsymbol{X}_1 = (n_2^{-1/2} \mathbf{1}_{n_2}' \otimes \boldsymbol{I}_{n_1}) \boldsymbol{Y}, \quad \boldsymbol{X}_2 = (\boldsymbol{K}_1' \otimes \boldsymbol{I}_{n_1}) \boldsymbol{Y}, \quad \boldsymbol{X}_2' = (\boldsymbol{X}_{21}', \ldots, \boldsymbol{X}_{2(n_2-1)}'),$$

$$\boldsymbol{S}_1 = \boldsymbol{X}_1 \boldsymbol{Q}_n \boldsymbol{X}_1', \quad \boldsymbol{S}_2 = \sum_{i=1}^{n_2-1} \boldsymbol{X}_{2i} \boldsymbol{X}_{2i}', \quad \boldsymbol{S}_3 = \boldsymbol{X}_1 \boldsymbol{P}_n \boldsymbol{X}_1'. \tag{7}$$

The next theorem establishes Λ_1.

Theorem 3.1. *The LRT statistic for testing H_1^0 versus H_1^a, defined in Section 2, is given by*

$$\Lambda_1^{2/n} = \frac{2^{n_2(n_1-1-\mathbb{1}_{\text{even}})} |\boldsymbol{S}_1| |\boldsymbol{S}_2|^{n_2-1}}{t_{21} \prod_{i=2}^{r} t_{1i}^{m_i} \prod_{i=r+1}^{2r} t_{2i}^{m_i}}, \tag{8}$$

where $\mathbb{1}_{\text{even}}$ is the indicator function that n_1 is even, matrices $\boldsymbol{S}_h$ are given in (7), $h = 1, 2$, $t_{1i} = \text{tr}((\boldsymbol{X}_1 \boldsymbol{X}_1')(\boldsymbol{v}_i \boldsymbol{v}_i' + \boldsymbol{v}_{n_1-i+2} \boldsymbol{v}_{n_1-i+2}'))$, $t_{21} = \text{tr}(\boldsymbol{S}_1 \boldsymbol{P}_{n_1})$, $t_{2,r+i} =$

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$\operatorname{tr}(\boldsymbol{S}_2(\boldsymbol{v}_i\boldsymbol{v}_i' + \boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}'))$, $i = 2, \ldots, r-1$, *and for* $i = r$,

$$
t_{1r} = \begin{cases} \operatorname{tr}((\boldsymbol{X}_1\boldsymbol{X}_1')(\boldsymbol{v}_r\boldsymbol{v}_r' + \boldsymbol{v}_{n_1-r+2}\boldsymbol{v}_{n_1-r+2}')), & \text{if } n_1 \text{ is odd,} \\ \operatorname{tr}((\boldsymbol{X}_1\boldsymbol{X}_1')(\boldsymbol{v}_r\boldsymbol{v}_r')), & \text{if } n_1 \text{ is even,} \end{cases}
$$

$$
t_{2,2r} = \begin{cases} \operatorname{tr}(\boldsymbol{S}_2(\boldsymbol{v}_r\boldsymbol{v}_r' + \boldsymbol{v}_{n_1-r+2}\boldsymbol{v}_{n_1-r+2}')), & \text{if } n_1 \text{ is odd,} \\ \operatorname{tr}(\boldsymbol{S}_2\boldsymbol{v}_r\boldsymbol{v}_r'), & \text{if } n_1 \text{ is even.} \end{cases}
$$

Proof. The likelihood function under the alternative hypothesis H_1^a is given by

$$
L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_I) = (2\pi)^{-\frac{pn}{2}} |\boldsymbol{\Sigma}_I|^{-\frac{n}{2}} e^{-\frac{1}{2}\operatorname{tr}\left\{\boldsymbol{\Sigma}_I^{-1}(\boldsymbol{Y}-(\mathbf{1}_{n_2}\otimes\boldsymbol{\mu})\mathbf{1}_n')(\boldsymbol{Y}-(\mathbf{1}_{n_2}\otimes\boldsymbol{\mu})\mathbf{1}_n')'\right\}}. \tag{9}
$$

Now it is convenient to use the reduction to a canonical form. Observe that $(\boldsymbol{X}_1'\vdots\boldsymbol{X}_2')' = (\boldsymbol{K}' \otimes \boldsymbol{I}_{n_1})\boldsymbol{Y}$, where $\boldsymbol{K}$ has been defined in (6). The expectations of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ equal

$$
(n_2^{-1/2}\mathbf{1}_{n_2}' \otimes \boldsymbol{I}_{n_1})(\mathbf{1}_{n_2} \otimes \boldsymbol{\mu})\mathbf{1}_n' = \sqrt{n_2}\boldsymbol{\mu}\mathbf{1}_n', \quad (\boldsymbol{K}_1' \otimes \boldsymbol{I}_{n_1})(\mathbf{1}_{n_2} \otimes \boldsymbol{\mu})\mathbf{1}_n' = \mathbf{0},
$$

respectively. For the dispersion matrices of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, first observe that

$$
\boldsymbol{K}'\boldsymbol{J}_{n_2}\boldsymbol{K} = \begin{pmatrix} n_2 & 0 \\ 0 & \mathbf{0}_{n_2-1} \end{pmatrix},
$$

and then using the properties of Kronecker product yields

$$
(\boldsymbol{K}' \otimes \boldsymbol{I}_{n_1})\boldsymbol{\Sigma}_I(\boldsymbol{K} \otimes \boldsymbol{I}_{n_1}) = \begin{pmatrix} \boldsymbol{\Sigma}^{(1)} + (n_2 - 1)\boldsymbol{\Sigma}^{(2)} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{n_2-1} \otimes (\boldsymbol{\Sigma}^{(1)} - \boldsymbol{\Sigma}^{(2)}) \end{pmatrix}.
$$

Thus, we have the following canonical form:

$$
\begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{pmatrix} \sim N_{p,n}\left[\begin{pmatrix} \sqrt{n_2}\boldsymbol{\mu}\mathbf{1}_n' \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Delta}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{n_2-1} \otimes \boldsymbol{\Delta}_2 \end{pmatrix} \right], \tag{10}
$$

where $\boldsymbol{\Delta}_1 = \boldsymbol{\Sigma}^{(1)} + (n_2 - 1)\boldsymbol{\Sigma}^{(2)}$ and $\boldsymbol{\Delta}_2 = \boldsymbol{\Sigma}^{(1)} - \boldsymbol{\Sigma}^{(2)}$. Moreover, since the co-variance between $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ equals $\mathbf{0}$, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independently distributed.

Let $L_1(\boldsymbol{\mu}, \boldsymbol{\Delta}_1)$ and $L_2(\boldsymbol{\Delta}_2)$ be the two likelihood functions corresponding to $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, respectively. Using (10), equation (9) can be rewritten as

$$
\begin{aligned}
L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_I) &= L_1(\boldsymbol{\mu}, \boldsymbol{\Delta}_1) L_2(\boldsymbol{\Delta}_2) \\
&= (2\pi)^{-\frac{n_1 n}{2}} |\boldsymbol{\Delta}_1|^{-\frac{n}{2}} e^{-\frac{1}{2}\text{tr}\left\{\boldsymbol{\Delta}_1^{-1}(\boldsymbol{X}_1 - \sqrt{n_2}\boldsymbol{\mu}\mathbf{1}_n')(\boldsymbol{X}_1 - \sqrt{n_2}\boldsymbol{\mu}\mathbf{1}_n')'\right\}} \\
&\quad \times (2\pi)^{-\frac{n_1(n_2-1)n}{2}} |\boldsymbol{\Delta}_2|^{-\frac{n(n_2-1)}{2}} e^{-\frac{1}{2}\text{tr}\left\{\boldsymbol{\Delta}_2^{-1}\sum_{i=1}^{n_2-1}\boldsymbol{X}_{2i}\boldsymbol{X}_{2i}'\right\}}.
\end{aligned}
\tag{11}
$$

From (11) it can be observed that the likelihood function in (9) is identical to the likelihood functions of two independent MANOVA (multivariate analysis of variance) models where in one the mean equals $\boldsymbol{0}$. Hence, the MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Delta}_1$ equal

$$
\sqrt{n_2}\hat{\boldsymbol{\mu}} = \frac{1}{n}\boldsymbol{X}_1\mathbf{1}_n, \quad \hat{\boldsymbol{\Delta}}_1 = \frac{1}{n}\boldsymbol{X}_1\boldsymbol{Q}_n\boldsymbol{X}_1' = \frac{1}{n}\boldsymbol{S}_1.
\tag{12}
$$

The MLE of $\boldsymbol{\Delta}_2$ is given by

$$
\hat{\boldsymbol{\Delta}}_2 = \frac{1}{n(n_2-1)}\sum_{i=1}^{n_2-1}\boldsymbol{X}_{2i}\boldsymbol{X}_{2i}' = \frac{1}{n(n_2-1)}\boldsymbol{S}_2.
$$

Hence, the likelihood function in (11) is maximized by replacing $\boldsymbol{\mu}$, $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ with their corresponding MLEs, i.e.,

$$
\max L(\boldsymbol{\mu}, \boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2) = L(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Delta}}_1, \hat{\boldsymbol{\Delta}}_2) = (2\pi)^{-\frac{pn}{2}} \left|\frac{\boldsymbol{S}_2}{n(n_2-1)}\right|^{-\frac{n(n_2-1)}{2}} \left|\frac{\boldsymbol{S}_1}{n}\right|^{-\frac{n}{2}} e^{-\frac{pn}{2}}.
\tag{13}
$$

The likelihood function under H_1^0, i.e. $\boldsymbol{Y} \sim N_{p,n}(\mu\mathbf{1}_p\mathbf{1}_n', \boldsymbol{\Sigma}_{II}, \boldsymbol{I}_n)$ can be written as

$$
L(\mu, \boldsymbol{\Sigma}_{II}) = (2\pi)^{-\frac{pn}{2}} |\boldsymbol{\Sigma}_{II}|^{-\frac{n}{2}} e^{-\frac{1}{2}\text{tr}\left\{\boldsymbol{\Sigma}_{II}^{-1}(\boldsymbol{Y} - \mu\mathbf{1}_p\mathbf{1}_n')(\boldsymbol{Y} - \mu\mathbf{1}_p\mathbf{1}_n')'\right\}}.
\tag{14}
$$

Furthermore,

$$
(\boldsymbol{I}_n \otimes \boldsymbol{K}' \otimes \boldsymbol{V}')\text{vec}\boldsymbol{Y} \sim N_{pn}\left[(\boldsymbol{I}_n \otimes \boldsymbol{K}'\mathbf{1}_{n_2} \otimes \boldsymbol{V}'\mathbf{1}_{n_1})\mu, \boldsymbol{I}_n \otimes \boldsymbol{D}(\boldsymbol{\eta})\right],
$$

where $\boldsymbol{K}'\mathbf{1}_{n_2} = (\sqrt{n_2}, 0, \ldots, 0)'$ and $\boldsymbol{V}'\mathbf{1}_{n_1} = (\sqrt{n_1}, 0, \ldots, 0)'$. The last equality follows because $\boldsymbol{K} \otimes \boldsymbol{V}$ is the matrix of eigenvectors of $\boldsymbol{\Sigma}_{II}$, i.e.,

$$
\boldsymbol{\Sigma}_{II} = (\boldsymbol{K} \otimes \boldsymbol{V})\boldsymbol{D}(\boldsymbol{\eta})(\boldsymbol{K}' \otimes \boldsymbol{V}'),
$$

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where $\boldsymbol{D}(\boldsymbol{\eta})$ is a $p \times p$ diagonal matrix with the eigenvalues of $\boldsymbol{\Sigma}_{II}$ (Liang et $al.$, 2012). Put $\boldsymbol{w}_i = (\boldsymbol{I}_n \otimes \boldsymbol{u}_i')\mathrm{vec}\boldsymbol{Y}$ and $\boldsymbol{u}_i$ is the ith column of $\boldsymbol{K} \otimes \boldsymbol{V}$, $i = 1, \ldots, p$. It follows that the model can be split into $2r$ independent models (Liang et $al.$, 2014, proof of Proposition 1), where $r = [n_1/2] + 1$ and $[\bullet]$ denotes the integer part. Each model has the following response vector:

$$\tilde{\mathbf{y}}_1 = \boldsymbol{w}_1, \quad \tilde{\mathbf{y}}_i = \mathrm{vec}(\boldsymbol{w}_i, \boldsymbol{w}_{n_1-i+2});$$

$$\tilde{\mathbf{y}}_r = \begin{cases} \mathrm{vec}(\boldsymbol{w}_r, \boldsymbol{w}_{n_1-r+2}), & \text{if } n_1 \text{ is odd}, \\ \boldsymbol{w}_r, & \text{if } n_1 \text{ is even}; \end{cases}$$

$$\tilde{\mathbf{y}}_{r+1} = \mathrm{vec}(\boldsymbol{w}_{n_1+1}, \boldsymbol{w}_{2n_1+1}, \ldots, \boldsymbol{w}_{(n_2-1)n_1+1}); \tag{15}$$

$$\tilde{\mathbf{y}}_{r+i} = \mathrm{vec}(\boldsymbol{w}_{n_1+i}, \boldsymbol{w}_{2n_1-i+2}, \ldots, \boldsymbol{w}_{(n_2-1)n_1+i}, \boldsymbol{w}_{n_2n_1-i+2}), \quad i = 2, \ldots, r-1;$$

$$\tilde{\mathbf{y}}_{2r} = \begin{cases} \mathrm{vec}(\boldsymbol{w}_{n_1+r}, \boldsymbol{w}_{2n_1-r+2}, \ldots, \boldsymbol{w}_{(n_2-1)n_1+r}, \boldsymbol{w}_{n_2n_1-r+2}), & \text{if } n_1 \text{ is odd}, \\ \mathrm{vec}(\boldsymbol{w}_{n_1+r}, \boldsymbol{w}_{2n_1+r}, \ldots, \boldsymbol{w}_{(n_2-1)n_1+r}), & \text{if } n_1 \text{ is even}. \end{cases}$$

It can be shown that $\tilde{\mathbf{y}}_1 \sim N_n(\sqrt{p}\mu\mathbf{1}_n, \eta_1\boldsymbol{I}_n)$ and $\tilde{\mathbf{y}}_l \sim N_{nm_l}(0, \eta_i\boldsymbol{I}_{nm_l})$, $l = 2, \ldots, 2r$, where η_l are the distinct eigenvalue of $\boldsymbol{\Sigma}_{II}$, of multiplicity m_l, $l = 1, \ldots, 2r$. Thus, the MLEs of μ and $\boldsymbol{\eta}$ are given by

$$\sqrt{p}\hat{\mu} = \frac{1}{n}\tilde{\mathbf{y}}_1'\mathbf{1}_n, \quad \hat{\eta}_1 = \frac{1}{n}\tilde{\mathbf{y}}_1'\boldsymbol{Q}_n\tilde{\mathbf{y}}_1, \quad \hat{\eta}_l = \frac{1}{nm_l}\tilde{\mathbf{y}}_l'\tilde{\mathbf{y}}_l, \quad l = 2, \ldots, 2r. \tag{16}$$

In order to related the maximum of the likelihoods under H_1^0 and H_1^a, we should rewrite the MLEs of η_l in terms of the matrices $\boldsymbol{S}_h$, $h = 1, 2, 3$. Since $\boldsymbol{w}_i = (\boldsymbol{I}_n \otimes \boldsymbol{u}_i')\mathrm{vec}\boldsymbol{Y} = \boldsymbol{Y}'\boldsymbol{u}_i$, i.e. $\boldsymbol{w}_i = \boldsymbol{Y}'(1/\sqrt{n_2}\mathbf{1}_{n_2} \otimes \boldsymbol{v}_i)$, and $\boldsymbol{w}_{jn_1+i} = \boldsymbol{Y}'(\boldsymbol{k}_j \otimes \boldsymbol{v}_i)$ (Liang et $al.$, 2012, Theorem 2.4), where $\boldsymbol{k}_j$ is the jth column of the matrix $\boldsymbol{K}_1$, $i = 1, \ldots, n_1, j = 1, \ldots, n_2 - 1$, after some manipulations we get the following expressions for $\hat{\eta}_l$:

$$\hat{\eta}_1 = \frac{1}{n}\tilde{\mathbf{y}}_1'\boldsymbol{Q}_n\tilde{\mathbf{y}}_1 = \frac{1}{n}\boldsymbol{u}_1'\boldsymbol{Y}\boldsymbol{Q}_n\boldsymbol{Y}'\boldsymbol{u}_1 = \frac{1}{n}\boldsymbol{v}_1'\boldsymbol{X}_1\boldsymbol{Q}_n\boldsymbol{X}_1'\boldsymbol{v}_1 = \frac{1}{n}\mathrm{tr}(\boldsymbol{S}_1\boldsymbol{P}_{n_1}), \tag{17}$$

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with the multiplicity $m_1 = 1$;

$$\hat{\eta}_{r+1} = \frac{1}{n(n_2 - 1)}\tilde{\boldsymbol{y}}'_{r+1}\tilde{\boldsymbol{y}}_{r+1} = \frac{1}{n(n_2 - 1)}\sum_{j=1}^{n_2-1}\boldsymbol{w}'_{jn_1+1}\boldsymbol{w}_{jn_1+1}$$

$$= \frac{1}{n(n_2 - 1)}\boldsymbol{v}'_1\sum_{j=1}^{n_2-1}\boldsymbol{X}_{2j}\boldsymbol{X}'_{2j}\boldsymbol{v}_1 = \frac{1}{n(n_2 - 1)}\text{tr}\left(\boldsymbol{S}_2\boldsymbol{P}_{n_1}\right), \qquad (18)$$

with $m_{r+1} = n_2 - 1$.

For $i = 2, \ldots, r - 1$, we have $m_i = 2$, and

$$\hat{\eta}_i = \frac{1}{2n}\tilde{\boldsymbol{y}}'_i\tilde{\boldsymbol{y}}_i = \frac{1}{2n}(\boldsymbol{w}'_i\boldsymbol{w}_i + \boldsymbol{w}'_{n_1-i+2}\boldsymbol{w}_{n_1-i+2})$$

$$= \frac{1}{2n}(\boldsymbol{v}'_i\boldsymbol{X}_1\boldsymbol{X}'_1\boldsymbol{v}_i + \boldsymbol{v}'_{n_1-i+2}\boldsymbol{X}_1\boldsymbol{X}'_1\boldsymbol{v}_{n_1-i+2})$$

$$= \frac{1}{2n}\text{tr}\left((\boldsymbol{S}_1 + \boldsymbol{S}_3)(\boldsymbol{v}_i\boldsymbol{v}'_i + \boldsymbol{v}_{n_1-i+2}\boldsymbol{v}'_{n_1-i+2})\right); \qquad (19)$$

$$\hat{\eta}_{r+i} = \frac{1}{2n(n_2 - 1)}\tilde{\boldsymbol{y}}'_i\tilde{\boldsymbol{y}}_i = \frac{1}{2n(n_2 - 1)}\sum_{j=1}^{n_2-1}(\boldsymbol{w}'_{jn_1+i}\boldsymbol{w}_{jn_1+i} + \boldsymbol{w}'_{(j+1)n_1-i+2}\boldsymbol{w}_{(j+1)n_1-i+2})$$

$$= \frac{1}{2n(n_2 - 1)}\sum_{j=1}^{n_2-1}\left[(\boldsymbol{k}'_j \otimes \boldsymbol{v}'_i)\boldsymbol{Y}\boldsymbol{Y}'(\boldsymbol{k}_j \otimes \boldsymbol{v}_i) + (\boldsymbol{k}'_j \otimes \boldsymbol{v}'_{n_1-i+2})\boldsymbol{Y}\boldsymbol{Y}'(\boldsymbol{k}_j \otimes \boldsymbol{v}_{n_1-i+2})\right]$$

$$= \frac{1}{2n(n_2 - 1)}\text{tr}\left(\boldsymbol{S}_2(\boldsymbol{v}_i\boldsymbol{v}'_i + \boldsymbol{v}_{n_1-i+2}\boldsymbol{v}'_{n_1-i+2})\right), \qquad (20)$$

with $m_{r+i} = 2(n_2 - 1)$.

Similarly, when n_1 is odd, we have $m_r = 2$ and $m_{2r} = 2(n_2 - 1)$; when n_1 is even, $m_r = 1$ and $m_{2r} = n_2 - 1$. Then,

$$\hat{\eta}_r = \begin{cases} \frac{1}{2n}\text{tr}\left((\boldsymbol{S}_1 + \boldsymbol{S}_3)(\boldsymbol{v}_r\boldsymbol{v}'_r + \boldsymbol{v}_{n_1-r+2}\boldsymbol{v}'_{n_1-r+2})\right), & \text{if } n_1 \text{ is odd,} \\ \frac{1}{n}\text{tr}\left((\boldsymbol{S}_1 + \boldsymbol{S}_3)(\boldsymbol{v}_r\boldsymbol{v}'_r)\right), & \text{if } n_1 \text{ is even.} \end{cases}$$

$$\hat{\eta}_{2r} = \begin{cases} \frac{1}{2n(n_2-1)}\text{tr}\left(\boldsymbol{S}_2(\boldsymbol{v}_r\boldsymbol{v}'_r + \boldsymbol{v}_{n_1-r+2}\boldsymbol{v}'_{n_1-r+2})\right), & \text{if } n_1 \text{ is odd,} \\ \frac{1}{n(n_2-1)}\text{tr}\left(\boldsymbol{S}_2(\boldsymbol{v}_r\boldsymbol{v}'_r)\right), & \text{if } n_1 \text{ is even.} \end{cases} \qquad (21)$$

Therefore, the likelihood function in (14) is maximized by

$$\max L(\mu, \boldsymbol{\Sigma}_{II}) = (2\pi)^{-\frac{pn}{2}}\prod_{i=1}^{2r}\hat{\eta}_i^{-\frac{nm_i}{2}}e^{-\frac{pn}{2}}, \qquad (22)$$

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where $\hat{\eta}_i$, $i = 1, \ldots, 2r$ is given in (17), (18), (19), (20) and (21). Using the expressions given in (13) and (22), Λ_1 has the form given in (8) and this completes the proof. ∎

Next we will show that under the null hypothesis H_1^0, the distribution of Λ_1 has the same distribution as a product of independent Beta random variables. The result is given in Theorem 3.4. We will use the notation "$\overset{d}{\sim}$" to indicate that "has the same distribution (of)" and $\beta(a, b)$ indicates a Beta random variable with parameters a and b. We first present two auxiliary lemmas which will be used in the subsequent proof.

Lemma 3.2. *(Muirhead, 1982, Theorem 3.2.15, p.100) If $\boldsymbol{A}$ is $W_m(n, \boldsymbol{\Sigma})$, where $n \geq m$ is an integer, then $|\boldsymbol{A}| \, / \, |\boldsymbol{\Sigma}|$ has the same distribution as the product of m independent χ^2 random variables, $\prod_{i=1}^m \chi_{n-i+1}^2$.*

Lemma 3.3. *(Olkin and Press, 1969, Lemma 2) Let $W_0, W_1, \ldots, W_m$ be independently distributed, $W_j \sim \chi_{2a_j}^2$, $j = 0, 1, \ldots, m$. If*

$$L = m^m \frac{\prod_{j=1}^m W_j}{(W_0 + \sum_{j=1}^m W_j)^m},$$

then $L \overset{d}{\sim} \prod_{j=1}^m X_j$, where $X_1, \ldots, X_m$ are independently distributed,

$$X_j \sim \beta\left(a_j, b_j\right), \quad a = \sum_{j=0}^m a_j, \quad b_j = \frac{a + j - 1}{m} - a_j.$$

Theorem 3.4. *Under the null hypothesis H_1^0, the distribution of Λ_1, given in (8), follows*

$$\Lambda_1^{2/n} \overset{d}{\sim} \prod_{i=1}^{n_1-1} B_{1i} B_{2i}^{n_2-1}, \tag{23}$$

where B_{1i} and B_{2i} are independently distributed, $i = 1, \ldots, n_1 - 1$,

$$B_{1i} \sim \begin{cases} \beta(\frac{n-i-1}{2}, \frac{i+1}{2}), & \textit{for } i = 1, \ldots, [n_1/2], \\ \beta(\frac{n-i-1}{2}, \frac{i+2}{2}), & \textit{for } i = [n_1/2] + 1, \ldots, n_1 - 1, \end{cases}$$

$$B_{2i} \sim \begin{cases} \beta(\frac{n(n_2-1)-i}{2}, \frac{i}{2}), & \textit{for } i = 1, \ldots, [n_1/2], \\ \beta(\frac{n(n_2-1)-i}{2}, \frac{i+1}{2}), & \textit{for } i = [n_1/2] + 1, \ldots, n_1 - 1. \end{cases}$$

Proof. To derive the distribution of Λ_1, we first need the distributions of $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_3$. Since $\boldsymbol{X}_1 \sim N_{n_1,n}(\sqrt{n_2}\boldsymbol{\mu}\boldsymbol{1}'_n, \boldsymbol{\Delta}_1, \boldsymbol{I}_n)$ and $(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n)$ is idempotent of rank $n - 1$, we have (Kollo and von Rosen, 2005, Corollary 2.4.3.1, p.240) $\boldsymbol{S}_1 \sim W_{n_1}(\boldsymbol{\Delta}_1, n - 1)$. By Definition 2.4.1 in Kollo and von Rosen (2005), $\boldsymbol{S}_3$ has a non-central Wishart distribution with the non-central parameter $n_2 n^2 \boldsymbol{\mu}\boldsymbol{\mu}'$, i.e., $\boldsymbol{S}_3 \sim W_{n_1}(\boldsymbol{\Delta}_1, 1, n_2 n^2 \boldsymbol{\mu}\boldsymbol{\mu}')$, and since $\boldsymbol{X}_{2i} \sim N_{n_1,n}(\boldsymbol{0}, \boldsymbol{\Delta}_2, \boldsymbol{I}_n)$, we have $\boldsymbol{X}_{2i}\boldsymbol{X}'_{2i} \sim W_{n_1}(\boldsymbol{\Delta}_2, n)$, $i = 1, \ldots, n_2 - 1$. Based on the property of the sum of independent Wishart variables (Anderson, 2003, Theorem 7.3.2, p.260), we have $\boldsymbol{S}_2 \sim W_{n_1}(\boldsymbol{\Delta}_2, n(n_2 - 1))$.

Using the results of Theorem 2.4. in Vaish and Chaganty (2004), it can be found that $\boldsymbol{S}_1$ and $\boldsymbol{S}_3$ are independent. Moreover, $\boldsymbol{S}_1$ and $\boldsymbol{S}_3$ are independent of $\boldsymbol{S}_2$ since they are the statistics from two separate MANOVA models, see equation (11). Now we start to derive the distribution of Λ_1. Recall that the orthogonal matrix $\boldsymbol{V}$ contains the orthonormal eigenvectors of the CT matrix of size $n_1 \times n_1$, we know that $|\boldsymbol{V}'\boldsymbol{S}_k\boldsymbol{V}| = |\boldsymbol{S}_k|$ holds, $k = 1, 2$. The statistics t_{21}, t_{1i} ($i = 2, \ldots, r$) and $t_{2,r+i}$ ($i = 1, \ldots, r$) are nothing but some functions of the ith diagonal element of the matrices $\boldsymbol{V}'\boldsymbol{S}_h\boldsymbol{V}$, $h = 1, \ldots, 3$. When n_1 is odd, based on Theorem 3.1, $\Lambda_1^{2/n}$ can be written as follows:

$$\frac{2^{n_2(n_1-1)}\left|\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V}\right|}{(\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{11} \prod_{i=2}^r \left[(\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{ii} + (\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{n_1-i+2,n_1-i+2} + (\boldsymbol{V}'\boldsymbol{S}_3\boldsymbol{V})_{ii} + (\boldsymbol{V}'\boldsymbol{S}_3\boldsymbol{V})_{n_1-i+2,n_1-i+2}\right]^2}$$
$$\times \left(\frac{\left|\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V}\right|}{(\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{11} \prod_{i=2}^r \left[(\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{ii} + (\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{n_1-i+2,n_1-i+2}\right]^2}\right)^{n_2-1},$$

$$(24)$$

where $(\boldsymbol{V}'\boldsymbol{S}_h\boldsymbol{V})_{ii}$ is the ith diagonal element of the matrix $\boldsymbol{V}'\boldsymbol{S}_h\boldsymbol{V}$. Due to the mutual independence of $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_3$, equation (24) actually contains two independent components

$$\frac{|\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V}|}{(\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{11}\prod_{i=2}^r\left[(\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{ii}+(\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{n_1-i+2,n_1-i+2}+(\boldsymbol{V}'\boldsymbol{S}_3\boldsymbol{V})_{ii}+(\boldsymbol{V}'\boldsymbol{S}_3\boldsymbol{V})_{n_1-i+2,n_1-i+2}\right]^2}$$

and

$$\left(\frac{|\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V}|}{(\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{11}\prod_{i=2}^r\left[(\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{ii}+(\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{n_1-i+2,n_1-i+2}\right]^2}\right)^{n_2-1}.$$

We first apply Barlett decomposition (Muirhead, 1982, Theorem 3.2.14, p.99). Decompose $\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V}=\boldsymbol{U}\boldsymbol{U}'$ and $\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V}=\boldsymbol{L}\boldsymbol{L}'$, where $\boldsymbol{U}$ and $\boldsymbol{L}$ are two different lower triangular matrices with positive diagonal elements and under the null hypothesis, we have $u_{11}^2\sim\eta_1\chi_{n-1}^2$, $u_{ii}^2\sim\eta_i\chi_{n-i}^2$, $u_{n_1-i+2,n_1-i+2}^2\sim\eta_i\chi_{n-(n_1-i+2)}^2$, $\sum_{\alpha=1}^{i-1}u_{i\alpha}^2\sim\eta_i\chi_{i-1}^2$ and $\sum_{\alpha=1}^{n_1-i+1}u_{n_1-i+2,\alpha}^2\sim\eta_i\chi_{n_1-i+1}^2$, for $i=2,\ldots,r$ and they are all independent. Similarly, it is noted that

$$l_{11}^2\sim\eta_{r+1}\chi_{n(n_2-1)}^2,\quad l_{ii}^2\sim\eta_{r+i}\chi_{n(n_2-1)-i+1}^2,\quad\sum_{\alpha=1}^{i-1}l_{i\alpha}^2\sim\eta_{r+i}\chi_{i-1}^2,$$

$$l_{n_1-i+2,n_1-i+2}^2\sim\eta_{r+i}\chi_{n(n_2-1)-(n_1-i+2)+1}^2,\quad\sum_{\alpha=1}^{n_1-i+1}l_{n_1-i+2,\alpha}^2\sim\eta_{r+i}\chi_{n_1-i+1}^2,$$

and mutual independence also holds. Then,

$$|\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V}|=\prod_{i=1}^{n_1}u_{ii}^2=u_{11}^2\prod_{i=2}^r u_{ii}^2 u_{n_1-i+2,n_1-i+2}^2,$$

$$|\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V}|=\prod_{i=1}^{n_1}l_{ii}^2=l_{11}^2\prod_{i=2}^r l_{ii}^2 l_{n_1-i+2,n_1-i+2}^2,$$

and under the null hypothesis H_1^0,

$$\prod_{i=1}^{n_1}u_{ii}^2\overset{d}{\sim}\eta_1\prod_{i=2}^r\eta_i^2\prod_{i=1}^{n_1}Z_{1i},\quad\prod_{i=1}^{n_1}l_{ii}^2\overset{d}{\sim}\eta_{r+1}\prod_{i=r+2}^{2r}\eta_i^2\prod_{i=1}^{n_1}Z_{2i},$$

where $Z_{1i}\sim\chi_{n-i}^2$, $Z_{2i}\sim\chi_{n(n_2-1)-i+1}^2$ are independently distributed, $i=1,\ldots,n_1$. Moreover, $(\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{11}=u_{11}^2$, $(\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{ii}=u_{ii}^2+\sum_{\alpha=1}^{i-1}u_{i\alpha}^2$, $(\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{11}=l_{11}^2$,

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$(\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{ii} = l_{ii}^2 + \sum_{\alpha=1}^{i-1} l_{i\alpha}^2$, and all u_{ij} and l_{ij} $(1 \le j \le i \le n_1)$ are independently distributed. Moreover, under H_1^0, $\boldsymbol{v}_i'\boldsymbol{S}_3\boldsymbol{v}_i$ and $\boldsymbol{v}_{n_1-i+2}'\boldsymbol{S}_3\boldsymbol{v}_{n_1-i+2}$ are both independently χ_1^2 distributed with the scale parameter η_i, $i = 2, \ldots, r$.

Since they are also independent of $\sum_{\alpha=1}^{i-1} u_{i\alpha}^2$ and $\sum_{\alpha=1}^{n_1-i+1} u_{n_1-i+2,\alpha}^2$, we have

$$\sum_{\alpha=1}^{i-1} u_{i\alpha}^2 + \sum_{\alpha=1}^{n_1-i+1} u_{n_1-i+2,\alpha}^2 + \boldsymbol{v}_i'\boldsymbol{S}_3\boldsymbol{v}_i + \boldsymbol{v}_{n_1-i+2}'\boldsymbol{S}_3\boldsymbol{v}_{n_1-i+2} \sim \eta_i \chi_{n_1+2}^2.$$

Similarly, we have $\sum_{\alpha=1}^{i-1} l_{i\alpha}^2 + \sum_{\alpha=1}^{n_1-i+1} l_{n_1-i+2,\alpha}^2 \sim \eta_{r+i}\chi_{n_1}^2$. Hence, equation (24) can be written as

$$\prod_{i=2}^{r} \frac{2^2 u_{ii}^2 u_{n_1-i+2,n_1-i+2}^2}{(u_{ii}^2 + u_{n_1-i+2,n_1-i+2}^2 + \sum_{\alpha=1}^{i-1} u_{i\alpha}^2 + \sum_{\alpha=1}^{n_1-i+1} u_{n_1-i+2,\alpha}^2 + \boldsymbol{v}_i'\boldsymbol{S}_3\boldsymbol{v}_i + \boldsymbol{v}_{n_1-i+2}'\boldsymbol{S}_3\boldsymbol{v}_{n_1-i+2})^2}$$

$$\times \left[\prod_{i=2}^{r} \frac{2^2 l_{ii}^2 l_{n_1-i+2,n_1-i+2}^2}{(l_{ii}^2 + l_{n_1-i+2,n_1-i+2}^2 + \sum_{\alpha=1}^{i-1} l_{i\alpha}^2 + \sum_{\alpha=1}^{n_1-i+1} l_{n_1-i+2,\alpha}^2)^2} \right]^{n_2-1}, \tag{25}$$

and its distribution can be illustrated:

$$\prod_{i=2}^{r} \frac{2^2 \eta_i^2 \chi_{n-i}^2 \chi_{n-(n_1-i+2)}^2}{\eta_i^2(\chi_{n_1+2}^2 + \chi_{n-i}^2 + \chi_{n-(n_1-i+2)}^2)^2} \left[\prod_{i=2}^{r} \frac{2^2 \eta_{r+i}^2 \chi_{n(n_2-1)-i+1}^2 \chi_{n(n_2-1)-(n_1-i+2)+1}^2}{\eta_{r+i}^2(\chi_{n_1}^2 + \chi_{n(n_2-1)-i+1}^2 + \chi_{n(n_2-1)-(n_1-i+2)+1}^2)^2} \right]^{n_2-1}.$$

Then we can use Lemma 3.3 to obtain the distribution of Λ_1. Put $W_{0u} = \sum_{\alpha=1}^{i-1} u_{i\alpha}^2 + \sum_{\alpha=1}^{n_1-i+1} u_{n_1-i+2,\alpha}^2 + \boldsymbol{v}_i'\boldsymbol{S}_3\boldsymbol{v}_i + \boldsymbol{v}_{n_1-i+2}'\boldsymbol{S}_3\boldsymbol{v}_{n_1-i+2}$, $W_{1u} = u_{ii}^2$ and $W_{2u} = u_{n_1-i+2,n_1-i+2}^2$ and in a similar way let $W_{0l} = \sum_{\alpha=1}^{i-1} l_{i\alpha}^2 + \sum_{\alpha=1}^{n_1-i+1} l_{n_1-i+2,\alpha}^2$, $W_{1l} = l_{ii}^2$ and $W_{2l} = l_{n_1-i+2,n_1-i+2}^2$. Then, (25) has the following form

$$\prod_{i=2}^{r} \frac{2^2 W_{1u} W_{2u}}{(W_{0u} + W_{1u} + W_{2u})^2} \left[\prod_{i=2}^{r} \frac{2^2 W_{1l} W_{2l}}{(W_{0l} + W_{1l} + W_{2l})^2} \right]^{n_2-1}.$$

Thus Lemma 3.3 yields that (25) is distributed as

$$\prod_{i=2}^{r} \beta\left(\frac{n-i}{2}, \frac{i}{2}\right) \beta\left(\frac{n-(n_1-i+2)}{2}, \frac{n_1-i+3}{2}\right)$$

$$\times \left[\prod_{i=2}^{r} \beta\left(\frac{n(n_2-1)-i+1}{2}, \frac{i-1}{2}\right) \beta\left(\frac{n(n_2-1)-(n_1-i+2)+1}{2}, \frac{n_1-i+2}{2}\right) \right]^{n_2-1},$$

which gives the result (23).

When n_1 is even, the distribution of $\Lambda_1^{2/n}$ can be obtained using exactly the same ideas and manipulations. ∎

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In the next theorem we are going to derive the expression of Λ_2. Let us first give a lemma which will be used in the proof (for the proof of the lemma we refer to Nahtman, 2006).

Lemma 3.5. *The matrix* $\boldsymbol{\Sigma}_{III} : p \times p$, $p = n_2 n_1$,

$$\boldsymbol{\Sigma}_{III} = \boldsymbol{I}_{n_2} \otimes \left[\sigma_{11}\boldsymbol{I}_{n_1} + \sigma_{12}(\boldsymbol{J}_{n_1} - \boldsymbol{I}_{n_1})\right] + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \left[\sigma_{21}\boldsymbol{I}_{n_1} + \sigma_{22}(\boldsymbol{J}_{n_1} - \boldsymbol{I}_{n_1})\right],$$

where σ_{h1}, $i = 1, 2$, *are constants, has the following four distinct eigenvalues*

$$\lambda_1 = \sigma_{11} + (n_1 - 1)\sigma_{12} + (n_2 - 1)\left[\sigma_{21} + (n_1 - 1)\sigma_{22}\right],$$
$$\lambda_2 = \sigma_{11} - \sigma_{12} + (n_2 - 1)\left(\sigma_{21} - \sigma_{22}\right),$$
$$\lambda_3 = \sigma_{11} + (n_1 - 1)\sigma_{12} - \left[\sigma_{21} + (n_1 - 1)\sigma_{22}\right],$$
$$\lambda_4 = \sigma_{11} - \sigma_{12} - \left(\sigma_{21} - \sigma_{22}\right),$$

of multiplicity $m_1 = 1$, $m_2 = n_1 - 1$, $m_3 = n_2 - 1$ *and* $m_4 = (n_2 - 1)(n_1 - 1)$, *respectively.*

Theorem 3.6. *The LRT statistic for testing* H_2^0 *versus* H_2^a, *defined in Section 2, is given by*

$$\Lambda_2^{2/n} = \frac{(n_1 - 1)^{n_2(n_1 - 1)} |\boldsymbol{S}_1| |\boldsymbol{S}_2|^{n_2 - 1}}{\prod_{\substack{j=1 \\ j \neq 2}}^{4} t_{3j}^{m_j} (t_{32} + \sum_{i=2}^{n_1} \boldsymbol{v}_i' \boldsymbol{S}_3 \boldsymbol{v}_i)^{n_1 - 1}} \tag{26}$$

where $\boldsymbol{S}_h$ *are given in* (7), $h = 1, 2, 3$, $m_j = 1, n_1 - 1, n_2 - 1, (n_2 - 1)(n_1 - 1)$ *for* $j = 1, \ldots, 4$, *respectively,* $t_{31} = \mathrm{tr}(\boldsymbol{S}_1 \boldsymbol{P}_{n_1})$, $t_{32} = \mathrm{tr}(\boldsymbol{S}_1 \boldsymbol{Q}_{n_1})$, $t_{33} = \mathrm{tr}(\boldsymbol{S}_2 \boldsymbol{P}_{n_1})$ *and* $t_{34} = \mathrm{tr}(\boldsymbol{S}_2 \boldsymbol{Q}_{n_1})$.

Proof. Since the maximum of the likelihood function under the alternative hypothesis H_2^a, i.e., $\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_I)$, has been given in (13), let us now consider the derivation of $\max L(\mu, \boldsymbol{\Sigma}_{III})$. The likelihood function under the null hypothesis H_2^0 is given by

$$L(\mu, \boldsymbol{\Sigma}_{III}) = (2\pi)^{-\frac{pn}{2}} |\boldsymbol{\Sigma}_{III}|^{-\frac{n}{2}} e^{-\frac{1}{2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_{III}^{-1}(\boldsymbol{Y} - \mu\mathbf{1}_p\mathbf{1}_n')(\boldsymbol{Y} - \mu\mathbf{1}_p\mathbf{1}_n')'\right\}}. \tag{27}$$

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Here it is convenient to use the canonical form with the help of Lemma 3.5. First,

$$(\boldsymbol{I}_n \otimes \boldsymbol{K}' \otimes \boldsymbol{V}')\mathrm{vec}\boldsymbol{Y} \sim N_{pn}\left[(\boldsymbol{I}_n \otimes \boldsymbol{K}'\boldsymbol{1}_{n_2} \otimes \boldsymbol{V}'\boldsymbol{1}_{n_1})\mu, \boldsymbol{I}_n \otimes (\boldsymbol{K}' \otimes \boldsymbol{V}')\boldsymbol{\Sigma}_{III}(\boldsymbol{K} \otimes \boldsymbol{V})\right],$$

which becomes $N_{pn}\left[(\boldsymbol{I}_n \otimes (\sqrt{n_2}, 0, \ldots, 0)' \otimes (\sqrt{n_1}, 0, \ldots, 0)')\mu, \boldsymbol{I}_n \otimes \boldsymbol{D}(\boldsymbol{\lambda}))\right]$, where $\boldsymbol{D}(\boldsymbol{\lambda})$ is the $p \times p$ diagonal matrix given in Lemma 3.5. It yields that the model can be split into 4 independent models and each model has the following response vector:

$$\tilde{\mathbf{y}}_1 = \boldsymbol{w}_1, \quad \tilde{\mathbf{y}}_2 = \mathrm{vec}(\boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n_1}), \quad \tilde{\mathbf{y}}_3 = \mathrm{vec}(\boldsymbol{w}_{n_1+1}, \ldots, \boldsymbol{w}_{(n_2-1)n_1+1}),$$
$$\tilde{\mathbf{y}}_4 = \mathrm{vec}(\boldsymbol{w}_{n_1+2}, \ldots, \boldsymbol{w}_{2n_1}, \boldsymbol{w}_{2n_1+2}, \ldots, \boldsymbol{w}_{3n_1}, \ldots, \boldsymbol{w}_{(n_2-1)n_1+2}, \ldots, \boldsymbol{w}_{n_2n_1}), \tag{28}$$

where $\boldsymbol{w}_i = (\boldsymbol{I}_n \otimes \boldsymbol{u}_i')\mathrm{vec}\boldsymbol{Y}$ and $\boldsymbol{u}_i$ is the ith column of $\boldsymbol{K} \otimes \boldsymbol{V}$, $i = 1, \ldots, p$. It can be shown that $\tilde{\mathbf{y}}_1 \sim N_n(\sqrt{p}\mu\boldsymbol{1}_n, \lambda_1\boldsymbol{I}_n)$ and $\tilde{\mathbf{y}}_j \sim N_{nm_j}(0, \lambda_j\boldsymbol{I}_{nm_j})$, for $j = 2, 3, 4$, where λ_j and m_j, $j = 1, \ldots, 4$ are given in Lemma 3.5. Thus, the MLEs of μ and $\boldsymbol{\lambda}$ are given by

$$\sqrt{p}\hat{\mu} = \frac{1}{n}\tilde{\mathbf{y}}_1'\boldsymbol{1}_n, \quad \hat{\lambda}_1 = \frac{1}{n}\tilde{\mathbf{y}}_1'\boldsymbol{Q}_n\tilde{\mathbf{y}}_1, \quad \hat{\lambda}_j = \frac{1}{nm_j}\tilde{\mathbf{y}}_j'\tilde{\mathbf{y}}_j, \quad j = 2, 3, 4. \tag{29}$$

Recall that there is a reformulation of the MLEs of $\boldsymbol{\eta}$ in the proof of Theorem 3.1. Here the expressions of $\boldsymbol{\lambda}$ in terms of $\boldsymbol{S}_h$, $h = 1, 2, 3$, are analogous. Observe that $\hat{\lambda}_1 = \hat{\eta}_1$ and $\hat{\lambda}_3 = \hat{\eta}_{r+1}$, where $\hat{\eta}_1$ and $\hat{\eta}_{r+1}$ are given in (16). As a result, $\hat{\lambda}_1$ and $\hat{\lambda}_3$ can be reexpressed as given in (17) and (18), respectively. Moreover, since

$$\tilde{\mathbf{y}}_2'\tilde{\mathbf{y}}_2 = \sum_{i=2}^{n_1} \boldsymbol{w}_i'\boldsymbol{w}_i = \sum_{i=2}^{n_1}(1/\sqrt{n_2}\boldsymbol{1}_{n_2}' \otimes \boldsymbol{v}_i')\boldsymbol{Y}\boldsymbol{Y}'(1/\sqrt{n_2}\boldsymbol{1}_{n_2} \otimes \boldsymbol{v}_i)$$
$$= \sum_{i=2}^{n_1} \boldsymbol{v}_i'\boldsymbol{X}_1\boldsymbol{X}_1'\boldsymbol{v}_i = \mathrm{tr}((\boldsymbol{S}_1 + \boldsymbol{S}_3)\boldsymbol{Q}_{n_1}),$$

$$\tilde{\mathbf{y}}_4'\tilde{\mathbf{y}}_4 = \sum_{j=1}^{n_2-1}\sum_{i=2}^{n_1} \boldsymbol{w}_{jn_1+i}'\boldsymbol{w}_{jn_1+i} = \sum_{j=1}^{n_2-1}\sum_{i=2}^{n_1}(\boldsymbol{k}_j' \otimes \boldsymbol{v}_i')\boldsymbol{Y}\boldsymbol{Y}'(\boldsymbol{k}_j \otimes \boldsymbol{v}_i)$$
$$= \sum_{i=2}^{n_1} \boldsymbol{v}_i'\left(\sum_{j=1}^{n_2-1} \boldsymbol{X}_{2j}\boldsymbol{X}_{2j}'\right)\boldsymbol{v}_i = \mathrm{tr}(\boldsymbol{S}_2\boldsymbol{Q}_{n_1}),$$

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the MLEs $\hat{\lambda}_2$ and $\hat{\lambda}_4$ can be rewritten as follows:

$$\hat{\lambda}_2 = \frac{1}{n(n_1-1)}\tilde{\mathbf{y}}_2'\tilde{\mathbf{y}}_2 = \frac{1}{n(n_1-1)}\mathrm{tr}((\mathbf{S}_1+\mathbf{S}_3)\mathbf{Q}_{n_1}), \tag{30}$$

$$\hat{\lambda}_4 = \frac{1}{n(n_2-1)(n_1-1)}\tilde{\mathbf{y}}_4'\tilde{\mathbf{y}}_4 = \frac{1}{n(n_2-1)(n_1-1)}\mathrm{tr}(\mathbf{S}_2\mathbf{Q}_{n_1}). \tag{31}$$

Hence, the maximum of the likelihood function in (27) is given by

$$\max L(\mu, \mathbf{\Sigma}_{III}) = (2\pi)^{-\frac{pn}{2}} \prod_{j=1}^{4} \hat{\lambda}_j^{-\frac{nm_j}{2}} e^{-\frac{pn}{2}}, \tag{32}$$

where $\hat{\lambda}_j$, $j = 1, \ldots, 4$, are given in (17), (30), (18) and (31). Using the expressions given in (13) and (32), Λ_2 has the form given in (26). This completes the proof. ∎

Theorem 3.7. *Under the null hypothesis H_2^0, the distribution of Λ_2, given in (26), equals*

$$\Lambda_2^{2/n} \overset{d}{\sim} \prod_{i=1}^{n_1-1} B_{1i} B_{2i}^{n_2-1}, \tag{33}$$

where B_{1i} and B_{2i} are independently distributed, $i = 1, \ldots, n_1 - 1$,

$$B_{1i} \sim \beta\left(\frac{n-i-1}{2}, \frac{i-1}{n_1-1} + \frac{i+1}{2}\right),$$
$$B_{2i} \sim \beta\left(\frac{n(n_2-1)-i}{2}, \frac{i-1}{n_1-1} + \frac{i}{2}\right).$$

Proof. The proof is similar to the one of Theorem 3.4. Based on Theorem 3.7, Λ_2 can be written as follows:

$$\Lambda_2^{2/n} = \frac{(n_1-1)^{n_2(n_1-1)} |\mathbf{V}'\mathbf{S}_1\mathbf{V}|}{(\mathbf{V}'\mathbf{S}_1\mathbf{V})_{11}\left[\sum_{i=2}^{n_1}(\mathbf{V}'\mathbf{S}_1\mathbf{V})_{ii} + \sum_{i=2}^{n_1}(\mathbf{V}'\mathbf{S}_3\mathbf{V})_{ii}\right]^{n_1-1}}$$
$$\times \left(\frac{|\mathbf{V}'\mathbf{S}_2\mathbf{V}|}{(\mathbf{V}'\mathbf{S}_2\mathbf{V})_{11}\left[\sum_{i=2}^{n_1}(\mathbf{V}'\mathbf{S}_2\mathbf{V})_{ii}\right]^{n_1-1}}\right)^{n_2-1}, \tag{34}$$

which contains two independent components $\dfrac{|\mathbf{V}'\mathbf{S}_1\mathbf{V}|}{(\mathbf{V}'\mathbf{S}_1\mathbf{V})_{11}\left[\sum_{i=2}^{n_1}(\mathbf{V}'\mathbf{S}_1\mathbf{V})_{ii}+\sum_{i=2}^{n_1}(\mathbf{V}'\mathbf{S}_3\mathbf{V})_{ii}\right]^{n_1-1}}$

and $\left(\dfrac{|\mathbf{V}'\mathbf{S}_2\mathbf{V}|}{(\mathbf{V}'\mathbf{S}_2\mathbf{V})_{11}\left[\sum_{i=2}^{n_1}(\mathbf{V}'\mathbf{S}_2\mathbf{V})_{ii}\right]^{n_1-1}}\right)^{n_2-1}$ because of the mutual independence of $\mathbf{S}_h$,

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$h = 1, 2, 3$. Applying the same Barlett decompositions as in the proof of Theorem 3.7, i.e., $\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V} = \boldsymbol{U}\boldsymbol{U}'$ and $\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V} = \boldsymbol{L}\boldsymbol{L}'$, where $\boldsymbol{U}$ and $\boldsymbol{L}$ are two different lower triangular matrices with positive diagonal elements. Under the null hypothesis, we have $u_{ii}^2 \sim \lambda_2 \chi_{n-i}^2$, $\sum_{\alpha=1}^{i-1} u_{i\alpha}^2 \sim \lambda_2 \chi_{i-1}^2$, $l_{ii}^2 \sim \lambda_4 \chi_{n(n_2-1)-i+1}^2$ and $\sum_{\alpha=1}^{i-1} l_{i\alpha}^2 \sim \lambda_4 \chi_{i-1}^2$, which are mutually independent. Then, $|\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V}| = \prod_{i=1}^{n_1} u_{ii}^2$ and $|\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V}| = \prod_{i=1}^{n_1} l_{ii}^2$. Moreover, under the null hypothesis, based on Lemma 3.2, we have

$$\prod_{i=1}^{n_1} u_{ii}^2 \overset{d}{\sim} \lambda_1 \lambda_2^{n_1-1} \prod_{i=1}^{n_1} Z_{1i} \quad \text{and} \quad \prod_{i=1}^{n_1} l_{ii}^2 \overset{d}{\sim} \lambda_3 \lambda_4^{n_1-1} \prod_{i=1}^{n_1} Z_{2i},$$

where $Z_{1i} \sim \chi_{n-i}^2$, $Z_{2i} \sim \chi_{n(n_2-1)-i+1}^2$ are independently distributed, $i = 1, \ldots, n_1$. Equation (34) can be written as

$$\frac{(n_1-1)^{n_1-1} \prod_{i=2}^{n_1} u_{ii}^2}{\left(\sum_{i=2}^{n_1}(u_{ii}^2 + \sum_{\alpha=1}^{i-1} u_{i\alpha}^2) + \sum_{i=2}^{n_1} \boldsymbol{v}_i'\boldsymbol{S}_3\boldsymbol{v}_i\right)^{n_1-1}} \left[\frac{(n_1-1)^{n_1-1} \prod_{i=2}^{n_1} l_{ii}^2}{\left(\sum_{i=2}^{n_1}(l_{ii}^2 + \sum_{\alpha=1}^{i-1} l_{i\alpha}^2)\right)^{n_1-1}}\right]^{n_2-1} \quad (35)$$

Moreover, $\sum_{i=2}^{n_1}(\sum_{\alpha=1}^{i-1} u_{i\alpha}^2 + \boldsymbol{v}_i'\boldsymbol{S}_3\boldsymbol{v}_i) \sim \lambda_2 \chi_{(n_1-1)i}^2$ and $\sum_{i=2}^{n_1} \sum_{\alpha=1}^{i-1} l_{i\alpha}^2 \sim \lambda_4 \chi_{(n_1-1)(i-1)}^2$ are independent. Defining $W_{0u} = \sum_{i=2}^{n_1}(\sum_{\alpha=1}^{i-1} u_{i\alpha}^2 + \boldsymbol{v}_i'\boldsymbol{S}_3\boldsymbol{v}_i)$ and $W_{iu} = u_{i-1,i-1}^2$, $i = 1, \ldots, n_1 - 1$. Similarly, put $W_{0l} = \sum_{i=2}^{n_1} \sum_{\alpha=1}^{i-1} l_{i\alpha}^2$ and $W_{il} = l_{i-1,i-1}^2$, $i = 1, \ldots, n_1 - 1$, the distribution of Λ_2 is obtained from Lemma 3.3. ∎

3.2. Testing hypotheses about means or patterned covariance matrices

In this section, we are going to derive the LRT statistics for testing patterned covariance matrices given a specific mean structure:

$$\Lambda_3 = \frac{\max L(\mu, \boldsymbol{\Sigma}_{III})}{\max L(\mu, \boldsymbol{\Sigma}_{II})}, \quad \Lambda_4 = \frac{\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{II})}{\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_I)}, \quad \Lambda_5 = \frac{\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{III})}{\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_I)},$$

Furthermore, we derive the LRT statistics for testing means given a specific covariance structure:

$$\Lambda_6 = \frac{\max L(\mu, \boldsymbol{\Sigma}_{II})}{\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{II})}, \quad \Lambda_7 = \frac{\max L(\mu, \boldsymbol{\Sigma}_{III})}{\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{III})}.$$

Theorem 3.8. *The LRT statistic for testing H_3^0 versus H_3^a, defined in Section 2, is given by*

$$\Lambda_3^{2/n} = \begin{cases} \left(\frac{n_1-1}{2}\right)^{n_2(n_1-1)} \frac{\prod_{i=2}^{r} t_{1i}^2}{\left(\sum_{i=2}^{r} t_{1i}\right)^{n_1-1}} \left(\frac{\prod_{i=2}^{r} t_{2,r+i}^2}{\left(\sum_{i=2}^{r} t_{2,r+i}\right)^{n_1-1}}\right)^{n_2-1}, & \text{if } n_1 \text{ is odd,} \\[3mm] \left(\frac{n_1-1}{2}\right)^{n_2(n_1-2)} \frac{t_{1r}\prod_{i=2}^{r-1} t_{1i}^2}{\left(\sum_{i=2}^{r} t_{1i}\right)^{n_1-1}} \left(\frac{t_{2,2r}\prod_{i=2}^{r-1} t_{2,r+i}^2}{\left(\sum_{i=2}^{r} t_{2,r+i}\right)^{n_1-1}}\right)^{n_2-1}, & \text{if } n_1 \text{ is even,} \end{cases} \tag{36}$$

where for $i = 2, \ldots, r-1$,

$$t_{1i} = \operatorname{tr}\left((\boldsymbol{X}_1\boldsymbol{X}_1')(\boldsymbol{v}_i\boldsymbol{v}_i' + \boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}')\right),$$

$$t_{2,r+i} = \operatorname{tr}(\boldsymbol{S}_2(\boldsymbol{v}_i\boldsymbol{v}_i' + \boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}')),$$

where $\boldsymbol{X}_1$ and $\boldsymbol{S}_2$ are given in (7), and for $i = r$,

$$t_{2,2r} = \begin{cases} \operatorname{tr}(\boldsymbol{S}_2(\boldsymbol{v}_r\boldsymbol{v}_r' + \boldsymbol{v}_{n_1-r+2}\boldsymbol{v}_{n_1-r+2}')), & \text{if } n_1 \text{ is odd,} \\[2mm] \operatorname{tr}(\boldsymbol{S}_2\boldsymbol{v}_r\boldsymbol{v}_r'), & \text{if } n_1 \text{ is even.} \end{cases}$$

Proof. The expressions of $\max L(\mu, \boldsymbol{\Sigma}_{III})$ and $\max L(\mu, \boldsymbol{\Sigma}_{II})$ have been given in (32) and (22), respectively. Observe that $t_{21} = t_{31}$ and $t_{2,r+1} = t_{33}$ with the corresponding multiplicities 1 and $n_2 - 1$, and therefore they can be cancelled out when calculating $\max L(\mu, \boldsymbol{\Sigma}_{III})/\max L(\mu, \boldsymbol{\Sigma}_{II})$. Hence, Λ_3 has the form given in (36). ∎

The result stated in the next lemma will be used when we derive the distribution of Λ_3 in Theorem 3.10.

Lemma 3.9. *(Olkin and Press, 1969, Lemma 1) Let $U_1, \ldots, U_{m_1}, V_1, \ldots, V_{m_2}$ be independently distributed, $U_i \sim \chi_n^2$, $V_j \sim \chi_{2n}^2$. If*

$$L = \frac{M^M \prod_{i=1}^{m_1} U_i \prod_{j=1}^{m_2} V_j^2}{2^{2m_2}(\sum_{i=1}^{m_1} U_i + \sum_{j=1}^{m_2} V_j)^M}, \quad M = m_1 + 2m_2,$$

then $L \stackrel{d}{\sim} \prod_{j=1}^{M-1} X_j$, where $X_1, \ldots, X_{M-1}$ are independently distributed,

$$X_j \sim \beta\left(\frac{n}{2}, \frac{j}{M}\right), \quad j = 1, \ldots, M - m_2 - 1,$$

$$X_j \sim \beta\left(\frac{n+1}{2}, \frac{j}{M} - \frac{1}{2}\right), \quad j = M - m_2, \ldots, M - 1. \tag{37}$$

Theorem 3.10. *Under the null hypothesis H_3^0, the distribution of Λ_3, given in (36), equals*

$$\Lambda_3^{2/n} \stackrel{d}{\sim} \prod_{i=1}^{n_1-2} B_{1i} B_{2i}^{n_2-1}, \qquad (38)$$

where B_{1i} and B_{2i} are independently distributed, $i = 1, \ldots, n_1 - 2$,

$$B_{1i} \sim \begin{cases} \beta(\frac{n}{2}, \frac{i}{n_1-1}) & \text{for } i = 1, \ldots, [n_1/2] - 1, \\ \beta(\frac{n+1}{2}, \frac{i}{n_1-1} - \frac{1}{2}) & \text{for } i = [n_1/2], \ldots, n_1 - 2, \end{cases}$$

$$B_{2i} \sim \begin{cases} \beta(\frac{n(n_2-1)}{2}, \frac{i}{n_1-1}) & \text{for } i = 1, \ldots, [n_1/2] - 1, \\ \beta(\frac{n(n_2-1)+1}{2}, \frac{i}{n_1-1} - \frac{1}{2}) & \text{for } i = [n_1/2], \ldots, n_1 - 2. \end{cases}$$

Proof. When n_1 is odd, (36) can be represented as

$$\Lambda_3^{2/n} \stackrel{d}{\sim} \frac{(n_1-1)^{n_1-1} \prod_{i=2}^{r} t_{1i}^2}{2^{2(r-1)} \left(\sum_{i=2}^{r} t_{1i}\right)^{n_1-1}} \left(\frac{(n_1-1)^{n_1-1} \prod_{i=2}^{r} t_{2,r+i}^2}{2^{2(r-1)} \left(\sum_{i=2}^{r} t_{2,r+i}\right)^{n_1-1}} \right)^{n_2-1}, \qquad (39)$$

where $t_{1i} \sim \chi_{2n}^2$ and $t_{2,r+i} \sim \chi_{2n(n_2-1)}^2$, $i = 1, \ldots, r - 1$, and they are independent. Define $M = n_1 - 1$ and $m_2 = r - 1$. We then apply Lemma 3.9 to both of components $\frac{(n_1-1)^{n_1-1} \prod_{i=2}^{r} t_{1i}^2}{2^{2(r-1)} \left(\sum_{i=2}^{r} t_{1i}\right)^{n_1-1}}$ and $\left(\frac{(n_1-1)^{n_1-1} \prod_{i=2}^{r} t_{2,r+i}^2}{2^{2(r-1)} \left(\sum_{i=2}^{r} t_{2,r+i}\right)^{n_1-1}} \right)^{n_2-1}$ in (39) separately by putting $V_i = t_{1i}$ for the first component and $V_i = t_{2,r+i}$ for the second component, $i = 2, \ldots, r$. Hence, we have the distributional result given in (38).

When n_1 is even, (36) can be represented as

$$\Lambda_3^{2/n} \stackrel{d}{\sim} \frac{(n_1-1)^{n_1-1} \prod_{i=2}^{r-1} t_{1i}^2 t_{1r}}{2^{2(r-2)} \left(\sum_{i=2}^{r-1} t_{1i} + t_{1r}\right)^{n_1-1}} \left(\frac{(n_1-1)^{n_1-1} \prod_{i=2}^{r-1} t_{2,r+i}^2 t_{2,2r}}{2^{2(r-1)} \left(\sum_{i=2}^{r-1} t_{2,r+i} + t_{2,2r}\right)^{n_1-1}} \right)^{n_2-1}, (40)$$

where $t_{1i} \sim \chi_{2n}^2$ and $t_{2,r+i} \sim \chi_{2n(n_2-1)}^2$, $i = 1, \ldots, r-1$, $t_{1r} \sim \chi_n^2$, $t_{2,2r} \sim \chi_{n(n_2-1)}^2$ and they are independent. Again, we can apply Lemma 3.9. Let $M = n_1 - 1$, $m_1 = 1$ and $m_2 = r - 2$. Defining $U_1 = t_{1r}$ and $V_i = t_{1i}$ to the first part $\frac{(n_1-1)^{n_1-1} \prod_{i=2}^{r-1} t_{1i}^2 t_{1r}}{2^{2(r-2)} \left(\sum_{i=2}^{r-1} t_{1i} + t_{1r}\right)^{n_1-1}}$ and for the second part $\frac{(n_1-1)^{n_1-1} \prod_{i=2}^{r-1} t_{2,r+i}^2 t_{2,2r}}{2^{2(r-1)} \left(\sum_{i=2}^{r-1} t_{2,r+i} + t_{2,2r}\right)^{n_1-1}}$ having $U_1 = t_{2,2r}$ and $V_i = t_{2,r+i}$, $i = 2, \ldots, r - 1$, we have the distributional result given in (38). ∎

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Theorem 3.11. *The LRT statistic for testing H_4^0 versus H_4^a, defined in Section [2](#), is given by*

$$\Lambda_4^{2/n} = \frac{2^{n_2(n_1-1-\mathbb{1}_{\text{even}})}\,|\boldsymbol{S}_1|\,|\boldsymbol{S}_2|^{n_2-1}}{\prod_{l=1}^{2r} t_{2l}^{m_l}}, \tag{41}$$

where $\mathbb{1}_{\text{even}}$ is the indicator function that n_1 is even, $\boldsymbol{S}_k$ are given in [(7)](#) and $t_{2,(k-1)r+i} = \text{tr}(\boldsymbol{S}_k(\boldsymbol{v}_i\boldsymbol{v}_i' + \boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}'))$, $i = 1,\dots,r$ and $k = 1,2$.

Proof. Since the maximum of the likelihood function under the alternative hypothesis H_4^a, i.e., $\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_I)$, has been given in [(13)](#), let us now consider the derivation of $\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{II})$. The likelihood function under the null hypothesis H_4^0 is given by

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{II}) = (2\pi)^{-\frac{pn}{2}}\,|\boldsymbol{\Sigma}_{II}|^{-\frac{n}{2}}\,e^{-\frac{1}{2}\text{tr}\left\{\boldsymbol{\Sigma}_{II}^{-1}\left[\boldsymbol{Y}-(\mathbf{1}_{n_2}\otimes\boldsymbol{\mu})\mathbf{1}_n'\right]\left[\boldsymbol{Y}-(\mathbf{1}_{n_2}\otimes\boldsymbol{\mu})\mathbf{1}_n'\right]'\right\}}. \tag{42}$$

We first proceed with the following canonical reduction:

$$(\boldsymbol{I}_n \otimes \boldsymbol{K}' \otimes \boldsymbol{V}')\text{vec}\boldsymbol{Y} \sim N_{pn}\left(\boldsymbol{I}_n \otimes \boldsymbol{K}'\mathbf{1}_{n_2} \otimes \boldsymbol{V}'\boldsymbol{\mu}, \boldsymbol{I}_n \otimes \boldsymbol{D}(\boldsymbol{\eta})\right),$$

where that $\boldsymbol{K}'\mathbf{1}_{n_2} = (\sqrt{n_2}, 0, \dots, 0)'$ and $\boldsymbol{V}'\boldsymbol{\mu} = (\boldsymbol{v}_1'\boldsymbol{\mu}, \dots, \boldsymbol{v}_{n_1}'\boldsymbol{\mu})'$. Put $\boldsymbol{w}_i = (\boldsymbol{I}_n \otimes \boldsymbol{u}_i')\text{vec}\boldsymbol{Y}$ and $\boldsymbol{u}_i$ is the ith column of $\boldsymbol{K} \otimes \boldsymbol{V}$, $i = 1, \dots, p$. It follows that the model can be split into $n_1 + r$ independent models with the responses $\boldsymbol{w}_l$, $l = 1, \dots, n_1$ and $\tilde{\boldsymbol{y}}_l$, $l = r+1, \dots, 2r$, respectively, where $r = [n_1/2] + 1$ and the r models are given in [(15)](#). It can be shown that $\boldsymbol{w}_i$ and $\tilde{\boldsymbol{y}}_i$, $i = 1, \dots, r$, are mutually independent and

$$\begin{aligned}
&\boldsymbol{w}_1 \sim N_n(\sqrt{n_2}\boldsymbol{v}_1'\boldsymbol{\mu}\mathbf{1}_n, \eta_1\boldsymbol{I}_n), \quad \boldsymbol{w}_i \sim N_n(\sqrt{n_2}\boldsymbol{v}_i'\boldsymbol{\mu}\mathbf{1}_n, \eta_i\boldsymbol{I}_n),\\
&\boldsymbol{w}_{n_1-i+2} \sim N_n(\sqrt{n_2}\boldsymbol{v}_{n_1-i+2}'\boldsymbol{\mu}\mathbf{1}_n, \eta_i\boldsymbol{I}_n), \quad i = 2, \dots, r, \\
&\tilde{\boldsymbol{y}}_i \sim N_{nm_{r+i}}(\mathbf{0}, \eta_{r+i}\boldsymbol{I}_{nm_{r+i}}), \quad i = 1, \dots, r,
\end{aligned} \tag{43}$$

where η_l are the distinct eigenvalues of $\boldsymbol{\Sigma}_{II}$ with multiplicities m_l, $l = 1, \dots, 2r$. It implies that the likelihood function in [(42)](#) is identical to the products of the

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likelihood functions of the $n_1 + r$ separate models, i.e.

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{II}) = L_1(\boldsymbol{v}_1'\boldsymbol{\mu}, \eta_1|\boldsymbol{\omega}_1) \prod_{i=2}^{r} L_i(\boldsymbol{v}_i'\boldsymbol{\mu}, \eta_i|\boldsymbol{\omega}_i) L_{n_1-i+2}(\boldsymbol{v}_{n_1-i+2}'\boldsymbol{\mu}, \eta_i|\boldsymbol{\omega}_{n_1-i+2})$$

$$\times \prod_{i=1}^{r} L_{r+i}(\eta_{r+i}|\tilde{\mathbf{y}}_i). \tag{44}$$

Since there is one-to-one transformation between the parameter sets $\{\boldsymbol{\mu}, \boldsymbol{\Sigma}_{II}\}$ and $\{\boldsymbol{V}'\boldsymbol{\mu}, \boldsymbol{\eta}\}$, the MLEs of $\theta_i = \boldsymbol{v}_i'\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ are given by

$$\sqrt{n_2}\hat{\theta}_i = \frac{1}{n}\boldsymbol{w}_i'\mathbf{1}_n, \quad i = 1, \ldots, n_1,$$

$$\hat{\eta}_1 = \frac{1}{n}\boldsymbol{w}_1'\boldsymbol{Q}_n\boldsymbol{w}_1, \quad \hat{\eta}_i = \frac{1}{2n}\text{tr}\left[\boldsymbol{Q}_n(\boldsymbol{w}_i\boldsymbol{w}_i' + \boldsymbol{w}_{n_1-i+2}\boldsymbol{w}_{n_1-i+2}')\right], \ i = 2, \ldots, r-1,$$

$$\hat{\eta}_r = \begin{cases} \frac{1}{2n}\left[\boldsymbol{Q}_n(\boldsymbol{w}_r\boldsymbol{w}_r' + \boldsymbol{w}_{n_1-r+2}\boldsymbol{w}_{n_1-r+2}')\right], & \text{if } n_1 \text{ is odd,} \\ \frac{1}{n}\boldsymbol{w}_r'\boldsymbol{Q}_n\boldsymbol{w}_r, & \text{if } n_1 \text{ is even,} \end{cases}$$

$$\tag{45}$$

and the MLEs of η_l, $l = r+1, \ldots, 2r$ are given in (16). Recall that $\boldsymbol{w}_i = \boldsymbol{Y}'(1/\sqrt{n_2}\mathbf{1}_{n_2} \otimes \boldsymbol{v}_i)$ and $\boldsymbol{w}_{jn_1+i} = \boldsymbol{Y}'(\boldsymbol{k}_j \otimes \boldsymbol{v}_i)$, where $\boldsymbol{k}_j$ is the jth column of the matrix $\boldsymbol{K}_1$, $i = 1, \ldots, n_1, j = 1, \ldots, n_2$. The MLEs of η_i could be reexpressed as follows:

$$\hat{\eta}_1 = \frac{1}{n}\text{tr}\left(\boldsymbol{S}_1\boldsymbol{v}_1\boldsymbol{v}_1'\right), \tag{46}$$

$$\text{for } i = 2, \ldots, r-1, \ \hat{\eta}_i = \frac{1}{2n}\text{tr}\left(\boldsymbol{S}_1(\boldsymbol{v}_i\boldsymbol{v}_i' + \boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}')\right), \tag{47}$$

$$\hat{\eta}_r = \begin{cases} \frac{1}{2n}\text{tr}\left(\boldsymbol{S}_1(\boldsymbol{v}_r\boldsymbol{v}_r' + \boldsymbol{v}_{n_1-r+2}\boldsymbol{v}_{n_1-r+2}')\right), & \text{if } n_1 \text{ is odd,} \\ \frac{1}{n}\text{tr}\left(\boldsymbol{S}_1(\boldsymbol{v}_r\boldsymbol{v}_r')\right), & \text{if } n_1 \text{ is even,} \end{cases} \tag{48}$$

and the MLEs of η_l, $l = r+1, \ldots, 2r$, have the expressions given in (18), (20) and (21). Therefore, the likelihood function in (42) is maximized by

$$\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{II}) = (2\pi)^{-\frac{pn}{2}} \prod_{i=1}^{2r} \left(\frac{t_{2i}}{nm_i}\right)^{-\frac{nm_i}{2}} e^{-\frac{pn}{2}}, \tag{49}$$

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where $t_{2,(k-1)r+i} = \operatorname{tr}(\boldsymbol{S}_k(\boldsymbol{v}_i\boldsymbol{v}_i' + \boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}'))$, $i = 1, \ldots, r$, $k = 1, 2$. Using the expressions given in (13) and (49), Λ_4 has the form given in (41) which completes the proof. ∎

Theorem 3.12. *Under the null hypothesis* H_4^0, *the distribution of* Λ_4, *given in* (41), *equals*

$$\Lambda_4^{2/n} \overset{d}{\sim} \prod_{i=1}^{n_1-1} B_{1i} B_{2i}^{n_2-1}, \tag{50}$$

where B_{1i} *and* B_{2i} *are independently distributed,* $i = 1, \ldots, n_1 - 1$,

$$B_{1i} \sim \begin{cases} \beta(\frac{n-i-1}{2}, \frac{i}{2}) & \text{for } i = 1, \ldots, [n_1/2], \\ \beta(\frac{n-i-1}{2}, \frac{i+1}{2}) & \text{for } i = [n_1/2] + 1, \ldots, n_1 - 1, \end{cases}$$

$$B_{2i} \sim \begin{cases} \beta(\frac{n(n_2-1)-i}{2}, \frac{i}{2}) & \text{for } i = 1, \ldots, [n_1/2], \\ \beta(\frac{n(n_2-1)-i}{2}, \frac{i+1}{2}) & \text{for } i = [n_1/2] + 1, \ldots, n_1 - 1. \end{cases}$$

Proof. The proof follows the same lines as for the proof of Theorem 3.4. We, therefore, briefly sketch the main steps. We begin by writing $\Lambda_4^{2/n}$ as

$$\Lambda_4^{2/n} = \frac{2^{n_2(n_1-1-\mathbb{1}_{\text{even}})}|\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V}|}{(\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{11} \prod_{i=2}^{r}\left[(\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{ii} + (\boldsymbol{V}'\boldsymbol{S}_1\boldsymbol{V})_{n_1-i+2,n_1-i+2}\right]^2}$$
$$\times \left(\frac{|\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V}|}{(\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{11} \prod_{i=2}^{r}\left[(\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{ii} + (\boldsymbol{V}'\boldsymbol{S}_2\boldsymbol{V})_{n_1-i+2,n_1-i+2}\right]^2}\right)^{n_2-1}, \tag{51}$$

where $\mathbb{1}_{\text{even}}$ is the indicator function that n_1 is even and $(\boldsymbol{V}'\boldsymbol{S}_k\boldsymbol{V})_{ii}$ is the ith diagonal element of the matrix $\boldsymbol{V}'\boldsymbol{S}_k\boldsymbol{V}$. After applying the same Barlett decomposition as we did in the proof of Theorem 3.4, under the null hypothesis, equation (51) has the same distribution as

$$\prod_{i=2}^{r} \frac{2^2\eta_i^2\chi_{n-i}^2\chi_{n-(n_1-i+2)}^2}{\eta_i^2(\chi_{n_1}^2 + \chi_{n-i}^2 + \chi_{n-(n_1-i+2)}^2)^2} \left[\prod_{i=2}^{r} \frac{2^2\eta_{r+i}^2\chi_{n(n_2-1)-i+1}^2\chi_{n(n_2-1)-(n_1-i+2)+1}^2}{\eta_{r+i}^2(\chi_{n_1}^2 + \chi_{n(n_2-1)-i+1}^2 + \chi_{n(n_2-1)-(n_1-i+2)+1}^2)^2}\right]^{n_2-1}.$$

We then use Lemma 3.3 to obtain the distribution of $\Lambda_4^{2/n}$ and it turns out that Λ_4 is distributed as the expression given in (50). ∎

Theorem 3.13. *The LRT statistic for testing H_5^0 versus H_5^a, defined in Section [2](#), is given by*

$$\Lambda_5^{2/n} = \frac{(n_1-1)^{n_2(n_1-1)} |\boldsymbol{S}_1| |\boldsymbol{S}_2|^{n_2-1}}{\prod\limits_{j=1}^{4} t_{3j}^{m_j}}, \tag{52}$$

where $\boldsymbol{S}_h$, $h = 1, 2$, are given in (7), t_{3j} and m_j, $j = 1, \ldots, 4$, are given in Theorem 3.6.

Proof. Since the maximum of the likelihood function under the alternative hypothesis H_5^a, i.e., $\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_I)$, has been given in (13), let us now consider the derivation of $\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{III})$. The likelihood function under the null hypothesis H_5^0 is given by

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{III}) = (2\pi)^{-\frac{pn}{2}} |\boldsymbol{\Sigma}_{III}|^{-\frac{n}{2}} e^{-\frac{1}{2}\text{tr}\left\{\boldsymbol{\Sigma}_{III}^{-1}\left[\boldsymbol{Y}-(\mathbf{1}_{n_2}\otimes\boldsymbol{\mu})\mathbf{1}_n'\right]\left[\boldsymbol{Y}-(\mathbf{1}_{n_2}\otimes\boldsymbol{\mu})\mathbf{1}_n'\right]'\right\}}.$$

Furthermore,

$$(\boldsymbol{I}_n \otimes \boldsymbol{K}' \otimes \boldsymbol{V}')\text{vec}\boldsymbol{Y} \sim N_{pn}\left(\boldsymbol{I}_n \otimes \boldsymbol{K}'\mathbf{1}_{n_2} \otimes \boldsymbol{V}'\boldsymbol{\mu}, \boldsymbol{I}_n \otimes (\boldsymbol{K}' \otimes \boldsymbol{V}')\boldsymbol{\Sigma}_{III}(\boldsymbol{K} \otimes \boldsymbol{V})\right),$$

which becomes $N_{pn}\left(\boldsymbol{I}_n \otimes (\sqrt{n_2}, 0, \ldots, 0)' \otimes (\boldsymbol{v}_1'\boldsymbol{\mu}, \ldots, \boldsymbol{v}_{n_1}'\boldsymbol{\mu})', \boldsymbol{I}_n \otimes \boldsymbol{D}(\boldsymbol{\lambda})\right)$, where $\boldsymbol{D}(\boldsymbol{\lambda})$ is the $p \times p$ diagonal matrix given in Lemma 3.5. It yields that the model can be split into $n_1 + 2$ independent models with the responses $\boldsymbol{w}_i$, $i = 1, \ldots, n_1$ and $\tilde{\boldsymbol{y}}_j$, $j = 3, 4$, where $\tilde{\boldsymbol{y}}_j$ are given in (28), $\boldsymbol{w}_i = (\boldsymbol{I}_n \otimes \boldsymbol{u}_i')\text{vec}\boldsymbol{Y}$ and $\boldsymbol{u}_i$ is the ith column of $\boldsymbol{K} \otimes \boldsymbol{V}$, $i = 1, \ldots, p$. It can be shown that $\boldsymbol{w}_1 \sim N_n(\sqrt{n_2}\boldsymbol{v}_1'\boldsymbol{\mu}, \lambda_1\boldsymbol{I}_n)$, $\boldsymbol{w}_i \sim N_n(\sqrt{n_2}\boldsymbol{v}_i'\boldsymbol{\mu}, \lambda_2\boldsymbol{I}_n)$, $i = 2, \ldots, n_1$ and $\boldsymbol{y}_j \sim N_{nm_j}(0, \lambda_j\boldsymbol{I}_{nm_j})$, for $j = 3, 4$, where λ_j and m_j, $j = 1, \ldots, 4$, are given in Lemma 3.5. Thus, the MLEs of $\theta_i = \boldsymbol{v}_i'\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are given by

$$\sqrt{n_2}\hat{\theta}_i = \frac{1}{n}\boldsymbol{w}_i'\mathbf{1}_n, \ i = 1, \ldots, n_1, \ \hat{\lambda}_1 = \frac{1}{n}\boldsymbol{w}_1'\boldsymbol{Q}_n\boldsymbol{w}_1, \ \hat{\lambda}_2 = \frac{1}{n(n_1-1)}\sum_{i=2}^{n_1}\boldsymbol{w}_i'\boldsymbol{Q}_n\boldsymbol{w}_i,$$

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and the MLEs of λ_3 and λ_4 are given in (29). Moreover, the estimators $\hat{\lambda}_j$, $j = 1, 3, 4$, can be expressed as given in (17), (18) and (31), respectively, and the MLE of λ_2 has also the following expression:

$$\hat{\lambda}_2 = \frac{1}{n(n_1 - 1)} tr(\boldsymbol{S}_1 \sum_{i=2}^{n_1} \boldsymbol{v}_i \boldsymbol{v}_i') = \frac{1}{n(n_1 - 1)} tr(\boldsymbol{S}_1 \boldsymbol{Q}_{n_1}).$$

Therefore,

$$\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{III}) = (2\pi)^{-\frac{pn}{2}} \prod_{j=1}^{4} (\frac{t_{3j}}{nm_j})^{-\frac{nm_j}{2}} e^{-\frac{pn}{2}}, \tag{53}$$

and Λ_5 has the form given in (52) and this completes the proof. ■

Theorem 3.14. *Under the null hypothesis H_5^0, the distribution of Λ_5, given in (52), equals*

$$\Lambda_5^{2/n} \overset{d}{\sim} \prod_{i=1}^{n_1 - 1} B_{1i} B_{2i}^{n_2 - 1}, \tag{54}$$

where B_{1i} and B_{2i} are independently distributed, $i = 1, \ldots, n_1 - 1$,

$$B_{1i} \sim \beta \left(\frac{n - i - 1}{2}, \frac{i - 1}{n_1 - 1} + \frac{i}{2} \right),$$
$$B_{2i} \sim \beta \left(\frac{n(n_2 - 1) - i}{2}, \frac{i - 1}{n_1 - 1} + \frac{i}{2} \right).$$

Proof. The proof follows the same lines as the proof of Theorem 3.7. Therefore, we only briefly sketch the main steps. We begin by writing $\Lambda_5^{2/n}$ as

$$\Lambda_5^{2/n} = \frac{(n_1 - 1)^{n_2(n_1 - 1)} |\boldsymbol{V}'\boldsymbol{S}_1 \boldsymbol{V}|}{(\boldsymbol{V}'\boldsymbol{S}_1 \boldsymbol{V})_{11} [\sum_{i=2}^{n_1} (\boldsymbol{V}'\boldsymbol{S}_1 \boldsymbol{V})_{ii}]^{n_1 - 1}}$$
$$\times \left(\frac{|\boldsymbol{V}'\boldsymbol{S}_2 \boldsymbol{V}|}{(\boldsymbol{V}'\boldsymbol{S}_2 \boldsymbol{V})_{11} [\sum_{i=2}^{n_1} (\boldsymbol{V}'\boldsymbol{S}_2 \boldsymbol{V})_{ii}]^{n_1 - 1}} \right)^{n_2 - 1}, \tag{55}$$

where $(\boldsymbol{V}'\boldsymbol{S}_k \boldsymbol{V})_{ii}$ is the ith diagonal element of the matrix $\boldsymbol{V}'\boldsymbol{S}_k \boldsymbol{V}$. After applying the same Barlett decomposition to $\boldsymbol{V}'\boldsymbol{S}_k \boldsymbol{V}$ as we did in the proof of Theorem 3.7,

under the null hypothesis H_5^0, equation (55) has the same distribution as

$$\frac{(n_1-1)^{n_1-1}\prod_{i=2}^{n_1}\chi_{n-i}^2}{\left(\chi_{(n_1-1)(i-1)}^2+\sum_{i=2}^{n_1}\chi_{n-i}^2\right)^{n_1-1}}\left[\frac{(n_1-1)^{n_1-1}\prod_{i=2}^{n_1}\chi_{n(n_2-1)-i+1}^2}{\left(\chi_{(n_1-1)(i-1)}^2+\sum_{i=2}^{n_1}\chi_{n(n_2-1)-i+1}^2\right)^{n_1-1}}\right]^{n_2-1}. \quad (56)$$

We then use Lemma 3.3 to obtain the distribution of $\Lambda_5^{2/n}$ and it comes out that Λ_5 is distributed as the expression given in (54). ∎

Theorem 3.15. *The LRT statistic for testing H_6^0 versus H_6^a, defined in Section 2, is given by*

$$\Lambda_6^{2/n}=\begin{cases}\prod_{i=2}^r G_i^2, & \text{if } n_1 \text{ is odd,}\\[2mm]\left(\prod_{i=2}^{r-1}G_i^2\right)\frac{\operatorname{tr}(\boldsymbol{S}_1\boldsymbol{v}_r\boldsymbol{v}_r')}{\operatorname{tr}((\boldsymbol{X}_1\boldsymbol{X}_1')(\boldsymbol{v}_r\boldsymbol{v}_r'))}, & \text{if } n_1 \text{ is even,}\end{cases} \quad (57)$$

where

$$G_i=\frac{\operatorname{tr}(\boldsymbol{S}_1(\boldsymbol{v}_i\boldsymbol{v}_i'+\boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}'))}{\operatorname{tr}((\boldsymbol{X}_1\boldsymbol{X}_1')(\boldsymbol{v}_i\boldsymbol{v}_i'+\boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}'))},$$

and $\boldsymbol{S}_1$ and $\boldsymbol{X}_1$ are given in (7).

Proof. Since the maximum of the likelihood function under the null hypothesis H_6^0, i.e. $\max L(\mu,\boldsymbol{\Sigma}_{II})$, has been derived in (22) and the maximum of the likelihood function under the alternative hypothesis H_6^a, i.e. $\max L(\boldsymbol{\mu},\boldsymbol{\Sigma}_{II})$ is given in (49), the likelihood ratio Λ_6 has the form presented in (57). ∎

Theorem 3.16. *Under the null hypothesis H_6^0, the distribution of Λ_6, given in (57), equals*

$$\Lambda_6^{2/n}\overset{d}{\sim}\begin{cases}\prod_{i=2}^r\beta(\frac{n-1}{2},1), & \text{if } n_1 \text{ is odd,}\\[2mm]\left(\prod_{i=2}^{r-1}\beta(\frac{n-1}{2},1)\right)\beta(\frac{n-1}{2},\frac{1}{2}), & \text{if } n_1 \text{ is even.}\end{cases} \quad (58)$$

Proof. We have already shown in the proof of Theorem 3.4 that $\boldsymbol{v}_i'\boldsymbol{S}_1\boldsymbol{v}_i\sim\eta_i\chi_{n-1}^2$ and $\boldsymbol{v}_{n_1-i+2}'\boldsymbol{S}_1\boldsymbol{v}_{n_1-i+2}\sim\eta_i\chi_{n-1}^2$. Therefore, $\operatorname{tr}(\boldsymbol{S}_1(\boldsymbol{v}_i\boldsymbol{v}_i'+\boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}'))\sim\eta_i\chi_{2(n-1)}^2$. And it has also been shown in the proof of Theorem 3.4 that $\boldsymbol{v}_i'\boldsymbol{S}_3\boldsymbol{v}_i\sim$

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$\eta_i \chi_1^2$ and $\boldsymbol{v}'_{n_1-i+2} \boldsymbol{S}_3 \boldsymbol{v}_{n_1-i+2} \sim \eta_i \chi_1^2$. Moreover, $\boldsymbol{S}_1$ and $\boldsymbol{S}_3$ are independent. Hence, when n_1 is odd,

$$G_i \stackrel{d}{\sim} \frac{\chi_{2(n-1)}^2}{\chi_{2(n-1)}^2 + \chi_2^2} \stackrel{d}{\sim} \beta(n-1, 1), \quad i = 2, \ldots, r,$$

and using the property that if $Z \sim \beta(a, 1)$ then $Z^2 \sim \beta(a/2, 1)$, we have obtained $\Lambda_6 \stackrel{d}{\sim} \left[\prod_{i=2}^r \beta(\frac{n-1}{2}, 1) \right]^{n/2}$. The situation when n_1 is even is analogous to the odd case. Hence, the distribution of Λ_6 has the expression in (58). ∎

Theorem 3.17. *The LRT statistic for testing H_7^0 versus H_7^a, defined in Section 2, is given by*

$$\Lambda_7^{2/n} = \left(\frac{\mathrm{tr}(\boldsymbol{S}_1 \boldsymbol{Q}_{n_1})}{\mathrm{tr}((\boldsymbol{X}_1 \boldsymbol{X}_1')\boldsymbol{Q}_{n_1})} \right)^{n_1-1}, \tag{59}$$

where $\boldsymbol{S}_1$ and $\boldsymbol{X}_1$ are given in (7).

Proof. Since the maximum of the likelihood function under the null hypothesis H_7^0, i.e. $\max L(\mu, \boldsymbol{\Sigma}_{III})$, has been derived in (32) and the maximum of the likelihood function under the alternative hypothesis H_7^a, i.e. $\max L(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{III})$ is given in (53). Hence, the likelihood ratio Λ_7 has the form in (59). ∎

Theorem 3.18. *Under the null hypothesis H_7^0, the distribution of Λ_7, given in (59), equals*

$$\Lambda_7^{\frac{2}{n(n_1-1)}} \stackrel{d}{\sim} B, \tag{60}$$

where $B \sim \beta \left(\frac{(n-1)(n_1-1)}{2}, \frac{n_1-1}{2} \right)$.

Proof. Recall $\boldsymbol{X}_1 \boldsymbol{X}_1' = \boldsymbol{S}_1 + \boldsymbol{S}_3$ in (7), we have $\mathrm{tr}((\boldsymbol{X}_1 \boldsymbol{X}_1')\boldsymbol{Q}_{n_1}) = \mathrm{tr}(\boldsymbol{S}_1 \boldsymbol{Q}_{n_1}) + \mathrm{tr}(\boldsymbol{S}_3 \boldsymbol{Q}_{n_1})$. Since $\mathrm{tr}(\boldsymbol{S}_1 \boldsymbol{Q}_{n_1}) \sim \chi_{(n-1)(n_1-1)}^2$, $\mathrm{tr}(\boldsymbol{S}_3 \boldsymbol{Q}_{n_1}) \sim \chi_{n_1-1}^2$ and it has been known that $\boldsymbol{S}_1$ and $\boldsymbol{S}_3$ are independent, we then have

$$\Lambda_7^{2/n} \stackrel{d}{\sim} \frac{\chi_{(n-1)(n_1-1)}^2}{\chi_{(n-1)(n_1-1)}^2 + \chi_{n_1-1}^2} \stackrel{d}{\sim} \left[\beta \left(\frac{(n-1)(n_1-1)}{2}, \frac{n_1-1}{2} \right) \right]^{n_1-1},$$

which yields the distribution of Λ_7 in (60). ∎

4. Testing parameters under a block circular Toeplitz structure

In this section, we consider testing hypotheses about parameters in model (2). Let $\boldsymbol{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$ be a random sample from model (2), i.e.,

$$\boldsymbol{Y} \sim N_{p,n}(\mu \mathbf{1}_p \mathbf{1}_n', \boldsymbol{V}(\boldsymbol{\theta}), \boldsymbol{I}_n). \tag{61}$$

After some manipulation, the covariance matrix $\boldsymbol{V}(\boldsymbol{\theta})$ in (3) can be rewritten as follows:

$$\boldsymbol{V}(\boldsymbol{\theta}) = \boldsymbol{I}_{n_2} \otimes \boldsymbol{V}^{(1)} + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{V}^{(2)}, \tag{62}$$

where $\boldsymbol{V}^{(h)}$, $h = 1, 2$, is a CT matrix and for $i, j = 1, \ldots, n_1$,

$$v_{ij}^{(h)} = \begin{cases} \sigma^2 \mathbb{1}_{(h=1)} + \sigma_h + \tau_{|j-i|+1+(h-1)r}, & \text{if } |j - i| \leq r - 1, \\ \sigma_h + \tau_{n_1 - |j-i|+1+(h-1)r}, & \text{otherwise,} \end{cases}$$

where $\mathbb{1}_{(\bullet)}$ is the indicator function, σ_h, τ_q and σ^2 are the parameters of the matrices $\boldsymbol{\Sigma}_1$, $\boldsymbol{\Sigma}_2$ and $\sigma^2 \boldsymbol{I}$ in (3), respectively, $q = 1, \ldots, 2r$. Let $\boldsymbol{\theta} = (\sigma^2, \sigma_1, \sigma_2, \tau_1, \ldots, \tau_{2r})'$ be the vector containing all unknown (co)variance parameters. It has been observed by Liang *et al.* (2012, 2014) that model (2) is overparametrized and we have to put at least three restrictions on the parameter space of $\boldsymbol{\theta}$ in order to estimate $\boldsymbol{\theta}$ uniquely. Three restricted models $\mathcal{M}_i$ considered by Liang *et al.* (2014) will be introduced in the following lemma. The matrices $\boldsymbol{K}_i$ and $\boldsymbol{L}$ in the lemma will be stated without explicit expressions, $i = 1, 2, 3$, and the proof will be omitted. For details, it is referred to Liang *et al.* (2014).

Lemma 4.1. *Let $\boldsymbol{\eta}$ be the vector of the $2r$ distinct eigenvalues of $\boldsymbol{V}(\boldsymbol{\theta})$. Then the restricted model $\mathcal{M}_i$ equals* (61) *but with the restriction $\boldsymbol{K}_i \boldsymbol{\theta} = \boldsymbol{0}$, $i = 1, 2, 3$. Equivalently $\mathcal{M}_i$ can be defined via the free parameters*

$$\boldsymbol{\theta}_i^* = (\boldsymbol{L}(\boldsymbol{K}_i')^o)^{-1} \boldsymbol{\eta},$$

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where

$$\boldsymbol{\theta}_1^* = (\sigma^2, \sigma_2, \tau_2, \ldots, \tau_r, \tau_{r+2}, \ldots, \tau_{2r})',$$

$$\boldsymbol{\theta}_2^* = (\sigma^2, \sigma_2, \tau_1, \ldots, \tau_r, \tau_{r+3}, \ldots, \tau_{2r})',$$

$$\boldsymbol{\theta}_3^* = (\sigma^2, \sigma_1, \sigma_2, \tau_2, \ldots, \tau_r, \tau_{r+3}, \ldots, \tau_{2r})'.$$

The matrix $(\boldsymbol{K}_i')^o : (2r+3) \times 2r$ *is a matrix which columns generate the orthogonal complement to the column vector space of* $\boldsymbol{K}_i'$ *and the explicit expressions of* $\boldsymbol{K}_i$, $(\boldsymbol{K}_i')^o$ *and* $\boldsymbol{L}$ *are given in* Liang et al. (2014), $i = 1, 2, 3$.

By using $\boldsymbol{L}$ and $(\boldsymbol{K}_i')^o$ the following important results for the three models $\mathcal{M}_i$, $i = 1, 2, 3$, can be studied (Liang *et al.*, 2014):

$$\mathcal{M}_1 : \eta_1 = \sigma^2, \quad \eta_{r+1} = \sigma^2 - n_1 n_2 \sigma_2,$$

$$\mathcal{M}_2 : \eta_l = \sigma^2, \quad \eta_{r+1} = \sigma^2 - n_1 n_2 \sigma_2, \text{ for some } l \in \{2, \ldots, r, r+2, \ldots, 2r\},$$

$$\mathcal{M}_3 : \eta_1 = \sigma^2 + n_1 [\sigma_1 + (n_2 - 1)\sigma_2], \quad \eta_{r+1} = \sigma^2 + n_1(\sigma_1 - \sigma_2),$$

$$\eta_l = \sigma^2, \text{ for some } l \in \{2, \ldots, r, r+2, \ldots, 2r\},$$

(63)

where $\boldsymbol{\eta} = (\eta_l)$ *are the distinct eigenvalues of* $\boldsymbol{V}(\boldsymbol{\theta})$, *given in* (62).

Lemma 4.2. *Let* $\boldsymbol{\eta} = (\eta_l)$, $l = 1, \ldots, 2r$, *be as in Lemma 4.1. Without loss of information, any of the models* $\mathcal{M}_i$, $i = 1, 2, 3$, *can be formulated equivalently as*

$$\tilde{\mathbf{y}}_1 \sim N_n(\sqrt{p}\mu \mathbf{1}_n, \eta_1 \boldsymbol{I}_n), \quad \tilde{\mathbf{y}}_l \sim N_{nm_l}(0, \eta_l \boldsymbol{I}_{nm_l}), \quad l = 2, \ldots, 2r.$$

Depending on which $\mathcal{M}_i$ *is considered,* η_l *is defined in Lemma 4.1, and it has multiplicity* m_l *which was given in* Liang et al. (2012, Table 2). *Moreover,* $\tilde{\mathbf{y}}_1$ *can be further decomposed as*

$$\frac{1}{\sqrt{n}}\mathbf{1}_n'\tilde{\mathbf{y}}_1 \sim N(\sqrt{pn}\mu, \eta_1), \quad \boldsymbol{z} \sim N_{n-1}(0, \eta_1 \boldsymbol{I}_{n-1}),$$

where $\boldsymbol{z} = \boldsymbol{T}\tilde{\mathbf{y}}_1$ *and* $\boldsymbol{T}$ *is an* $(n-1) \times n$ *semiorthogonal matrix such that* $\boldsymbol{TT}' = \boldsymbol{I}_{n-1}$ *and* $\boldsymbol{T}\mathbf{1}_n = \mathbf{0}$.

For each restricted model given in (63), we will consider the following hypothesis testing problems for the parameters of $\boldsymbol{V}(\boldsymbol{\theta})$. In model $\mathcal{M}_1$ and $\mathcal{M}_2$ we will test

$$H_{01} : \sigma_2 = 0 \quad \text{versus} \quad H_{a1} : \sigma_2 < 0,$$

and in $\mathcal{M}_3$ we will test

$$H_{02} : \sigma_2 = 0 \quad \text{versus} \quad H_{a2} : -\frac{\sigma_1}{n_2 - 1} \leq \sigma_2 \leq \sigma_1.$$

The hypothesis H_{01} implies that there is no random effect $\boldsymbol{\gamma}_1$ in model $\mathcal{M}_1$ and $\mathcal{M}_2$, while hypothesis H_{02} means that the factor levels of $\boldsymbol{\gamma}_2$ are uncorrelated.

Remark: The alternative hypothesis H_{a1} is due to the fact $\sigma_1 = -(n_2 - 1)\sigma_2$ and σ_2 must be negative in order to make sure that $\boldsymbol{\Sigma}_1$ is positive semidefinite. In model $\mathcal{M}_3$, there is no restriction imposed on the eigenvalues of $\boldsymbol{\Sigma}_1$. Hence the alternative hypothesis H_{a2} is $\sigma_1 \geq \sigma_2 \geq -\sigma_1/(n_2 - 1)$ in order to preserve the semidefiniteness of $\boldsymbol{\Sigma}_1$.

From (63) it can be directly observed that testing H_{01} versus H_{a1} in model $\mathcal{M}_1$ is equivalent to test

$$H_{01,\mathcal{M}_1} : \eta_1 = \eta_{r+1} \text{ versus } H_{a1,\mathcal{M}_1} : \eta_1 < \eta_{r+1}, \tag{64}$$

and testing H_{01} versus H_{a1} in model $\mathcal{M}_2$ is equivalent to test

$$H_{01,\mathcal{M}_2} : \eta_l = \eta_{r+1} \text{ versus } H_{a1,\mathcal{M}_2} : \eta_l < \eta_{r+1}, \tag{65}$$
$$\text{for some } l \in \{2, \ldots, r, r + 2, \ldots, 2r\}.$$

In model $\mathcal{M}_3$, testing H_{02} versus H_{a2} in model $\mathcal{M}_3$ is equivalent to test

$$H_{02} : \eta_1 = \eta_{r+1} \text{ versus } H_{a2} : \eta_1 \neq \eta_{r+1}. \tag{66}$$

Next, concerning the covariance matrix $\boldsymbol{\Sigma}_2$ in (5), we will test for $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$

$$H_{03} : \tau_1 = 0 \quad \text{versus} \quad H_{a3} : \tau_1 \neq 0.$$

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The hypothesis H_{03} means that there is no random effect $\boldsymbol{\gamma}_2$ in all restricted models, since under the null hypothesis $\tau_1 = 0$, the only possibility to preserve the semidefiteness of $\boldsymbol{\Sigma}_2$ is that $\boldsymbol{\Sigma}_2$ is the zero matrix. Then the linear function $\eta_l(\sigma^2, \tau_q)$ (see Liang *et al.*, 2014, for all coefficients), becomes $\eta_l(\sigma^2, 0) = \sigma^2$ for some l, and q depending on which $\boldsymbol{\theta}_i^*$ in Lemma 4.1 is considered. Consequently, testing H_{03} versus H_{a3} in the restricted models $\mathcal{M}_1$ and $\mathcal{M}_2$ is equivalent to test

$$H_{03,\mathcal{M}_i} : \eta_l = \sigma^2 \quad \text{versus } H_{a3,\mathcal{M}_i} : \text{ none of } \eta_l \text{ is equal, } i \in \{1,2\}, l \neq r+1, \quad (67)$$

and in model $\mathcal{M}_3$, the testing problem can be formulated as

$$H_{03,\mathcal{M}_3} : \eta_l = \sigma^2 \quad \text{versus } H_{a3,\mathcal{M}_3} : \text{ none of } \eta_l \text{ is equal, } \quad l \neq 1, r+1. \quad (68)$$

In the subsequent two subsections we will propose the test procedures for testing the null hypotheses H_{0i}, $i = 1, 2, 3$. It is seen from Lemma 4.2 and (64)-(68) that to test each hypothesis is nothing but testing the equality of several variances of some of the $2r$ independent models of Lemma 4.2.

Bartlett's test is a test procedure which is often used to test the null hypothesis that population variances are equal under a normality assumption (Bartlett, 1937). In the context of zero means for k populations, the test statistic has the following expression (Bartlett, 1937, p.274):

$$B = \frac{N \ln\left(\frac{\sum_{i=1}^k n_i S_i^2}{N}\right) - \sum_{i=1}^k n_i \ln S_i^2}{1 + \frac{1}{3(k-1)}\left(\sum_{i=1}^k \frac{1}{n_i} - \frac{1}{N}\right)}, \quad (69)$$

where S_i^2 is the sample variance of the i-th population with sample size n_i, $i = 1, \ldots, k$, and N is the total sample sizes, i.e. $N = \sum_{i=1}^k n_i$. The test statistic in (69) is a modification of the LRT statistic, which is here denoted as W, and with zero means we have

$$B = -2c \ln W, \text{ where } c^{-1} = 1 + \frac{1}{3(k-1)}\left(\sum_{i=1}^k \frac{1}{n_i} - \frac{1}{N}\right).$$

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The asymptotic null distribution of B in (69) is the χ^2 distribution with $(k-1)$ degrees of freedom.

4.1. Test for the hypotheses of parameters in $\boldsymbol{\Sigma}_1$ under different restricted models

In this subsection, we consider testing the null hypotheses H_{01} and H_{02}, respectively, and construct the corresponding Bartlett's test statistics.

According to the equivalent hypothesis (64), constructing a test statistic for (64) is actually to test equal variances in the following two populations of Lemma 4.2:

$$\boldsymbol{z} \sim N_{n-1}(0, \eta_1 \boldsymbol{I}_{n-1}), \quad \tilde{\boldsymbol{y}}_{r+1} \sim N_{n(n_2-1)}(0, \eta_{r+1} \boldsymbol{I}_{n(n_2-1)}).$$

Thus, it is straightforward to construct a LRT or the corresponding Bartlett's test for testing equality of two variances based on the two populations $\boldsymbol{z}$ and $\tilde{\boldsymbol{y}}_{r+1}$ with zero means and sample sizes $n-1$ and $n(n_2-1)$. The result is presented in the following theorem. The proof is omitted since it is a direct application of the results from the previous section.

Theorem 4.3. *In model $\mathcal{M}_1$, the LRT statistic for testing $H_{01,\mathcal{M}_1} : \sigma_2 = 0$ or equivalently $\eta_1 = \eta_{r+1}$ against $H_{a1,\mathcal{M}_1} : \sigma_2 < 0$ or equivalently $\eta_1 < \eta_{r+1}$ is given by*

$$W_1 = \frac{(nn_2-1)^{\frac{nn_2-1}{2}}}{(n-1)^{\frac{n-1}{2}} \left[n(n_2-1)\right]^{\frac{n(n_2-1)}{2}}} \frac{(\boldsymbol{z}'\boldsymbol{z})^{\frac{n-1}{2}} (\tilde{\boldsymbol{y}}'_{r+1}\tilde{\boldsymbol{y}}_{r+1})^{\frac{n(n_2-1)}{2}}}{\left(\boldsymbol{z}'\boldsymbol{z} + \tilde{\boldsymbol{y}}'_{r+1}\tilde{\boldsymbol{y}}_{r+1}\right)^{\frac{nn_2-1}{2}}}, \tag{70}$$

and the corresponding Bartlett's test statistic with the asymptotic null distribution is given by

$$B_1 = -2c \ln W_1 \overset{H_{01,\mathcal{M}_1}}{\sim} \chi_1^2, \tag{71}$$

where $c^{-1} = 1 + \frac{1}{3} \left[\frac{1}{n-1} + \frac{1}{n(n_2-1)} - \frac{1}{nn_2-1}\right].$

Theorem 4.3 suggests to reject the null hypothesis $H_{01,\mathcal{M}_1}$ for small values of W_1 or for large values of B_1, which implies to reject the null hypothesis of no random effect $\boldsymbol{\gamma}_1$ in (2).

Next we construct a test statistic for the testing problem given in (65), which can be considered as testing the equal variances of two populations with respective sample sizes $n(n_2 - 1)$ and nm_l as follows:

$$\tilde{\mathbf{y}}_{r+1} \sim N_{n(n_2-1)}(0, \eta_{r+1}\boldsymbol{I}_{n(n_2-1)}), \quad \tilde{\mathbf{y}}_l \sim N_{nm_l}(0, \eta_l \boldsymbol{I}_{nm_l}), \tag{72}$$

for some $l \in \{2, \ldots, r, r+2, \ldots, 2r\}$.

Theorem 4.4. *In model $\mathcal{M}_2$, the LRT statistic for testing $H_{01,\mathcal{M}_2} : \sigma_2 = 0$ or equivalently $\eta_l = \eta_{r+1}$ against $H_{a1,\mathcal{M}_2} : \sigma_2 < 0$ or equivalently $\eta_l < \eta_{r+1}$, $l \in \{2, \ldots, r, r+2, \ldots, 2r\}$, is given by*

$$W_2^{2/n} = \frac{(m_l + n_2 - 1)^{m_l+n_2-1}}{m_l^{m_l}(n_2 - 1)^{n_2-1}} \frac{(\tilde{\mathbf{y}}_l'\tilde{\mathbf{y}}_l)^{m_l}(\tilde{\mathbf{y}}_{r+1}'\tilde{\mathbf{y}}_{r+1})^{n_2-1}}{(\tilde{\mathbf{y}}_l'\tilde{\mathbf{y}}_l + \tilde{\mathbf{y}}_{r+1}'\tilde{\mathbf{y}}_{r+1})^{m_l+n_2-1}}, \tag{73}$$

and the corresponding Bartlett's test statistic with the asymptotic null distribution is given by

$$B_2 = -2c \ln W_2 \overset{H_{01,\mathcal{M}_2}}{\sim} \chi_1^2, \tag{74}$$

where $c^{-1} = 1 + \frac{1}{3}\left[\frac{1}{nm_l} + \frac{1}{n(n_2-1)} - \frac{1}{n(m_l+n_2-1)}\right]$.

Similarly to Theorem 4.3, in Theorem 4.4 we will reject the null hypothesis $H_{01,\mathcal{M}_2}$ for small values of W_2 or for large values of B_2, which implies rejection of the null hypothesis of no random effect $\boldsymbol{\gamma}_1$. Differently from $\mathcal{M}_1$ and $\mathcal{M}_2$, it can be seen from (63), in model $\mathcal{M}_3$, there is no restriction imposed on the parameters (or equivalently eigenvalues) of $\boldsymbol{\Sigma}_1$, which results in rejecting the null hypothesis that the factor levels of $\boldsymbol{\gamma}_1$ are uncorrelated. However, the testing problem in (66) is in fact a test for equality of two variances based on two populations $\mathbf{z}$ and $\tilde{\mathbf{y}}_{r+1}$, which leads to exactly the same test statistic as given in Theorem 4.3.

Theorem 4.5. *In model $\mathcal{M}_3$, the LRT statistic for testing $H_{02} : \sigma_2 = 0$ or equivalently $\eta_1 = \eta_{r+1}$ against $H_{a2} : \sigma_1 \geq \sigma_2 \geq -\sigma_1/(n_2-1)$ or equivalently $\eta_1 \neq \eta_{r+1}$ is given by*

$$W_3 = \frac{(nn_2 - 1)^{\frac{nn_2-1}{2}}}{(n-1)^{\frac{n-1}{2}}\left[n(n_2-1)\right]^{\frac{n(n_2-1)}{2}}} \frac{(z'z)^{\frac{n-1}{2}}\left(\tilde{\mathbf{y}}'_{r+1}\tilde{\mathbf{y}}_{r+1}\right)^{\frac{n(n_2-1)}{2}}}{\left(z'z + \tilde{\mathbf{y}}'_{r+1}\tilde{\mathbf{y}}_{r+1}\right)^{\frac{nn_2-1}{2}}}, \tag{75}$$

and the corresponding Bartlett's test statistic with the asymptotic null distribution is given by

$$B_3 = -2c \ln W_3 \overset{H_{02}}{\sim} \chi_1^2, \tag{76}$$

where $c^{-1} = 1 + \frac{1}{3}\left[\frac{1}{n-1} + \frac{1}{n(n_2-1)} - \frac{1}{nn_2-1}\right].$

4.2. Test for the hypotheses of parameters in $\boldsymbol{\Sigma}_2$ under different restricted models

In this subsection, we consider testing the null hypotheses H_{03} under different restricted models, i.e. $H_{03,\mathcal{M}_i}$, $i = 1, 2, 3$, respectively. The testing problem in (67) can be considered to test equality of variances coming from the $2r - 1$ following populations:

$$\boldsymbol{z} \sim N_{n-1}(0, \eta_1 \boldsymbol{I}_{n-1}), \quad \tilde{\mathbf{y}}_l \sim N_{nm_l}(0, \eta_l \boldsymbol{I}_{nm_l}),$$

for some $l = 2, \ldots, 2r$ but $l \neq r + 1$.

Theorem 4.6. *In model $\mathcal{M}_1$ and $\mathcal{M}_2$, the LRT statistic for testing $H_{03,\mathcal{M}_i} : \eta_l = \sigma^2$ against $H_{a3,\mathcal{M}_i} : $ "none of η_l is equal", $i \in \{1, 2\}$, $l \neq r + 1$, is given by*

$$W_4 = \frac{\left[n(p - n_2 + 1) - 1\right]^{\frac{n(p-n_2+1)-1}{2}}}{(n-1)^{\frac{n-1}{2}} \prod_{\substack{l=2 \\ l \neq r+1}}^{2r} (nm_l)^{\frac{nm_l}{2}}} \frac{(z'z)^{\frac{n-1}{2}} \prod_{\substack{l=2 \\ l \neq r+1}}^{2r} (\tilde{\mathbf{y}}'_l\tilde{\mathbf{y}}_l)^{\frac{nm_l}{2}}}{\left(z'z + \sum_{\substack{l=2 \\ l \neq r+1}}^{2r} \tilde{\mathbf{y}}'_l\tilde{\mathbf{y}}_l\right)^{\frac{n(p-n_2+1)-1}{2}}}, \tag{77}$$

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and the corresponding Bartlett's test statistic with the asymptotic distribution is given by

$$B_4 = -2c \ln W_4 \overset{H_{03,\mathcal{M}_i}}{\sim} \chi^2_{2r-2}, \quad i \in \{1,2\}, \tag{78}$$

where $r = [n_1/2] + 1$ and

$$c^{-1} = 1 + \frac{1}{3(2r-2)} \left(\frac{1}{n-1} + \sum_{\substack{l=2 \\ l \neq r+1}}^{2r} \frac{1}{nm_l} - \frac{1}{n(p-n_2+1)-1} \right).$$

In Theorem 4.6, we reject the null hypothesis $H_{a3,\mathcal{M}_i}$ for small values of W_4 or large values of B_4 and it means that we reject the hypothesis of no random effect $\boldsymbol{\gamma}_2$ under $\mathcal{M}_1$ exist, or $\mathcal{M}_2$. The testing problem in (68) can be considered to be a test for equal variances for the $2r-2$ populations:

$$\tilde{\mathbf{y}}_l \sim N_{nm_l}(0, \eta_l \boldsymbol{I}_{nm_l}),$$

where $l = 2, \ldots, 2r$ but $l \neq r+1$.

Theorem 4.7. *In model $\mathcal{M}_3$, the LRT statistic for testing $H_{03,\mathcal{M}_3} : \eta_l = \sigma^2$ against $H_{a3,\mathcal{M}_3} :$ "none of η_l is equal", $l \neq 1, r+1$, is given by*

$$W_5^{2/n} = \frac{(p-n_2)^{p-n_2}}{\prod\limits_{\substack{l=2 \\ l \neq r+1}}^{2r} m_l^{m_l}} \frac{\prod\limits_{\substack{l=2 \\ l \neq r+1}}^{2r} (\tilde{\mathbf{y}}_l' \tilde{\mathbf{y}}_l)^{m_l}}{\left(\sum\limits_{\substack{l=2 \\ l \neq r+1}}^{2r} \tilde{\mathbf{y}}_l' \tilde{\mathbf{y}}_l \right)^{p-n_2}}, \tag{79}$$

and the corresponding Bartlett's test statistic with the asymptotic distribution is given by

$$B_5 = -2c \ln W_5 \overset{H_{03,\mathcal{M}_3}}{\sim} \chi^2_{2r-3}, \tag{80}$$

where $r = [n_1/2] + 1$ and

$$c^{-1} = 1 + \frac{1}{3(2r-3)} \left(\sum_{\substack{l=2 \\ l \neq r+1}}^{2r} \frac{1}{nm_l} - \frac{1}{n(p-n_2)} \right).$$

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