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Statistical Analysis of Multigraphs

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Abstract

This article analyzes multigraphs by performing statistical tests of multigraph models obtained by random stub matching (RSM) and by independent edge assignments (IEA). The tests are performed using goodness-of-fit measures between the multiplicity sequence of an observed multigraph and the expected multiplicity sequence according to a simple or composite IEA hypothesis. Test statistics of Pearson type and of information divergence type are used. The expected values of the Pearson goodness-of-fit statistic under different multigraph models are derived, and some approximations of the test statistics with adjusted χ^2-distributions are considered. Illustrations of test performances are presented for all models, and the results indicate that even for very small number of edges, the null distributions of both statistics are well approximated by their asymptotic χ^2-distribution. This holds true for testing simple as well as composite hypotheses with different asymptotic distributions. The non-null distributions of the test statistics can be well approximated by adjusted χ^2-distributions which can be used for power approximations. The influence of RSM on both test statistics is substantial for small number of edges and implies a shift of their distributions towards smaller values compared to what holds true for the null distributions under IEA.

Keywords: multigraph, multiplicity, goodness-of-fit, information divergence.

1 Introduction

A random multigraph model is given by a probability distribution over some class of multigraphs. In this article multigraphs are analyzed by performing statistical tests of some multigraph models presented in Frank and Shafie (2012) and Shafie (2012). Two main multigraph models are considered. The first is obtained by random stub matching with fixed degrees (RSM) so that edge assignments to sites are dependent, and the second is obtained by independent edge assignments (IEA) according to a common probability distribution. Further, we present two different methods for obtaining an approximate IEA model from an RSM model. This is done by assuming that the stubs are randomly generated

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and independently assigned to vertices (ISA) and can be viewed as a Bayesian model for the stub frequencies under RSM. Another way of obtaining an approximate IEA model is to ignore the dependency between edges in the RSM model and assume independent edge assignments of stubs (IEAS). The tests are performed using goodness-of-fit measures between the multiplicity sequence of an observed multigraph and the expected multiplicity sequence according to a simple or composite IEA hypothesis. The exact distributions of the test statistics are investigated and compared to different approximations given by adjusted χ^2-distributions.

In the next section, multigraph data structures are described and exemplified. It is shown how they can be obtained by different kinds of vertex and edge aggregations. These kinds of aggregations are powerful methods to analyze structures in very large graphs. In Section 3, some basic notations are introduced, and the different multigraph models mentioned above are defined.

Statistical tests of simple hypotheses are considered in Section 4 where the hypotheses are fully specified IEA models. For an IEAS model, the edge probability parameters are functions of a specified degree sequence $\mathbf{d}$, and for an ISA model these parameters are functions of a specified stub selection probability sequence $\mathbf{p}$. The Pearson goodness-of-fit statistic S and the divergence statistic T for these tests are defined. The expected value of S is derived under different multigraph models, and in particular it is shown that for the null distribution under RSM, this expected value only depends on the numbers of vertices and edges. Test illustrations for IEAS, ISA and RSM models are presented where the moments and cumulative distribution functions of the test statistics are used to compare and evaluate their performances. The convergence of the null distributions of S and T to their asymptotic χ^2-distributions is rapid and even for small number of edges m, a good fit is seen between the null distributions and the asymptotic χ^2-distribution. For cases when flat $\mathbf{d}$ or $\mathbf{p}$ is tested against skew $\mathbf{d}$ or $\mathbf{p}$ (or vice versa), both statistics have good powers of rejecting a simple hypothesis about a false model. The non-null distributions of S and T needed for determining power are approximated by adjusted χ^2-distributions. The influence of RSM on the distributions of S and T is substantial for small m and implies a shift towards smaller values of the statistics compared to what holds true for the null distributions under IEA.

In Section 5, statistical tests of composite multigraph hypotheses are illustrated for IEAS, ISA and RSM models. Moments and cumulative distribution functions of the test statistics are used for comparisons and evaluations of their performances. The composite multigraph hypotheses might be unspecified IEAS or ISA where the parameters have to be estimated from data. For composite IEAS or ISA hypotheses including the correct model, the following results are noted. The null distributions of S and T converge faster to their asymptotic χ^2-distributions for flat $\mathbf{d}$ or $\mathbf{p}$ than for skew $\mathbf{d}$ or $\mathbf{p}$, but even for rather small m, there is a good fit between these distributions and their asymptotic χ^2-distributions. Further, both statistics have very poor powers of detecting differences between IEAS and ISA hypotheses for small as well as for large m.

2 Data Structures and Possible Applications

A multigraph is defined as a graph where multiple edges and edge-loops are permitted. Such data structures are common in contexts when several edges can be mapped on the same vertex pair, but they are also obtained by different types of aggregation. Several simple graphs representing different binary relations can be aggregated to a multigraph, or an initial very large graph can be transformed to a multigraph by aggregating vertices into special subsets. Such possibilities are illustrated by some examples.

Consider a social network of friendships between 15 school children consisting of 12 pairs of mutual friendships. The children are categorized by two attributes, gender with categories labeled G (girl) and B (boy), and living area with categories labeled N (north) and S (south). Thus, there are four vertex categories BN, BS, GN and GS which are displayed together with mutual friendships in Figure 1. By aggregating vertices in the same category, we obtain a multigraph on 4 new vertices corresponding to the categories, and it has the same number of edges as the initial graph. This is shown in Figure 2. By performing this kind of transformation, we reduce the number of vertices but increase the number of multiple edges and edge loops. Generally, social networks of contacts between individuals can be transformed to multigraphs on vertices corresponding to combined categories of individual attributes, and edge multiplicities represent frequencies of contacts within and between these categories.

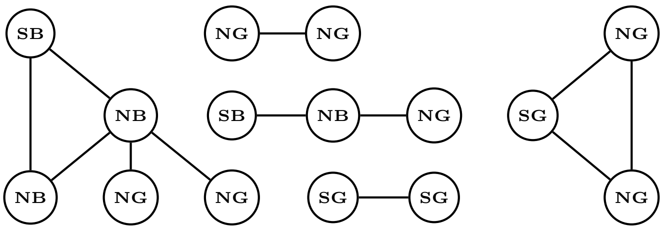

Figure 1: Initial graph of friendships between 15 children in a school and 12 pairs of mutually good friends. The children are categorized by gender, girl (G) or boy (B), and living area, north (N) or south (S).

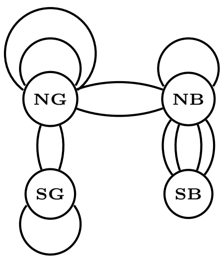

Figure 2: Final multigraph of the friendships in Figure 1 categorized by gender and living area. The edges represent pairwise friendships within and between categories.

As another illustration of vertex aggregation, consider network of co-operations between business companies which are categorized by branch. Figure 3 shows 20 co-operation pairs between 25 companies belonging to branch A, B or C. The multigraph on the three branches is given in Figure 4 and has edge multiplicities that represent co-operations within and between the branches. It is conveniently presented in table format.

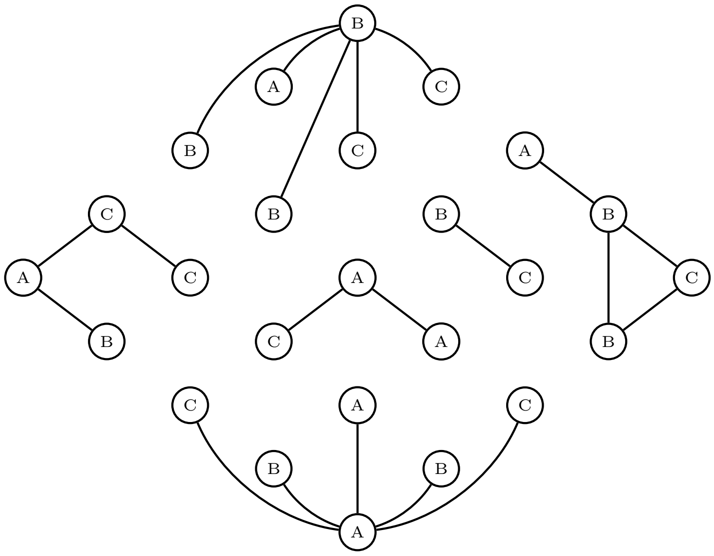

Figure 3: Initial graph of 20 co-operations between 25 companies. The companies are categorized by three different branches labeled A, B and C.

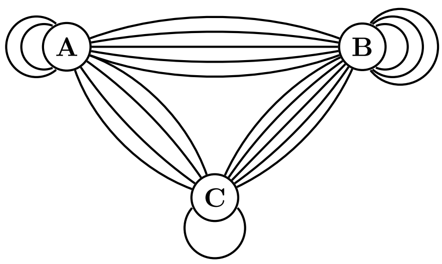

Figure 4: Final multigraph of the co-operations in Figure 3 categorized by branch. The edges represent pairwise co-operations within and between the three branches.

Vertex aggregation is a powerful method to analyze structure in very large graphs. Another type of aggregation that is often useful is edge aggregation, which is illustrated by the following example of a time series. Assume that we are studying a graph with a fixed number of vertices and different categories of pairwise contacts. Further, assume that we study this graph over a period of time, i.e. how the different contacts vary over a time period. For example, let the initial graph have 5 vertices representing 5 different departments in a company and we study the variation of 3 different edge categories representing pairwise contact types between and within the departments. These contact types are phone call, video call or meeting. The connections between the five departments have been observed every day for a total time period of three days. This is illustrated in Figure 5 where the edge attributes are labeled with the colors blue, red and green.

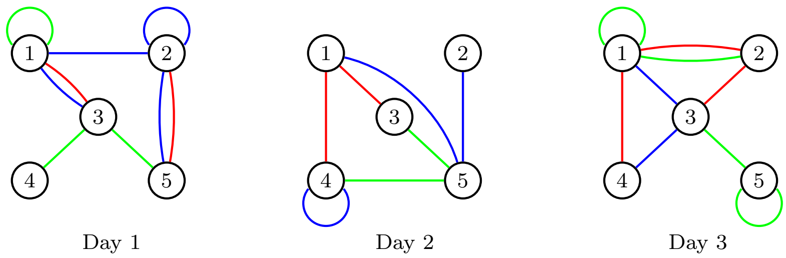

Figure 5: Initial daily graphs on 5 different departments of a company showing three connection types labeled blue (phone call), red (video call) and green (meeting).

The transformation with respect to edge attributes of the graphs in Figure 5 can be done in different ways. If we aggregate over time periods, we obtain for each edge category a multigraph for the total time period of three days, which is shown in Figure 6.

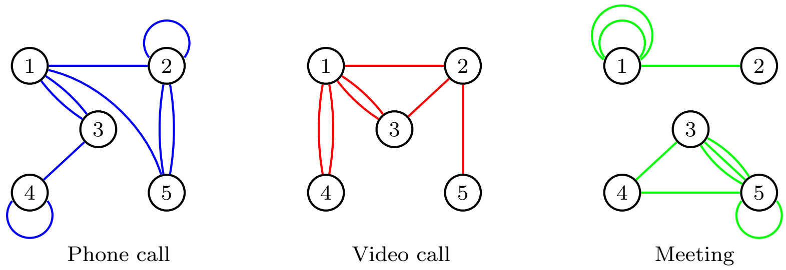

Figure 6: Multigraphs obtained by aggregating each of the edge categories in Figure 5 over all three days.

Another way of transforming the initial graphs in Figure 5 to multigraphs is by aggregating over contact types (ignoring edge colors), to get one multigraph for each time period. If we also aggregate over the three time periods we obtain a multigraph with 5 vertices and a total of 25 edges, shown in Figure 7.

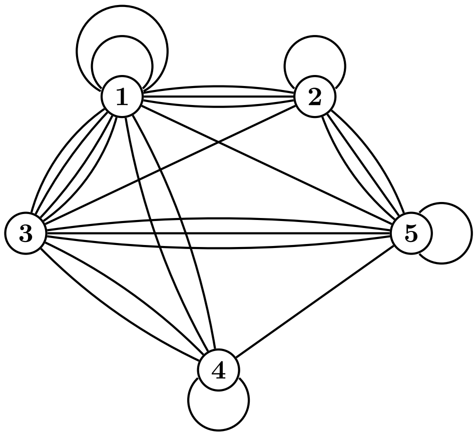

Figure 7: Final multigraph of the total number of connections during 3 days within and between the five departments in Figure 5.

3 Some Random Multigraph Models

In order to analyze multigraphs, we perform statistical tests of some random multigraph models considered in Frank and Shafie (2012) and Shafie (2012). First we introduce some basic notations. A finite graph g with n labeled vertices and m labeled edges associates with each edge an ordered or unordered vertex pair. Let $V = \{1, \ldots, n\}$ and $E = \{1, \ldots, m\}$ be the sets of vertices and edges labeled by integers, and let R denote the set of available sites for the edges. For directed graphs the site space is $R = V^2$ and the number of sites is given by $r = n^2$. For undirected graphs we use the site space $R = \{(i, j) \in V^2 : i \leq j\}$ where we consider (i, j) with $i \leq j$ as a canonical representation for the unordered vertex pair. The number of sites for undirected graphs is given by $r = \binom{n+1}{2}$. The graph is thus an injective map $g : E \to R \subseteq V^2$.

A random multigraph is given by a probability distribution over some class of multigraphs. A multigraph with labeled vertices and undistinguished edges is represented by the edge multiplicity sequence $\mathbf{m} = (m_{ij} : (i, j) \in R)$ where the edge multiplicity m_{ij} denotes the number of multiple edges at site $(i, j) \in R$. For undirected multigraphs, the edge sites are listed in the canonical order

$$(1, 1) < (1, 2) < \cdots < (1, n) < (2, 2) < (2, 3) < \cdots < (n, n) \ ,$$

so that m_{ii} is the number of loops at vertex i, and m_{ij} for $i < j$ is the number of edges between vertices i and j. In this case it is convenient to define $m_{ij} = 0$ for $i > j$. The edge multiplicity sequence $\mathbf{m}$ has total

$$m_{\cdot\cdot} = \sum\sum_{i\leq j} m_{ij} = m \quad \text{and} \quad m_{i\cdot} + m_{\cdot i} = \sum_{j=1}^{n} m_{ij} + \sum_{j=1}^{n} m_{ji} = d_i$$

is the degree of vertex i, which can also be interpreted as the number of edge-stubs or half-edges at vertex i for $i = 1, \ldots, n$. The stub multiplicity sequence $\mathbf{d} = (d_1, \ldots, d_n)$ has total $\sum_{i=1}^{n} d_i = 2m$.

Consider a random undirected multigraph model where the edges are independently assigned to sites according to a common probability model. Let Q_{ij} denote the probability of assigning an edge to site $(i, j) \in R$ so that $\sum\sum_{i\leq j} Q_{ij} = 1$. This independent edge assignment (IEA) model has edge multiplicity sequence $\mathbf{m}(\text{IEA})$ that is multinomially distributed with parameters m and $\mathbf{Q} = (Q_{ij} : (i, j) \in R)$ so that edge sequences $\mathbf{m}$ have probabilities

$$P(\mathbf{m}(\text{IEA}) = \mathbf{m}) = \binom{m}{\mathbf{m}} \mathbf{Q}^{\mathbf{m}} = \frac{m!}{\prod_{i\leq j} m_{ij}!} \prod_{i\leq j} Q_{ij}^{m_{ij}} .$$

Another random multigraph model is obtained by assuming that the edges are formed by random matching of pairs of edge-stubs in a given sequence of edge-stubs. This random stub matching (RSM) model has fixed stub multiplicity sequence $\mathbf{d} = (d_1, \ldots, d_n)$. Under RSM, the edge assignments to sites are dependent. The probability that an edge is assigned to site $(i, j) \in R$ is given by

$$Q_{ij} = \begin{cases} \binom{d_i}{2} / \binom{2m}{2} & \text{for } i = j \\ d_i d_j / \binom{2m}{2} & \text{for } i < j \ , \end{cases}$$

so that the edge probability sequence $\mathbf{Q} = \mathbf{Q}(\mathbf{d})$ is a function of the stub multiplicity sequence $\mathbf{d}$. The probability of edge multiplicity sequence $\mathbf{m}$ under RSM is shown in Shafie (2012) to be given by

$$P(\mathbf{m}(\text{RSM}) = \mathbf{m}) = \frac{2^{m_2}\binom{m}{\mathbf{m}}}{\binom{2m}{\mathbf{d}}} = \frac{2^{m_2} m! \prod_{i=1}^{n} d_i!}{(2m)! \prod_{i\leq j} m_{ij}!} \ ,$$

where $m_2 = \sum\sum_{i<j} m_{ij}$.

A Bayesian version of the RSM model is obtained by assuming that the stubs are independently assigned to vertices according to a probability distribution $\mathbf{p} = (p_1, ..., p_n)$. The stub multiplicity sequence under independent stub assignments (ISA) is multinomially distributed with parameters $2m$ and $\mathbf{p}$. This multinomial distribution can be viewed as a

Bayesian model for the stub multiplicities and leads to independent edge assignments. Thus by the Bayesian assumption the RSM model is turned into a special IEA model with edge probability sequence $\mathbf{Q}$ defined as a function of $\mathbf{p}$ according to

$$Q_{ij} = \begin{cases} p_i^2 & \text{for } i = j \\ 2p_ip_j & \text{for } i < j \ . \end{cases}$$

Another way to get an approximate IEA model from an RSM model is to ignore the dependency between the edge assignments in the RSM model. The edge probability sequence $\mathbf{Q} = \mathbf{Q}(\mathbf{d})$ of the RSM model is used to define a model with independent edge assignment of stubs (IEAS). Note that the IEAS model, like other IEA models, has $\binom{m+r-1}{m}$ different outcomes of $\mathbf{m}$, while the RSM models are restricted to outcomes that are consistent with stub multiplicity sequence $\mathbf{d}$ only.

The following notations will be used for the models presented in this section. Independent edge assignment is denoted IEA($\mathbf{Q}$), random stub matching is denoted RSM($\mathbf{d}$), independent stub assignments is denoted ISA($\mathbf{p}$), and independent edge assignments of stubs is denoted IEAS($\mathbf{d}$).

4 Statisticial Tests of a Simple Multigraph Hypothesis

4.1 Test Statistics

A simple multigraph hypothesis H_0 is defined as a fully specified IEA($\mathbf{Q_0}$) which can be an ISA($\mathbf{p_0}$) or an IEAS($\mathbf{d_0}$) with $\mathbf{Q_0}$ specified as a function of $\mathbf{d_0}$ or $\mathbf{p_0}$. The tests are performed using goodness-of-fit measures between the multiplicity sequence $\mathbf{m}$ of an observed multigraph and the expected multiplicity sequence according to H_0.

Asymptotic theory for likelihood ratios and goodness-of-fit statistics is given for instance by Anderson (1980) and Cox and Hinkley (1974). The Pearson goodness-of-fit statistic is given by

$$S_0 = \sum\sum_{i \leq j} \frac{(m_{ij} - mQ_{0ij})^2}{mQ_{0ij}} = \sum\sum_{i \leq j} \frac{m_{ij}^2}{mQ_{0ij}} - m \ ,$$

which is asymptotically χ^2-distributed with $df = r - 1$ degrees of freedom if the multiplicity sequence is obtained according to IEA($\mathbf{Q}$) and the correct model $\mathbf{Q_0} = \mathbf{Q}$ is tested. We denote a random variable with this distribution χ^2_{r-1}. The divergence statistic is given by

$$D_0 = \sum\sum_{i \leq j} \frac{m_{ij}}{m} \log \frac{m_{ij}}{mQ_{0ij}} \ ,$$

and an asymptotic χ^2_{r-1}-statistic can be obtained as

$$T_0 = \frac{2m}{\log \mathrm{e}} D_0 \ .$$

Divergence statistics are used as goodness-of-fit statistics for instance by Kullback (1959) and Frank (2011). For good asymptotics it is normally assumed that m is large and mQ_{ij} is not too small (for instance $mQ_{ij} \geq 5$ and $m \geq 5r$). By approximation of the logarithm function it can be shown that $S_0 \approx T_0$ for large m. The critical region for the tests is taken as values of S_0 and T_0 above a critical value cv given by

$$cv = df + 2\sqrt{2df} = r - 1 + \sqrt{8(r-1)} \ ,$$

which has a significance level approximately equal to 5% given by

$$\alpha = P(\chi^2_{r-1} > cv) \ .$$

The power functions

$$P(S_0 > cv) = 1 - \beta_{S_0}(\mathbf{Q}) \quad \text{and} \quad P(T_0 > cv) = 1 - \beta_{T_0}(\mathbf{Q})$$

are calculated using the distributions of S_0 and T_0 when $\mathbf{m}$ is multinomially distributed with parameters m and $\mathbf{Q}$, for $\mathbf{Q} = \mathbf{Q_0}$ and for $\mathbf{Q} \neq \mathbf{Q_0}$. Specifically, S_0 and T_0 are compared to χ^2_{r-1} via moments and cumulative distribution functions. For instance, the expected value of S_0 reveals how far from $E(\chi^2_{r-1}) = r - 1$ the distribution of S_0 is. This expected value is given by

$$E(S_0) = \sum\sum_{i \leq j} \frac{E(m_{ij}^2)}{mQ_{0ij}} - m = \sum\sum_{i \leq j} \frac{Q_{ij} + (m-1)Q_{ij}^2}{Q_{0ij}} - m \ ,$$

where m_{ij} is binomially distributed with parameters m and Q_{ij} so that

$$E(m_{ij}^2) = \mathrm{Var}(m_{ij}) + [E(m_{ij})]^2 = mQ_{ij}(1 - Q_{ij}) + m^2 Q_{ij}^2 = mQ_{ij} + m(m-1)Q_{ij}^2 \ .$$

In particular, if $\mathbf{Q} = \mathbf{Q_0}$ so that $Q_{ij} = Q_{0ij}$ for $i \leq j$, the null distribution of S_0 has expected value

$$E(S_0) = \sum\sum_{i \leq j} [1 + (m-1)Q_{ij}] - m = r - 1 \ .$$

Under the ISA($\mathbf{p}$) model and ISA($\mathbf{p_0}$) hypothesis, the expected value of S_0 is given as

$$\begin{aligned} E(S_0) &= \sum_{i=1}^{n} L_i^2 \left[1 + (m-1)p_i^2\right] + \sum\sum_{i \neq j} \frac{L_i L_j}{2} \left[1 + (m-1)2p_i p_j\right] - m \\ &= \sum_{i=1}^{n} L_i^2 + (m-1)\sum_{i=1}^{n} (L_i p_i)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{L_i L_j}{2} \\ &\qquad + (m-1)\sum_{i=1}^{n}\sum_{j=1}^{n} L_i L_j p_i p_j - \sum_{i=1}^{n} \frac{L_i^2}{2} - (m-1)\sum_{i=1}^{n} (L_i p_i)^2 - m \\ &= \frac{\sum_{i=1}^{n} L_i^2 + \left(\sum_{i=1}^{n} L_i\right)^2}{2} - m + (m-1)\left(\sum_{i=1}^{n} L_i p_i\right)^2 \ , \end{aligned}$$

where $L_i = p_i/p_{0i}$ is the likelihood ratio for stub assignments. As seen, the variation of $E(S_0)$ depends on $\sum_{i=1}^n L_i$, $\sum_{i=1}^n L_i^2$ and $\sum_{i=1}^n L_i p_i$. In particular, for a uniform ISA($\mathbf{p_0}$) hypothesis where $p_{oi} = 1/n$,

$$E(S_0) = \frac{n^2 \sum_{i=1}^n p_i^2 + n^2}{2} - m + (m-1)n^2 \left(\sum_{i=1}^n p_i^2 \right)^2 ,$$

which by letting $s_2 = \sum_{i=1}^n p_i^2$ can be simplified to

$$E(S_0) = m(n^2 s_2^2 - 1) + \frac{n^2}{2}(1 + s_2 - 2s_2^2) .$$

From this we see that $E(S_0)$ grows linearly with m having coefficients depending on n and s_2. By using

$$E(S_0) = s_2^2 n^2 (m-1) + s_2 \frac{n^2}{2} + \frac{n^2}{2} - m$$

and $1/n \leq s_2 \leq 1$, it follows that

$$r - 1 \leq E(S_0) \leq m(n^2 - 1) .$$

We also note that if $\mathbf{p} = \mathbf{p_0}$ so that $p_i = p_{0i}$, the null distibution has

$$E(S_0) = \frac{n + n^2}{2} - m + (m-1) = \binom{n+1}{2} - 1 = r - 1$$

which is consistent with the result shown previously for $\mathbf{Q} = \mathbf{Q_0}$.

The expected value of S_0 can also be considered for the RSM($\mathbf{d}$) model when H_0 is RSM($\mathbf{d_0}$) or IEAS($\mathbf{d_0}$) since $\mathbf{Q_0}$ of IEAS and RSM are identical. Shafie (2012) gives the moments of m_{ij} under RSM as

$$E(m_{ij}) = mQ_{ij} \quad \text{for } i \leq j ,$$

and

$$\mathrm{Var}(m_{ij}) = \sigma_{ij}^2 + \Delta_{ij} \quad \text{for } i \leq j ,$$

where $\sigma_{ij}^2 = mQ_{ij}(1 - Q_{ij})$ is the variance under IEA, and Δ_{ij} is the difference between the variances of m_{ij} under RSM and IEA:

$$\Delta_{ij} = m(m-1)(Q_{ijij} - Q_{ij}^2) ,$$

where

$$Q_{ijij} = \begin{cases} Q_{ii} \left(\frac{(d_i-2)(d_i-3)}{(2m-2)(2m-3)} \right) & \text{for } i = j \\ \\ Q_{ij} \left(\frac{2(d_i-1)(d_j-1)}{(2m-2)(2m-3)} \right) & \text{for } i < j . \end{cases}$$

A general expression for the expected value of S_0 under RSM is here obtained as

$$
\begin{aligned}
E(S_0) &= \sum_{i \le j}\sum \frac{E(m_{ij}^2)}{mQ_{0ij}} - m \\
&= \sum_{i \le j}\sum \frac{\sigma_{ij}^2 + \Delta_{ij} + m^2 Q_{ij}^2}{mQ_{0ij}} - m \\
&= \sum_{i \le j}\sum \frac{mQ_{ij}(1-Q_{ij}) + \Delta_{ij} + m^2 Q_{ij}^2}{mQ_{0ij}} - m \ .
\end{aligned}
$$

For $\mathbf{Q} = \mathbf{Q_0}$ so that $Q_{ij} = Q_{0ij}$ for $i \le j$, this simplifies to

$$
\begin{aligned}
E(S_0) &= r - 1 + \sum_{i \le j}\sum \frac{\Delta_{ij}}{mQ_{ij}} \\
& r - 1 + \sum_{i \le j}\sum \frac{m(m-1)(Q_{ijij} - Q_{ij}^2)}{mQ_{ij}} \\
&= r - 1 + (m-1)\left[\sum_{i \le j}\sum \frac{Q_{ijij}}{Q_{ij}} - \sum_{i \le j}\sum Q_{ij}\right] \\
&= r - m + (m-1)\left[\sum_{i \le j}\sum \frac{Q_{ijij}}{Q_{ij}}\right] \\
&= r - m + (m-1)\left[\sum_{i < j}\sum \frac{2(d_i-1)(d_j-1)}{(2m-2)(2m-3)} + \sum_{i=1}^{n} \frac{(d_i-2)(d_i-3)}{(2m-2)(2m-3)}\right] \\
&= r - m + \frac{1}{2(2m-3)}\left[\sum_{i \ne j}\sum (d_i-1)(d_j-1) + \sum_{i=1}^{n}(d_i-2)(d_i-3)\right] \\
&= r - m + \frac{1}{2(2m-3)}\left[4m^2 + 4mn + n^2 - 6m + 5n\right] \\
&= \frac{(m-1)n(n-1)}{2m-3} \ ,
\end{aligned}
$$

which implies that the expected value of the null distribution only depends on the number of vertices and edges. Using this expression we can now show for which values of m and n the expected value of S_0 under RSM is smaller than $r-1$, i.e.

$$
E(S_0) = \frac{(m-1)n(n-1)}{2m-3} < (r-1) = \frac{n(n+1)}{2} - 1 \ .
$$

Solving the inequality for m gives the following results:

$$E(S_0) < r - 1 \quad \text{for} \quad m > \frac{n+6}{4} \ ,$$

$$E(S_0) = r - 1 \quad \text{if} \quad m = \frac{n+6}{4} \quad \text{is integer} \ ,$$

and

$$E(S_0) > r - 1 \quad \text{for} \quad m < \frac{n+6}{4} \ .$$

Note that the restriction $2m \geq n$ imposed by no isolated vertices implies that $E(S_0) > r-1$ only for some degenerate cases ($n = 2$, $m = 1$) and the extreme cases $n = 3$ or 4, and $m = 2$. Therefore, under RSM the null distribution of the test statistic S_0 has for all other cases an expected value below $r - 1$, and its cumulative distribution function will tend to lie on or above that of χ^2_{r-1} for all practical useful cases. Exceptional cases with $m < (n + 6)/4$ have so few stubs to be matched that they are unlikely to be useful in practice. Compare the requirement of large m needed for good χ^2 asymptotics. Note however that the test statistics may not have asymptotic χ^2-distributions under RSM due to dependency between edges.

Any test statistic S, like S_0 or T_0, can be approximated by an adjusted χ^2-distribution given by

$$S^* = \frac{\mu}{k}\chi^2_k \ ,$$

where $\mu = E(S)$. For any positive integer k the approximation S^* has the same mean as S and a variance given by

$$Var(S^*) = \frac{2\mu^2}{k} \ .$$

Two particular approximations $S^{'}$ and $S^{''}$ are given by S^* for k chosen as the integer part of μ and for $k = r - 1$, respectively. Their variances are

$$Var(S^{'}) = \frac{2\mu^2}{\lfloor \mu \rfloor} \quad \text{and} \quad Var(S^{''}) = \frac{2\mu^2}{r-1} \ ,$$

and the preferred approximation is the one with variance closest to $Var(S) = \sigma^2$. Equivalently, the preferred adjusted χ^2-distribution is the one with degrees of freedom closest to $2\mu^2/\sigma^2$. A good approximation is useful for power calculations.

4.2 Test Illustrations for IEAS Models

We consider multigraphs with 4 vertices and 10 edges and test IEAS($\mathbf{d_0}$) hypotheses against IEAS($\mathbf{d}$) models. The degree sequences are chosen to include both skew and flat (uniform and close to uniform) cases. The number of edge sites is here given by $r = 10$ and the test statistics S_0 and T_0 are thus asymptotically χ^2_9-distributed when the correct model with $\mathbf{d_0} = \mathbf{d}$ is being tested. The critical value is $cv = 17.49$ and $\alpha = P(\chi^2_9 > cv) = 0.04$. The powers of these tests according to S_0 and T_0 are given in Table 1, where the diagonal representing $\mathbf{d_0} = \mathbf{d}$ is shaded. Note that there is one case where the order between the components in $\mathbf{d_0}$ is switched. For this special case, the large deviations between the degree values in models and hypotheses result in powers being close or equal to one for both statistics. When $\mathbf{d_0} = \mathbf{d}$, $\alpha_{T_0} = 1-\beta_{T_0} < \alpha \leq 1-\beta_{S_0} = \alpha_{S_0}$. For flat $\mathbf{d_0} = \mathbf{d}$, both statistics have significance levels equal or close to α, but for skew $\mathbf{d_0} = \mathbf{d}$, the significance level of T_0 is much below α and that of S_0 is much above α. For the majority of cases with not too skew $\mathbf{d_0} \neq \mathbf{d}$, both statistics have fairly good powers, but the inequalities between them persist indicating that their cumulative distribution functions can approach an asymptotic distribution from either below or above. To illustrate the fit of the distributions of the statistics S_0 and T_0 to χ^2_9, their cumulative distribution functions are shown in Figure 8. For flat $\mathbf{d_0} = \mathbf{d}$, the null distribution of S_0 almost coincides with that of χ^2_9. For skew $\mathbf{d_0} = \mathbf{d}$, the null distributions of both statistics give poor fit to χ^2_9-distribution. This poor fit is also noted for both flat and skew $\mathbf{d_0} \neq \mathbf{d}$. Both S_0 and T_0 seem to have distributions that would be better approximated by χ^2 with degrees of freedom chosen to be higher than $r - 1$ in cases with $\mathbf{d_0} \neq \mathbf{d}$.

The speed of the convergence of the cumulative distribution functions of S_0 and T_0 is illustrated in Figures 9 and 10 where both flat and skew $\mathbf{d_0} = \mathbf{d}$ are considered. The number of edges m increases as multiples of the chosen degree sequences. We see that even for small m, the null distributions of both statistics are fairly well approximated by their asymptotic χ^2-distribution. A similar investigation of the non-null distributions of S_0 and T_0 is shown in Figure 11 for flat $\mathbf{d_0} \neq \mathbf{d}$ and in Figure 12 for skew $\mathbf{d_0} \neq \mathbf{d}$, where $\mathbf{d_0}$ is kept fixed and $\mathbf{d}$ is varied. For both flat and skew $\mathbf{d_0}$, the deviations between the non-null distributions of S_0 and T_0 and their asymptotic null distribution increase with the number of edges, and even for $m = 12$ this deviation is clearly notable. Thus even for the rather small $m = 12$, it is easy to detect simple hypotheses about false models.

Two cases in Table 2 illustrate how test statistics can be approximated by adjusted χ^2-distributions. The approximated goodness-of-fit statistics are $S_0^{'}$ and $S_0^{''}$, and the approximated divergence statistics are $T_0^{'}$ and $T_0^{''}$. These approximations are evaluated by comparing their variances to $Var(S_0)$ and $Var(T_0)$. The expected values and variances of all versions of the test statistics are presented in Table 2 where the versions that are not preferred are shaded so that it is easier to compare preferences in different cases. For the first case, $S_0^{''}$ is preferred to $S_0^{'}$ while $T_0^{'}$ is preferred to $T_0^{''}$. Equivalently, the preferred adjusted χ^2-distribution for S_0 has $df = r - 1 = 9$ since it is closer than $df = \lfloor \mu \rfloor = 13$ to $2E(S_0)/Var(S_0) = 7.31$, and the adjusted χ^2-distribution for T_0 has $df = \lfloor \mu \rfloor = 13$ since it is closer than $df = r - 1 = 9$ to $2E(T_0)/Var(T_0) = 18.19$. For the second case, $S_0^{'}$ is preferred to $S_0^{''}$, while $T_0^{''}$ is preferred to $T_0^{'}$.

Table 1: Power according to S_0 (upper value) and T_0 (value below) when model is IEAS($\mathbf{d}$) and hypothesis is IEAS($\mathbf{d_0}$) for $n = 4$ and $m = 10$. $\alpha = 0.04$.

$\mathbf{d_0}$ \ $\mathbf{d}$	(14, 2, 2, 2)	(12, 3, 3, 2)	(9, 7, 2, 2)	(8, 8, 2, 2)	(6, 6, 6, 2)	(6, 5, 5, 4)	(5, 5, 5, 5)
(14, 2, 2, 2)	0.19 0.01	0.42 0.06	0.87 0.52	0.94 0.70	0.98 0.87	0.97 0.82	0.99 0.92
(2, 2, 14, 2)	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	0.98 0.87	0.99 0.92	0.99 0.92
(12, 3, 3, 2)	0.06 0.01	0.10 0.01	0.50 0.24	0.66 0.40	0.75 0.52	0.77 0.50	0.89 0.69
(9, 7, 2, 2)	0.34 0.25	0.31 0.14	0.13 0.01	0.14 0.02	0.74 0.40	0.78 0.44	0.87 0.60
(8, 8, 2, 2)	0.58 0.47	0.44 0.26	0.13 0.02	0.13 0.01	0.73 0.38	0.78 0.44	0.87 0.58
(6, 6, 6, 2)	0.85 0.74	0.54 0.39	0.24 0.19	0.21 0.18	0.08 0.02	0.36 0.14	0.53 0.27
(6, 5, 5, 4)	0.78 0.69	0.46 0.36	0.26 0.22	0.28 0.23	0.07 0.06	0.05 0.03	0.07 0.05
(5, 5, 5, 5)	0.91 0.63	0.70 0.63	0.48 0.43	0.46 0.41	0.13 0.12	0.05 0.04	0.04 0.03

Table 2: Moments of S_0, $S_0^{'}$, $S_0^{''}$, T_0, $T_0^{'}$ and $T_0^{''}$ for some IEAS($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) hypotheses with $n = 4$ and $m = 10$. The unshaded columns correspond to the best approximations to the test statistics.

Case 1: $\mathbf{d_0} = (6, 6, 6, 2)$, $\mathbf{d} = (8, 8, 2, 2)$

	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	13.62	13.62	13.62	13.40	13.40	13.40
Variance	50.74	28.52	41.20	19.74	27.64	39.93

Case 2: $\mathbf{d_0} = (12, 3, 3, 2)$, $\mathbf{d} = (14, 2, 2, 2)$

	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	7.51	7.51	7.51	7.82	7.82	7.82
Variance	31.79	16.14	12.55	10.61	17.47	13.59

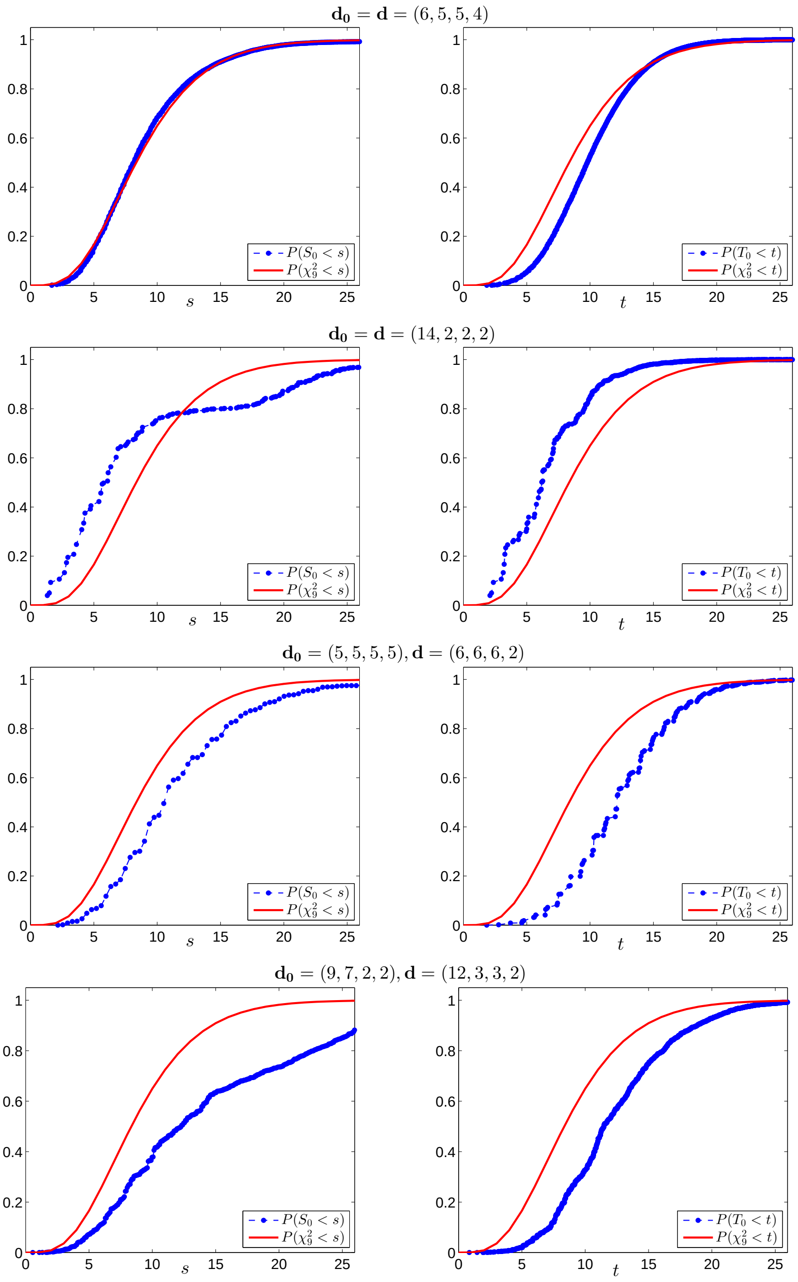

Figure 8: Distributions of S_0, T_0, and χ^2_9 for some IEAS($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) hypotheses with $n = 4$ and $m = 10$.

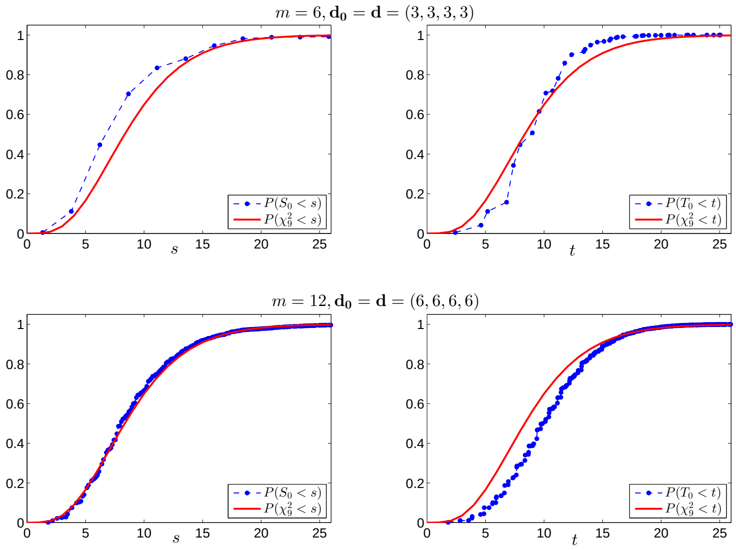

Figure 9: Null distributions of S_0 and T_0 for some IEAS($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) hypotheses with flat $\mathbf{d_0} = \mathbf{d}$ when m increases.

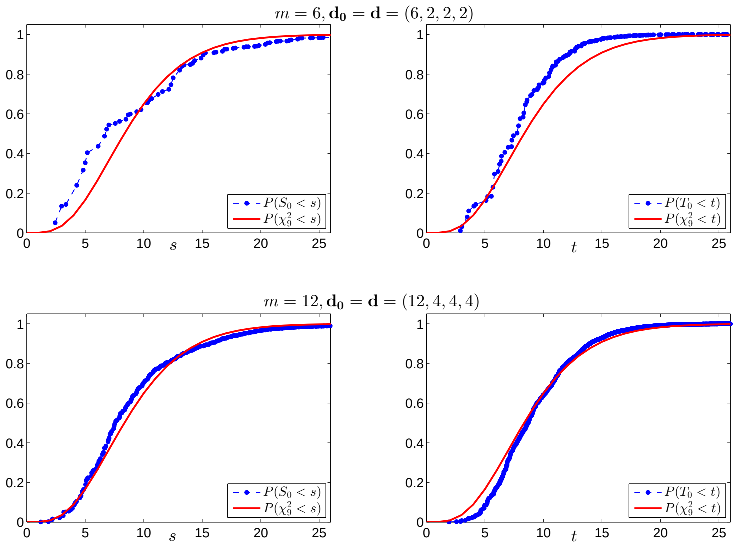

Figure 10: Null distributions of S_0 and T_0 for some IEAS($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) hypotheses with skew $\mathbf{d_0} = \mathbf{d}$ when m increases.

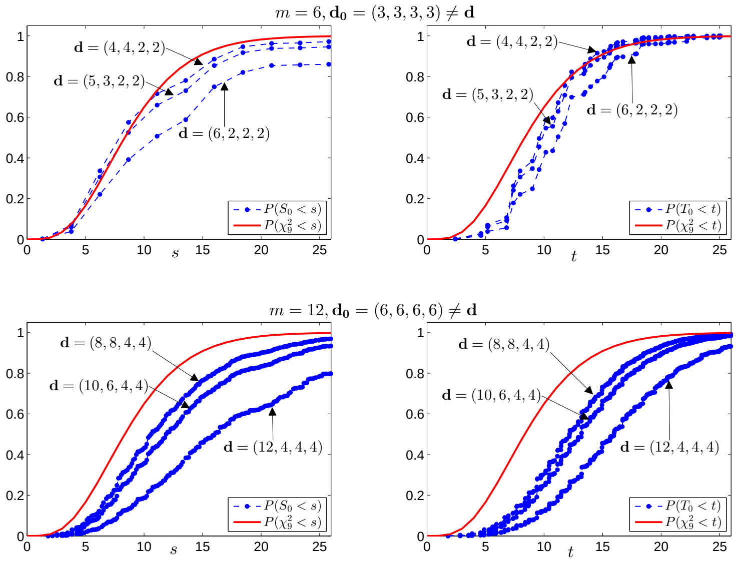

Figure 11: Non-null distributions of S_0 and T_0 for some IEAS($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) hypotheses with flat $\mathbf{d_0}$ and different $\mathbf{d}$ when m increases.

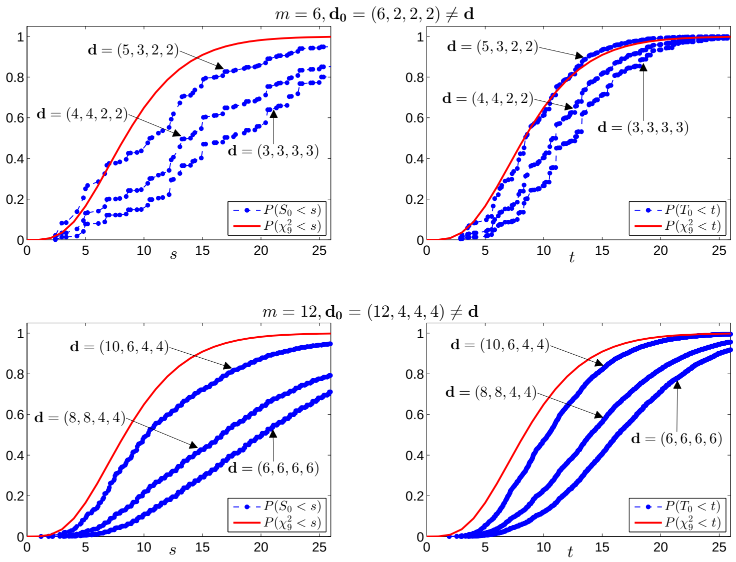

Figure 12: Non-null distributions of S_0 and T_0 for some IEAS($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) hypotheses with skew $\mathbf{d_0}$ and different $\mathbf{d}$ when m increases.

4.3 Test Illustrations for ISA Models

We now turn our attention to ISA($\mathbf{p_0}$) hypotheses tested against ISA($\mathbf{p}$) models and consider tests of different multigraphs of the same size. The stub selection probability sequences are chosen to illustrate both skew and flat cases. Note that there is one case where the order between the components in $\mathbf{p_0}$ is switched. The powers of these tests according to S_0 and T_0 are given in Table 3. We see that most results are consistent with those seen in Table 1 for IEAS models. For $\mathbf{p_0} = \mathbf{p}$, α_{S_0} and α_{T_0} are on opposite sides of $\alpha = 0.04$ but they are both close to α except for very skew cases. For the majority of cases with $\mathbf{p_0} \neq \mathbf{p}$, both test statistics have reasonable powers unless $\mathbf{p_0}$ and $\mathbf{p}$ are too close. In Figure 13, the fit of the distributions of the statistics S_0 and T_0 to that of χ^2_9 are illustrated for some selected cases. Overall, we see that even for these examples with small m, we have fairly good fit for all illustrated cases with both flat and skew $\mathbf{p_0} = \mathbf{p}$, and $\mathbf{p_0} \neq \mathbf{p}$.

The impact on the null and non-null distributions of S_0 and T_0 for skew and flat $\mathbf{p_0}$ when m increases is illustrated in Figures 14 to 17 where similar results as those for IEAS models are noted. The convergence to the asymptotic distribution is rapid for null distributions of both statistics, and the deviations between the non-null distributions of both statistics and their asymptotic null distribution increase with m. The latter result implies that adjusted χ^2-distributions should be used to approximate the non-null distributions.

Two cases from Table 3 are chosen to illustrate the performance of the approximate test statistics. By comparing variances we obtain the results presented in Table 4, where non-preferred statistics are shaded.

Table 3: Power according to S_0 (upper value) and T_0 (value below) when model is ISA($\mathbf{p}$) and hypothesis is ISA($\mathbf{p_0}$) for $n = 4$ and $m = 10$. $\alpha = 0.04$.

$\mathbf{p_0}$ \ $\mathbf{p}$	$(\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$	$(\frac{3}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{10})$	$(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$	$(\frac{4}{9}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9})$	$(\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
$(\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$	0.10 0.01	0.33 0.07	0.57 0.19	0.80 0.47	0.92 0.67	0.99 0.88
$(\frac{1}{10}, \frac{1}{10}, \frac{7}{10}, \frac{1}{10})$	1.00 1.00	1.00 1.00	1.00 0.99	1.00 1.00	1.00 1.00	0.99 0.88
$(\frac{3}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{10})$	0.06 0.01	0.08 0.01	0.36 0.11	0.32 0.11	0.50 0.24	0.92 0.72
$(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$	0.04 0.05	0.04 0.03	0.06 0.02	0.27 0.14	0.42 0.25	0.53 0.35
$(\frac{4}{9}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9})$	0.27 0.24	0.09 0.05	0.29 0.14	0.06 0.02	0.11 0.04	0.74 0.49
$(\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8})$	0.54 0.47	0.19 0.14	0.29 0.19	0.04 0.02	0.05 0.02	0.58 0.38
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	0.88 0.86	0.66 0.63	0.32 0.28	0.35 0.33	0.25 0.23	0.03 0.03

Table 4: Moments of S_0, $S_0^{'}$, $S_0^{''}$, T_0, $T_0^{'}$ and $T_0^{''}$ for some ISA($\mathbf{p}$) models and ISA($\mathbf{p_0}$) hypotheses with $n = 4$ and $m = 10$. The unshaded columns correspond to the best approximations to the test statistics.

Case 1: $\mathbf{p_0} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, $\mathbf{p} = (\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8})$	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	14.56	14.56	14.56	14.43	14.43	14.43
Variance	50.98	30.30	47.13	22.05	29.74	46.27

Case 2: $\mathbf{p_0} = (\frac{3}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{10})$, $\mathbf{p} = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	17.08	17.08	17.08	11.20	11.20	11.20
Variance	167.82	34.33	64.85	25.83	22.82	27.89

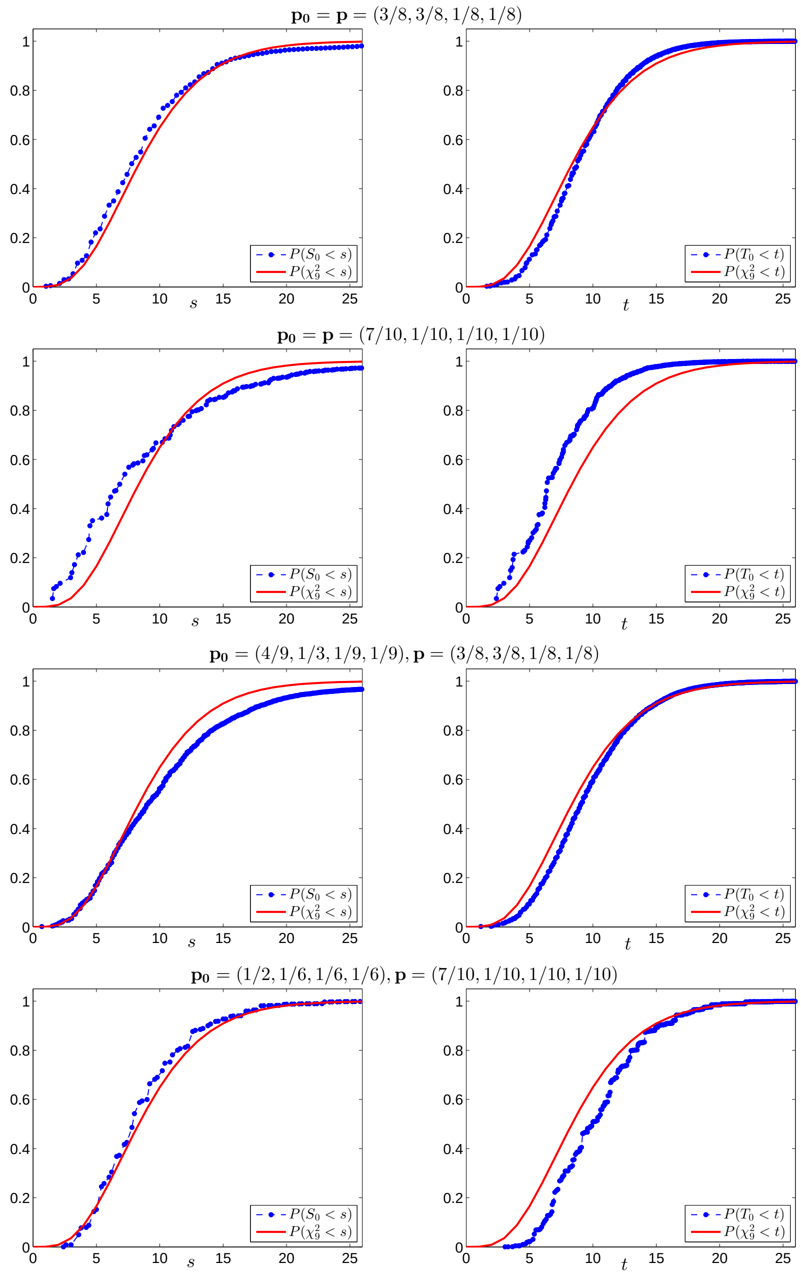

Figure 13: Distributions of S_0, T_0 and χ^2_9 for some ISA($\mathbf{p}$) models and ISA($\mathbf{p_0}$) hypotheses with $n = 4$ and $m = 10$.

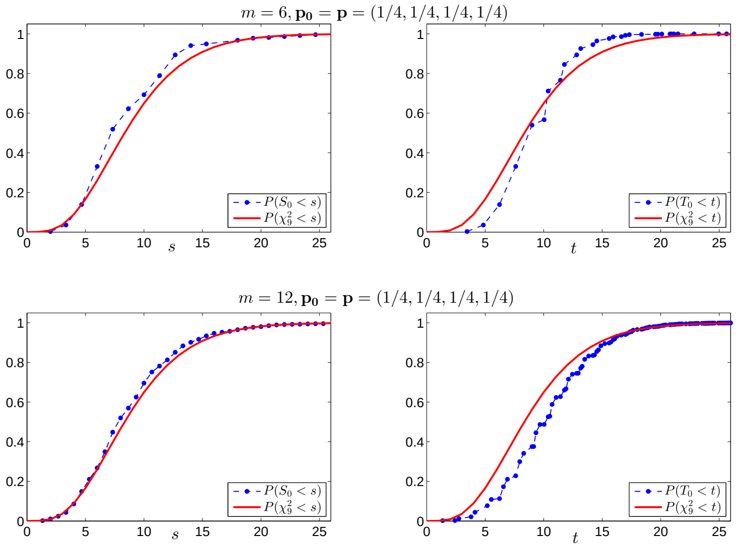

Figure 14: Null distributions of S_0 and T_0 for some ISA($\mathbf{p}$) models and ISA($\mathbf{p_0}$) hypotheses with flat $\mathbf{p_0} = \mathbf{p}$ when m increases.

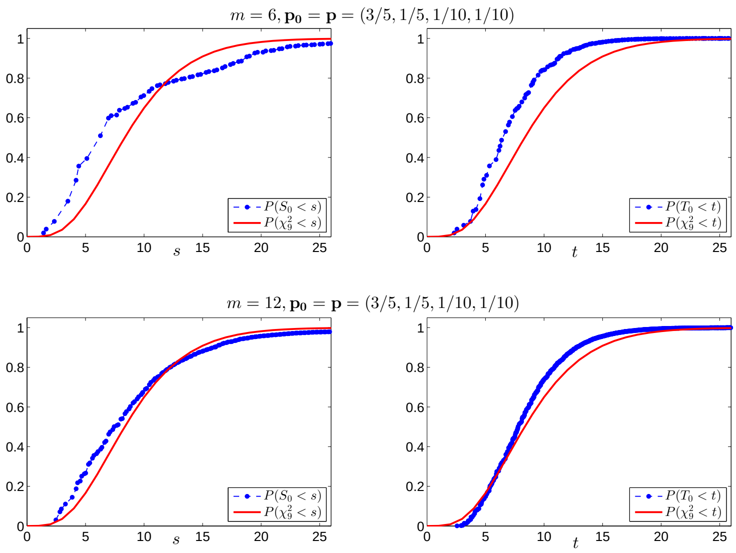

Figure 15: Null distributions of S_0 and T_0 for some ISA($\mathbf{p}$) models and ISA($\mathbf{p_0}$) hypotheses with skew $\mathbf{p_0} = \mathbf{p}$ when m increases.

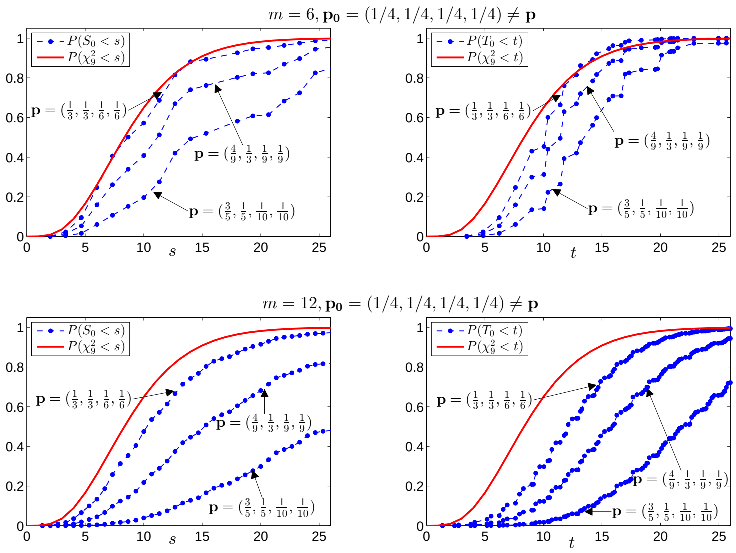

Figure 16: Non-null distributions of S_0 and T_0 for some ISA($\mathbf{p}$) models and ISA($\mathbf{p_0}$) hypotheses with flat $\mathbf{p_0}$ and different $\mathbf{p}$ when m increases.

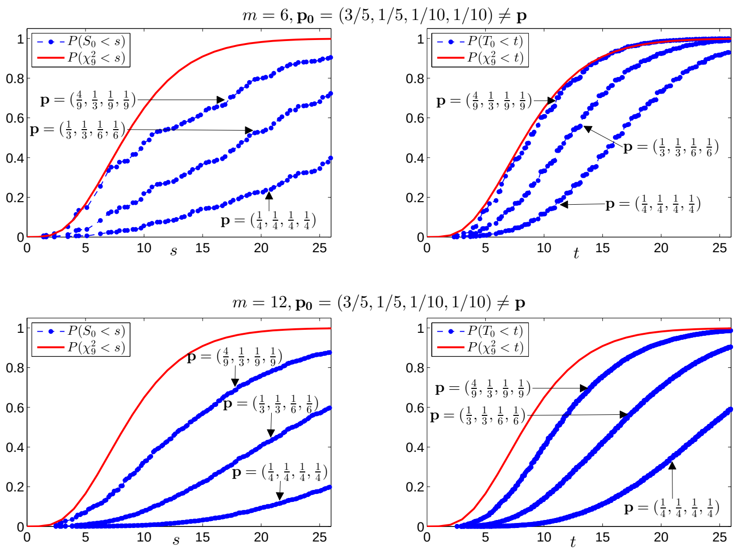

Figure 17: Non-null distributions of S_0 and T_0 for some ISA($\mathbf{p}$) models and ISA($\mathbf{p_0}$) hypotheses with skew $\mathbf{p_0}$ and different $\mathbf{p}$ when m increases.

4.4 Test Illustrations for RSM Models

When performing tests of IEA models, multigraphs are known to have multiplicity sequences that are multinomially distributed, which implies that the distributions of the test statistics S_0 and T_0 are asymptotically χ^2-distributed when the correct model is being tested. However, for RSM models, there is dependence between edges, and the distributions of S_0 and T_0 are unknown. In this section, we illustrate some of the consequences of using the previously described tests of simple hypotheses against a false IEA model when the true model is RSM. Here, both IEAS($\mathbf{d_0}$) and ISA($\mathbf{p_0}$) hypotheses are tested for flat and skew $\mathbf{d_0}$ and $\mathbf{p_0}$. The true model is RSM($\mathbf{d}$) so that only non-null distributions of S_0 and T_0 are considered.

We start by testing multigraphs with 4 vertices and 12 edges. The powers of these tests according to S_0 and T_0 are presented in Table 5. For IEAS($\mathbf{d_0}$) hypotheses, the diagonal representing $\mathbf{d_0} = \mathbf{d}$ is shaded, and for ISA($\mathbf{p_0}$) hypotheses, the diagonal representing $\mathbf{p_0} = \mathbf{d}/2m$ is shaded. For these shaded cases, both α_{S_0} and α_{T_0} are much below $\alpha = 0.04$, except for the very skew $\mathbf{d_0} = \mathbf{d} = (18,\ 2,\ 2,\ 2)$ where α_{S_0} is much above α. For the majority of cases with $\mathbf{d_0} \neq \mathbf{d}$ or $\mathbf{p_0} \neq \mathbf{d}/2m$ both test statistics have good or reasonable powers, unless $\mathbf{d}$ is too close to $\mathbf{d_0}$ or $2m\mathbf{p_0}$. To illustrate the fit of the distributions of the statistics S_0 and T_0 to that of χ^2_9, their cumulative distribution functions for some selected cases are shown in Figure 18. We see similar trends as those for IEAS models in Figure 8 and ISA models in Figure 13 which generally makes it hard to detect differences between how the models RSM, IEAS and ISA effect the test statistics.

Four cases are chosen to illustrate adjusted χ^2-approximations to the distributions of the test statistics. Table 6 shows the expected values and variances for test statistics and approximations, and the approximations that are not preferred are shaded. Thus we see that the preferences can vary in all different ways.

Let us now increase the number of edges and consider multigraphs with 3 vertices and 45 edges. The powers of these tests according to S_0 and T_0 are presented in Table 7 where the following is noted. For IEAS and ISA hypotheses with $\mathbf{d_0} = \mathbf{d}$ and $\mathbf{p_0} = \mathbf{d}/2m$, the significance levels of both S_0 and T_0 are much smaller than α and also equal or almost equal. There are some cases of powers below α implying that it is difficult to detect hypotheses about wrong models. For IEAS hypotheses where $\mathbf{d_0} \neq \mathbf{d}$, and ISA hypotheses where $\mathbf{p_0} \neq \mathbf{d}/2m$, the powers are equal or close to one another in the majority of cases. This is a consequence of similarities between IEAS and ISA models for large m. Other results concerning the powers in Table 7 are similar to those seen in Table 5. Figure 19 illustrates the fit of the distributions of S_0 and T_0 to that of χ^2_5 for some different cases with $m = 45$ and should be compared to Figure 18 which illustrates $m = 12$. We see strong deviations from χ^2_5 for both S_0 and T_0, and for flat $\mathbf{d_0}$, the distributions are either to the left of the χ^2_5-distribution or close to it. This is further illustrated in Figures 20-23 and is in contrast to the finding for IEAS models in Figure 9-12 and for ISA models in Figures 14-17.

In Figures 20 to 23, the non-null distribution of S_0 and T_0 for some RSM($\mathbf{d}$) models are illustrated where m increases as multiples of different specified $\mathbf{d}$. Figures 20 and 21 illustrate IEAS hypotheses with flat and skew $\mathbf{d_0}$, and Figures 22 and 23 illustrate ISA hypotheses with flat and skew $\mathbf{p_0}$. When $\mathbf{d_0} = \mathbf{d}$ or $\mathbf{p_0} = \mathbf{d}/2m$, the non-null distributions of both S_0 and T_0 lie above the asymptotic null distributions. This is consistent with results shown in Section 5.1. Further for these cases we see that as m increases, these distributions still lie above the asymptotic null distribution, and a χ^2-distribution with lower degrees of freedom seem to better approximate these distributions. For cases with $\mathbf{d_0} \neq \mathbf{d}$ or $\mathbf{p_0} \neq \mathbf{d}/2m$, the non-null distributions of both statistics move further away from the asymptotic null distribution as m increases, implying a need to use adjusted χ^2-distributions for better fit.

Three cases to illustrate the approximations by adjusted χ^2-distributions are given in Table 8. For all three cases, $S_0^{'}$ and $T_0^{'}$ are preferred. For the second and third case with $\mathbf{d_0} = \mathbf{d}$, the variances of S_0 and T_0 are roughly twice their expected values which are approximately equal to 3. Thus, the adjusted χ^2-distribution for both test statistics seem to be closer to $r - n$ rather than $r - 1$ degrees of freedom under RSM. This is also supported by the expected value of S_0 which according to the result in Section 4.1 is $(m-1)n(n-1)/(2m-3)$ which is about $r - n = n(n-1)/2$.

So far in this section we have considered the consequences of replacing IEA models with RSM models, but have only tested IEA hypotheses. We conclude this section with a comment about testing RSM hypotheses. A simple RSM($\mathbf{d_0}$) hypothesis has the same $\mathbf{Q_0}$ as the IEAS($\mathbf{d_0}$) hypothesis, and S_0 and T_0 can not distinguish between these two hypotheses. Should the model be RSM($\mathbf{d}$), there is a dependency between edges when they are assigned to sites, which could be used to distinguish between the two hypotheses. This requires a test not using S_0 or T_0, but a test using the full potential of $\mathbf{m}$ having as its critical region the set $\overline{M(\mathbf{d_0})}$ consisting of all outcomes $\mathbf{m}$ that are not compatible with $\mathbf{d_0}$. This test has zero probability of false rejection of RSM($\mathbf{d_0}$), and its power can be determined as the sum of the probabilities according to RSM($\mathbf{d}$) of the outcomes in the critical region. Shafie (2012) gives the RSM($\mathbf{d}$) probabilities and specifies outcomes of $\mathbf{m}$ compatible with a fixed degree sequence. We will not pursue details of this test further here.

Table 5: Power according to S_0 (upper value) and T_0 (value below) when model is RSM($\mathbf{d}$) and hypothesis is IEAS($\mathbf{d_0}$) or ISA($\mathbf{p_0}$) for $n = 4$ and $m = 12$. $\alpha = 0.04$.

	d						
$\mathbf{d_0}$	(18, 2, 2, 2)	(16, 3, 3, 2)	(13, 5, 4, 2)	(8, 8, 4, 4)	(7, 7, 7, 3)	(7, 6, 6, 5)	(6, 6, 6, 6)
(18, 2, 2, 2)	0.14 0.00	0.33 0.00	0.74 0.08	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00
(16, 3, 3, 2)	0.05 0.00	0.05 0.00	0.15 0.01	1.00 0.79	1.00 0.96	1.00 0.95	1.00 1.00
(13, 5, 4, 2)	0.05 0.02	0.04 0.00	0.05 0.00	0.47 0.10	0.49 0.14	0.79 0.28	0.99 0.70
(8, 8, 4, 4)	1.00 1.00	0.57 0.36	0.07 0.03	0.01 0.01	0.07 0.03	0.09 0.03	0.15 0.04
(7, 7, 7, 3)	1.00 1.00	1.00 1.00	0.15 0.07	0.02 0.02	0.00 0.01	0.05 0.02	0.12 0.04
(7, 6, 6, 5)	1.00 1.00	1.00 1.00	0.15 0.14	0.01 0.02	0.01 0.01	0.00 0.01	0.01 0.01
(6, 6, 6, 6)	1.00 1.00	1.00 1.00	0.52 0.37	0.02 0.02	0.02 0.02	0.01 0.01	0.01 0.01
$\mathbf{p_0}$							
$(\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$	0.02 0.00	0.00 0.00	0.79 0.04	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00
$(\frac{2}{3}, \frac{1}{8}, \frac{1}{8}, \frac{1}{12})$	0.01 0.00	0.01 0.00	0.15 0.00	0.98 0.77	1.00 0.92	1.00 0.95	1.00 0.99
$(\frac{13}{24}, \frac{5}{24}, \frac{1}{6}, \frac{1}{12})$	0.02 0.02	0.02 0.02	0.01 0.00	0.42 0.09	0.43 0.07	0.78 0.27	0.99 0.76
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$	1.00 1.00	1.00 1.00	0.02 0.03	0.00 0.00	0.03 0.02	0.04 0.03	0.08 0.04
$(\frac{7}{24}, \frac{7}{24}, \frac{7}{24}, \frac{1}{8})$	1.00 1.00	0.83 1.00	0.11 0.06	0.02 0.02	0.00 0.01	0.05 0.02	0.12 0.04
$(\frac{7}{24}, \frac{1}{4}, \frac{1}{4}, \frac{5}{24})$	1.00 1.00	0.83 0.84	0.11 0.08	0.01 0.02	0.01 0.01	0.00 0.00	0.01 0.01
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	1.00 1.00	1.00 1.00	0.52 0.34	0.02 0.02	0.01 0.02	0.00 0.01	0.00 0.01

Table 6: Moments of S_0, $S_0^{'}$, $S_0^{''}$, T_0, $T_0^{'}$ and $T_0^{''}$ for some RSM($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) or ISA($\mathbf{d_0}/2m$) hypotheses with $n = 4$ and $m = 12$. The unshaded columns correspond to the best approximations to the test statistics.

Case 1: ISA $\mathbf{d_0} = \mathbf{d} = (7, 6, 6, 5)$	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	6.18	6.18	6.18	7.90	7.90	7.90
Variance	8.70	12.71	8.47	11.32	17.83	13.87

Case 2: IEAS $\mathbf{d_0} = \mathbf{d} = (16, 3, 3, 2)$	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	6.29	6.29	6.29	5.08	5.08	5.08
Variance	32.41	13.17	8.78	7.25	10.32	5.73

Case 3: ISA $\mathbf{d_0} = (13, 5, 4, 2)$, $\mathbf{d} = (8, 8, 4, 4)$	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	16.98	16.98	16.98	12.71	12.71	12.71
Variance	39.45	36.02	64.04	10.84	26.93	35.91

Case 4: IEAS $\mathbf{d_0} = (7, 7, 7, 3)$, $\mathbf{d} = (6, 6, 6, 6)$	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	13.28	13.28	13.28	10.85	10.85	10.85
Variance	47.97	27.12	39.17	11.73	23.53	26.14

Table 7: Power according to S_0 (upper value) and T_0 (value below) when model is RSM($\mathbf{d}$) and hypothesis is IEAS($\mathbf{d_0}$) or ISA($\mathbf{p_0}$) for $n = 3$ and $m = 45$. $\alpha = 0.05$.

	$\mathbf{d}$						
$\mathbf{d_0}$	(70, 10, 10)	(65, 15, 10)	(50, 20, 20)	(45, 35, 10)	(40, 30, 20)	(35, 30, 25)	(30, 30, 30)
(70, 10, 10)	0.01 0.01	0.13 0.03	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00
(65, 15, 10)	0.01 0.01	0.01 0.01	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00
(50, 20, 20)	1.00 1.00	0.85 1.00	0.01 0.02	1.00 1.00	0.37 0.23	0.97 0.90	1.00 1.00
(45, 35, 10)	1.00 1.00	1.00 1.00	1.00 1.00	0.01 0.01	1.00 0.56	1.00 1.00	1.00 1.00
(40, 30, 20)	1.00 1.00	1.00 1.00	0.12 0.24	0.13 0.32	0.01 0.01	0.04 0.03	0.42 0.28
(35, 30, 25)	1.00 1.00	1.00 1.00	0.92 0.90	1.00 1.00	0.02 0.03	0.01 0.01	0.03 0.03
(30, 30, 30)	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	0.23 0.25	0.02 0.03	0.01 0.01
$\mathbf{p_0}$							
$(\frac{7}{9}, \frac{1}{9}, \frac{1}{9})$	0.01 0.01	0.12 0.03	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00
$(\frac{13}{18}, \frac{1}{6}, \frac{1}{9})$	0.01 0.01	0.01 0.01	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00
$(\frac{5}{9}, \frac{2}{9}, \frac{2}{9})$	1.00 1.00	0.85 1.00	0.01 0.02	1.00 1.00	0.34 0.21	0.91 0.84	1.00 1.00
$(\frac{1}{2}, \frac{7}{18}, \frac{1}{9})$	1.00 1.00	1.00 1.00	1.00 1.00	0.01 0.01	1.00 0.54	1.00 1.00	1.00 1.00
$(\frac{4}{9}, \frac{1}{3}, \frac{2}{9})$	1.00 1.00	1.00 1.00	0.11 0.24	0.13 0.32	0.01 0.01	0.04 0.03	0.39 0.28
$(\frac{7}{18}, \frac{1}{3}, \frac{5}{18})$	1.00 1.00	1.00 1.00	0.90 0.90	1.00 1.00	0.02 0.03	0.01 0.01	0.03 0.02
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1.00 1.00	1.00 1.00	1.00 1.00	1.00 1.00	0.19 0.24	0.02 0.03	0.01 0.01

Table 8: Moments of S_0, $S_0^{'}$, $S_0^{''}$, T_0, $T_0^{'}$ and $T_0^{''}$ for some RSM($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) or ISA($\mathbf{d_0}/2m$) hypotheses with $n = 3$ and $m = 45$. The unshaded columns correspond to the best approximations to the test statistics.

Case 1: IEAS $\mathbf{d_0} = (70, 10, 10), \mathbf{d} = (65, 15, 10)$						
	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	7.43	7.43	7.43	6.05	6.05	6.05
Variance	15.96	15.77	22.07	5.28	12.19	14.63

Case 2: ISA $\mathbf{d_0} = \mathbf{d} = (50, 20, 20)$						
	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	3.01	3.01	3.01	3.32	3.32	3.32
Variance	5.50	6.05	3.63	7.45	7.35	4.41

Case 3: IEAS $\mathbf{d_0} = \mathbf{d} = (30, 30, 30)$						
	S_0	$S_0^{'}$	$S_0^{''}$	T_0	$T_0^{'}$	$T_0^{''}$
Expected value	3.03	3.03	3.03	3.14	3.14	3.14
Variance	5.83	6.14	3.68	6.66	6.57	3.94

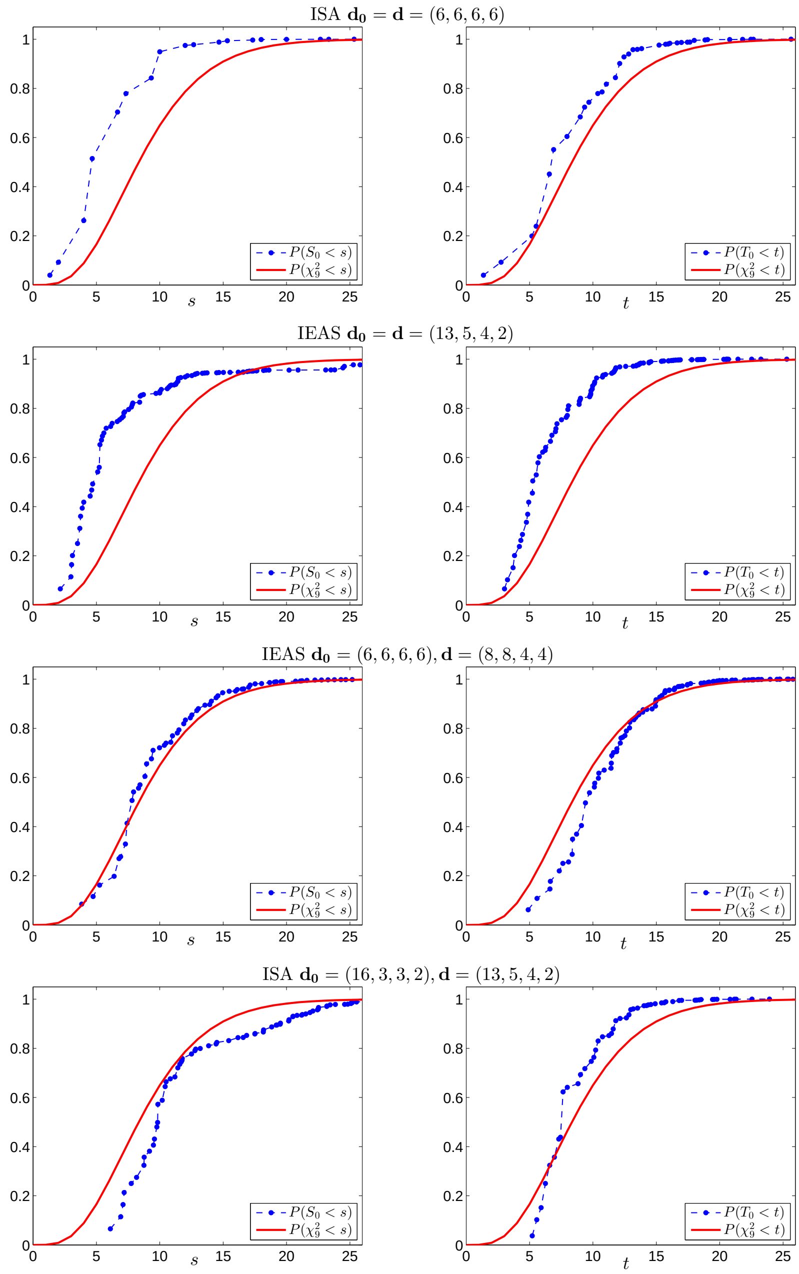

Figure 18: Distributions of S_0, T_0 and χ^2_9 for some RSM($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) or ISA($\mathbf{d_0}/2m$) hypotheses with $n = 4$ and $m = 12$.

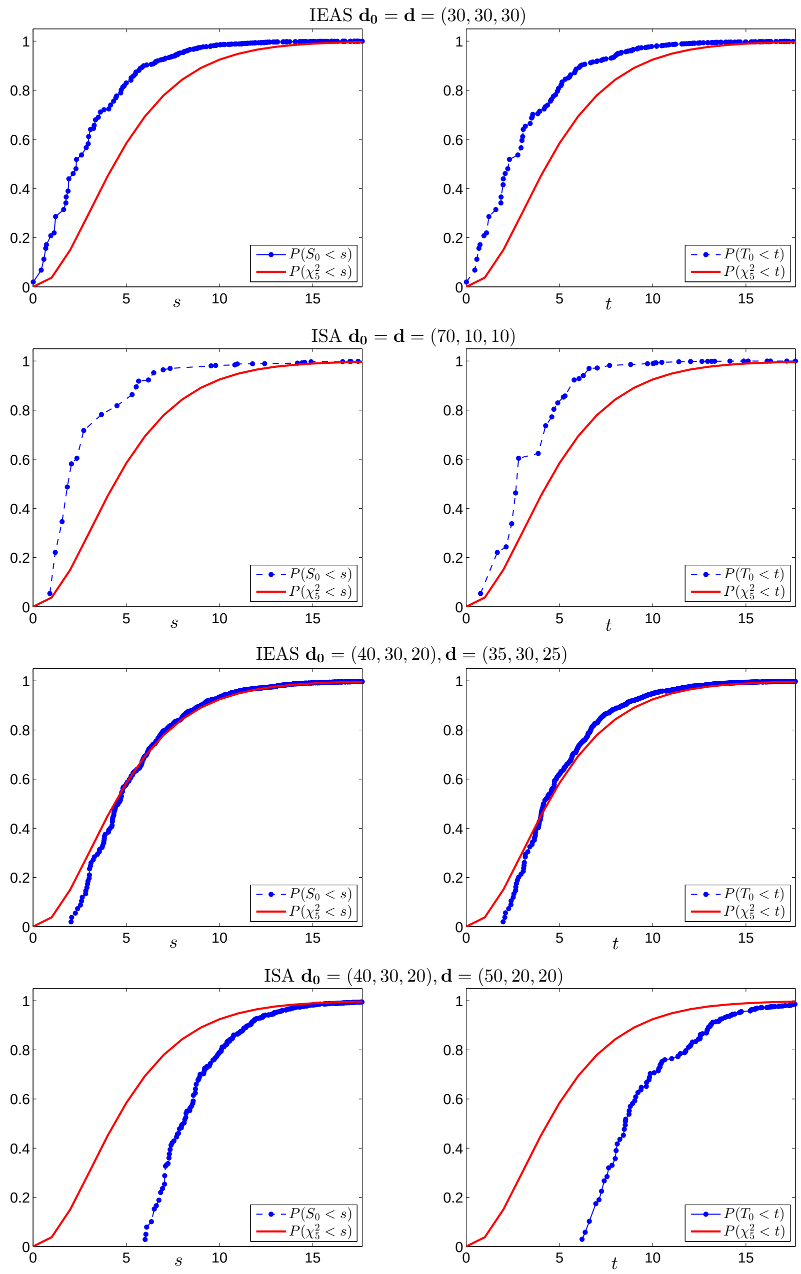

Figure 19: Distributions of S_0, T_0 and χ^2_5 for some RSM($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) or ISA($\mathbf{d_0}/2m$) hypotheses with $n = 3$ and $m = 45$.

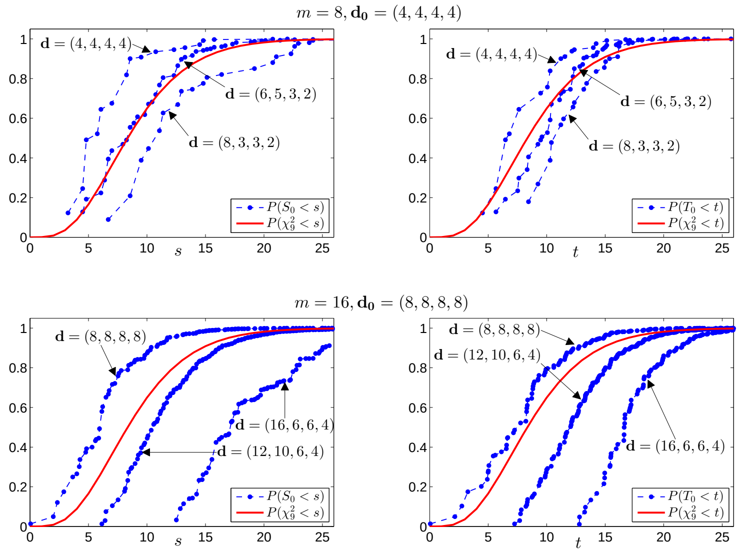

Figure 20: Non-null distributions of S_0 and T_0 for some RSM($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) hypotheses with flat $\mathbf{d_0}$ and different $\mathbf{d}$ when m increases.

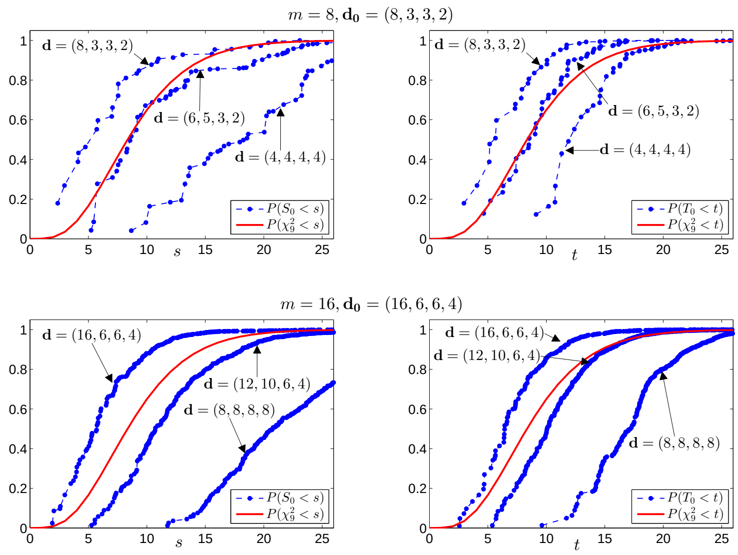

Figure 21: Non-null distributions of S_0 and T_0 for some RSM($\mathbf{d}$) models and IEAS($\mathbf{d_0}$) hypotheses with skew $\mathbf{d_0}$ and different $\mathbf{d}$ when m increases.

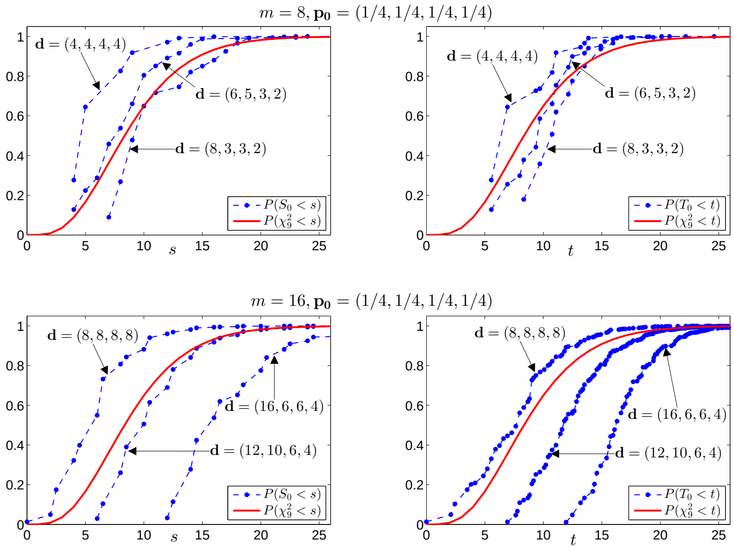

Figure 22: Non-null distributions of S_0 and T_0 for some RSM($\mathbf{d}$) models and ISA($\mathbf{p_0}$) hypotheses with flat $\mathbf{p_0}$ and different $\mathbf{d}$ when m increases.

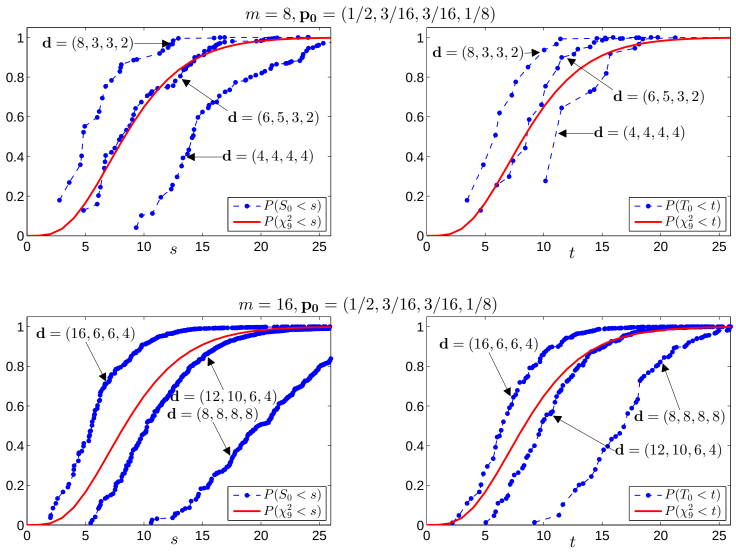

Figure 23: Non-null distributions of S_0 and T_0 for some RSM($\mathbf{d}$) models and ISA($\mathbf{p_0}$) hypotheses with skew $\mathbf{p_0}$ and different $\mathbf{d}$ when m increases.

5 Statisticial Tests of a Composite Multigraph Hypothesis

5.1 Test Statistics

The composite multigraph hypothesis might be ISA for unknown $\mathbf{p}$ or IEAS for unknown $\mathbf{d}$. The parameters have to be estimated from data $\mathbf{m}$. These estimates are denoted $\hat{\mathbf{p}} = \hat{\mathbf{p}}(\mathbf{m})$ and $\hat{\mathbf{d}} = \hat{\mathbf{d}}(\mathbf{m})$, and they are related according to

$$\hat{\mathbf{p}} = \frac{\hat{\mathbf{d}}}{2m} \ ,$$

where

$$\hat{d}_i = \sum_{j=1}^{n}(m_{ij} + m_{ji}) = m_{i\cdot} + m_{\cdot i} \quad \text{for } i = 1, \ldots, n \ ,$$

and $m_{ij} = 0$ for $i > j$. Thus, we have estimated sequences $\hat{\mathbf{Q}} = (\hat{Q}_{ij} : (i,j) \in R)$ in the two cases with composite ISA and IEAS hypotheses. Note that for ISA

$$\hat{Q}_{ij} = \begin{cases} \hat{p}_i^2 & \text{for } i = j \\ 2\hat{p}_i\hat{p}_j & \text{for } i < j \ , \end{cases}$$

and for IEAS

$$\hat{Q}_{ij} = \begin{cases} \binom{\hat{d}_i}{2}/\binom{2m}{2} & \text{for } i = j \\ \hat{d}_i\hat{d}_j/\binom{2m}{2} & \text{for } i < j \ . \end{cases}$$

The Pearson goodness-of-fit and divergence statistics are here given as

$$\hat{S} = \sum\sum_{i \leq j} \frac{(m_{ij} - m\hat{Q}_{ij})^2}{m\hat{Q}_{ij}} = \sum\sum_{i \leq j} \frac{m_{ij}^2}{m\hat{Q}_{ij}} - m \ ,$$

and

$$\hat{D} = \sum\sum_{i \leq j} \frac{m_{ij}}{m} \log \frac{m_{ij}}{m\hat{Q}_{ij}} \ .$$

Here, $\hat{S}$ and

$$\hat{T} = \frac{2m}{\log \mathrm{e}}\hat{D}$$

are asymptotically $\chi^2_{\binom{n}{2}}$-distributed when the correct model is tested. Note that the number of degrees of freedom here is given as the difference in numbers of estimated free parameters without and with the hypothesis, i.e. $df = (r-1) - (n-1) = r - n = \binom{n}{2}$. The critical regions for these tests are given by values of $\hat{S}$ and $\hat{T}$ above a critical value cv which can be chosen as

$$cv = df + 2\sqrt{2df} = \binom{n}{2} + \sqrt{8\binom{n}{2}}$$

to get a significance level close to 5% given by

$$\alpha = P(\chi^2_{\binom{n}{2}} > cv) \ .$$

The power functions $P(\hat{S} > cv)$ and $P(\hat{T} > cv)$ are functions of $\mathbf{p}$ or $\mathbf{d}$ depending on whether an ISA($\mathbf{p}$) or IEAS($\mathbf{d}$) model is considered. The error probabilities of false rejection and false acceptance are denoted by α and β indexed by $\hat{S}$ and $\hat{T}$.

Similar to the test statistic approximations described in Section 4.1, $S^{'}$ and $S^{''}$ are here given by S^{*} for k chosen as the integer part of μ and $r - n$, respectively. These approximations can be used as alternative test statistics provided the expected values of $\hat{S}$ and $\hat{T}$ are known. Formal expressions for the expected values are complicated to obtain due to that $\mathbf{m}$ is involved also via $\hat{\mathbf{Q}}$ that depends on $\hat{\mathbf{d}}$ which is determined by $\mathbf{m}$. However, for our theoretical investigation we use complete enumerations of all outcomes of $\mathbf{m}$ and find the expected values and variances numerically. Under an RSM($\mathbf{d}$) model the estimated $\hat{\mathbf{d}}$ is always (for any data $\mathbf{m}$) equal to the $\mathbf{d}$ specified in the model which implies that

$$E(\hat{S}) = E(S_0) = \frac{(m-1)n(n-1)}{2m-3} \ ,$$

as shown in Section 4.1. The preferences between approximations to the test statistics under IEA models are determined by comparing variances, as mentioned in Section 4.1.

5.2 Test Illustrations for IEAS Models

Consider composite IEAS hypotheses against IEAS($\mathbf{d}$) models for multigraphs with 4 vertices and 10 edges. Here, the composite hypotheses include the correct model and the probabilities of false rejection according to $\hat{S}$ and $\hat{T}$ are given in Table 9. For flat $\mathbf{d}$, both $\alpha_{\hat{S}}$ and $\alpha_{\hat{T}}$ are close or equal to $\alpha = 0.04$ and for skew $\mathbf{d}$, $\alpha_{\hat{S}}$ remains close or equal to α while $\alpha_{\hat{T}}$ is below. If the composite ISA hypothesis is instead tested against the IEAS($\mathbf{d}$) model, the powers of $\hat{S}$ and $\hat{T}$ are almost equal to the values of $\alpha_{\hat{S}}$ and $\alpha_{\hat{T}}$ in Table 9. Thus, both statistics have very poor powers of detecting differences between composite ISA and IEAS hypotheses. Figure 24 illustrates the fit of the distributions of $\hat{S}$ and $\hat{T}$ to that of χ^2_6. For skew $\mathbf{d}$ there are larger deviations from χ^2_6 for both $\hat{S}$ and $\hat{T}$ than there are for flat $\mathbf{d}$.

Figures 25 and 26 shows the null and non-null distributions of $\hat{S}$ and $\hat{T}$ for some IEAS($\mathbf{d}$) models with flat and skew $\mathbf{d}$ when m increases as multiples of the specified $\mathbf{d}$. The null distributions correspond to composite IEAS hypotheses while non-null distributions correspond to composite ISA hypotheses. The convergence of the null distributions for flat $\mathbf{d}$ is rapid towards the asymptotic distribution, especially for $\hat{S}$. However, the convergence of the null distributions for skew **d** is slower for both statistics. The non-null distributions of both statistics also seem to converge to the asymptotic null distributions. Thus, for small and large m, it is difficult to detect differences between composite ISA and IEAS hypotheses.

The expected values and variances of $\hat{S}$ and $\hat{T}$, and of their approximations $\hat{S}'$, $\hat{S}''$, $\hat{T}'$ and $\hat{T}''$ are presented in Table 10, where the versions that are not preferred are shaded. For flat **d**, the variances of $\hat{S}$ are roughly twice their expected values, which are approximately equal to 6. This indicates a good fit to the χ^2_6-distribution in terms of the first two moments. This is also noted by $\hat{S}''$ being preferred to $\hat{S}'$. Further, for flat **d**, $\hat{S}''$ and $\hat{T}'$ are preferred, while for the majority of cases with skew **d**, $\hat{T}''$ and $\hat{S}'$ are preferred. Two particular cases are **d** =(6, 6, 6, 2) and **d** =(6, 5, 5, 4) where the variances of the approximations are equal so that any one of them can be preferred.

Table 9: Probabilities of false rejection according to $\hat{S}$ and $\hat{T}$ when model is IEAS(**d**) and a composite IEAS hypothesis is tested for $n = 4$ and $m = 10$. $\alpha = 0.04$.

d	(14, 2, 2, 2)	(12, 3, 3, 2)	(9, 7, 2, 2)	(8, 8, 2, 2)	(6, 6, 6, 2)	(6, 5, 5, 4)	(5, 5, 5, 5)
$\hat{S}$	0.04	0.04	0.03	0.03	0.02	0.03	0.03
$\hat{T}$	0.00	0.01	0.01	0.01	0.03	0.04	0.04

Table 10: Moments of $\hat{S}$, $\hat{S}'$, $\hat{S}''$, $\hat{T}$, $\hat{T}'$ and $\hat{T}''$ when model is IEAS(**d**) and a composite IEAS hypothesis is tested for $n = 4$ and $m = 10$. The unshaded rows correspond to the best approximation to the test statistics.

	d	(14, 2, 2, 2)	(12, 3, 3, 2)	(9, 7, 2, 2)	(8, 8, 2, 2)	(6, 6, 6, 2)	(6, 5, 5, 4)	(5, 5, 5, 5)
$\hat{S}$	Mean	3.69	4.59	4.56	4.57	5.31	5.88	5.94
	Variance	18.50	16.12	12.79	12.71	10.57	11.32	11.08
$\hat{S}'$	Mean	3.69	4.59	4.56	4.57	5.31	5.88	5.94
	Variance	9.09	10.52	10.39	10.42	11.29	13.85	14.11
$\hat{S}''$	Mean	3.69	4.59	4.56	4.57	5.31	5.88	5.94
	Variance	4.54	7.02	6.92	6.95	9.41	11.54	11.76
$\hat{T}$	Mean	3.30	4.57	5.04	5.07	6.23	6.96	7.07
	Variance	6.71	7.43	8.29	8.43	8.97	9.20	9.22
$\hat{T}'$	Mean	3.30	4.57	5.04	5.07	6.23	6.96	7.07
	Variance	7.26	10.45	10.15	10.29	12.95	16.16	14.28
$\hat{T}''$	Mean	3.30	4.57	5.04	5.07	6.23	6.96	7.07
	Variance	3.63	6.96	8.46	8.57	12.95	16.16	16.66

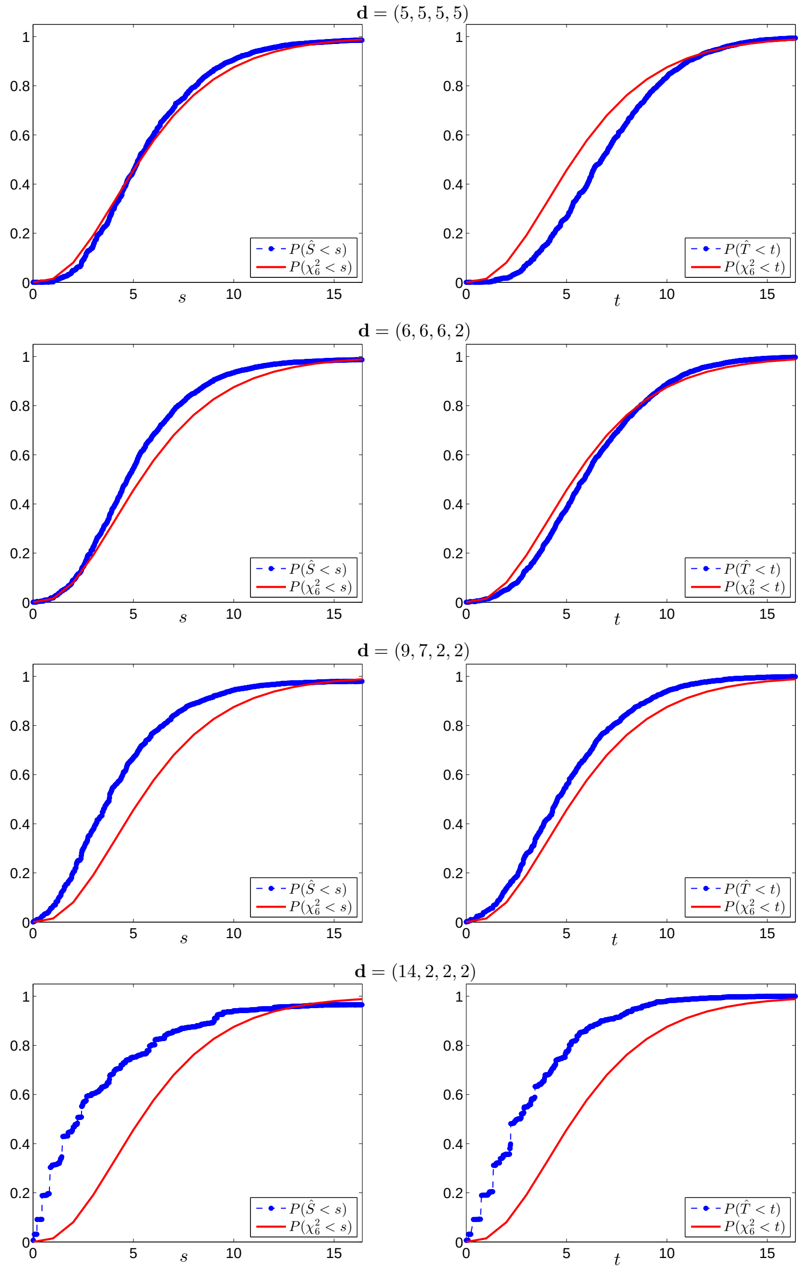

Figure 24: Distributions of $\hat{S}$, $\hat{T}$ and χ^2_6 for some IEAS($\mathbf{d}$) models and composite IEAS hypothesis with $n = 4$ and $m = 10$.

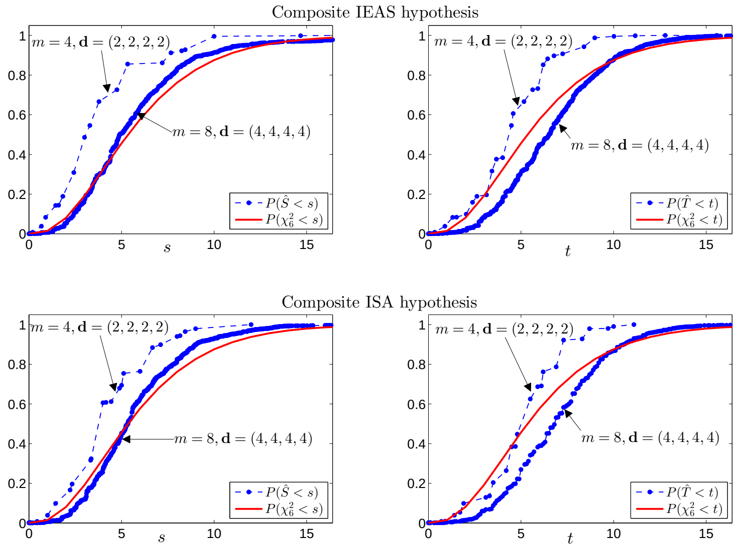

Figure 25: Null and non-null distributions of $\hat{S}$ and $\hat{T}$ for some IEAS($\mathbf{d}$) models with flat $\mathbf{d}$ and composite IEAS and ISA hypotheses when m increases.

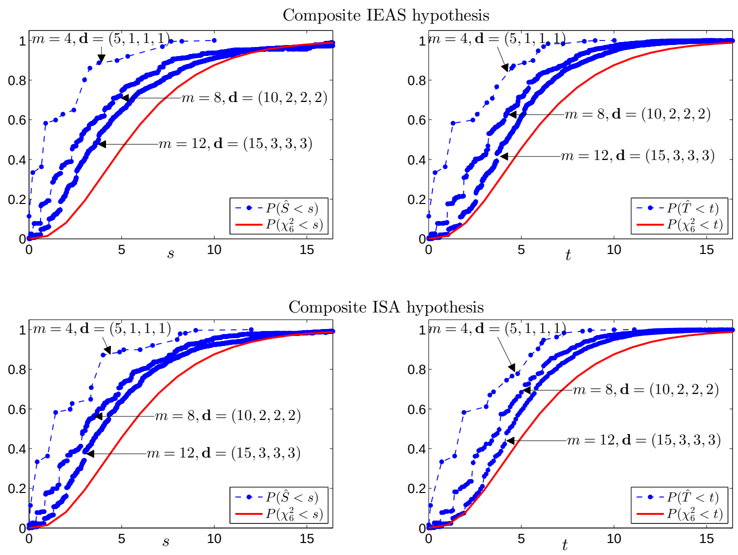

Figure 26: Null and non-null distributions of $\hat{S}$ and $\hat{T}$ for some IEAS($\mathbf{d}$) models with skew $\mathbf{d}$ and composite IEAS and ISA hypotheses when m increases.

5.3 Test Illustrations for ISA Models

We now turn to composite ISA hypotheses against ISA($\mathbf{p}$) models, and consider tests of multigraphs with 4 vertices and 10 edges. The probabilities of false rejection according to $\hat{S}$ and $\hat{T}$ are given in Table 11 where similar results are seen as for IEAS models in Table 9. For skew $\mathbf{p}$, $\alpha_{\hat{T}}$ is much below $\alpha = 0.04$, while $\alpha_{\hat{S}}$ is always close to α. If a composite hypothesis IEAS instead of composite ISA is tested against the ISA($\mathbf{p}$) model, the powers of $\hat{S}$ and $\hat{T}$ are approximately equal to $\alpha_{\hat{S}}$ and $\alpha_{\hat{T}}$ given in Table 11. As noted before, these poor powers are due to the resemblances between ISA and IEAS models.

Some selected cases from Table 11 are illustrated in Figure 27 where the cumulative distribution functions of $\hat{S}$ and $\hat{T}$ are given. We see that we have fairly good fit between the distributions of both statistics and that of χ^2_6, except for the very skew $\mathbf{p}$ =(7/10, 1/10, 1/10, 1/10).

In Figures 28 and 29 we illustrate the effect of increasing m on the null and non-null distributions of $\hat{S}$ and $\hat{T}$ for some ISA($\mathbf{p}$) models with flat and skew $\mathbf{p}$. Here, the resemblance between IEAS and ISA models gives similar results as those for composite hypotheses against IEAS models shown in Section 5.2. The convergence of the null distributions is faster for flat $\mathbf{p}$ than for skew $\mathbf{p}$, and the detection of a composite hypothesis not including the correct model is more difficult as m is increased.

The expected values and variances of all the versions of the test statistics are presented in Table 12 where the unshaded rows correspond to the best approximations. For the majority of cases $\hat{S}''$ and $\hat{T}''$ are preferred, except for very skew $\mathbf{p}$ where $\hat{S}'$ and $\hat{T}'$ are preferred.

Table 11: Probabilities of false rejection according to $\hat{S}$ and $\hat{T}$ when model is ISA($\mathbf{p}$) and a composite ISA hypothesis is tested for $n = 4$ and $m = 10$. $\alpha = 0.04$.

$\mathbf{p}$	$(\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$	$(\frac{3}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{10})$	$(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$	$(\frac{4}{9}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9})$	$(\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
$\hat{S}$	0.03	0.03	0.03	0.03	0.03	0.03
$\hat{T}$	0.01	0.01	0.02	0.02	0.02	0.05

Table 12: Moments of $\hat{S}$, $\hat{S}'$, $\hat{S}''$, $\hat{T}$, $\hat{T}'$ and $\hat{T}''$ when model is ISA($\mathbf{p}$) and a composite ISA hypothesis is tested for $n = 4$ and $m = 10$. The unshaded rows correspond to the best approximation to the test statistics.

	$\mathbf{p}$	$(\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$	$(\frac{3}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{10})$	$(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$	$(\frac{4}{9}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9})$	$(\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
$\hat{S}$	Mean	4.00	4.59	5.42	4.99	5.22	5.96
	Variance	15.50	11.92	9.81	10.19	9.28	8.41
$\hat{S}'$	Mean	4.00	4.59	5.42	4.99	5.22	5.96
	Variance	8.01	10.52	11.73	12.43	10.91	14.23
$\hat{S}''$	Mean	4.00	4.59	5.42	4.99	5.22	5.96
	Variance	5.34	7.01	9.78	8.29	9.09	11.86
$\hat{T}$	Mean	3.70	4.73	6.01	5.56	5.93	7.24
	Variance	7.17	7.48	8.15	8.72	9.00	9.21
$\hat{T}'$	Mean	3.70	4.73	6.01	5.56	5.93	7.24
	Variance	9.14	11.18	12.05	12.38	14.06	14.99
$\hat{T}''$	Mean	3.70	4.73	6.01	5.56	5.93	7.24
	Variance	4.57	7.45	12.05	10.32	11.72	17.49

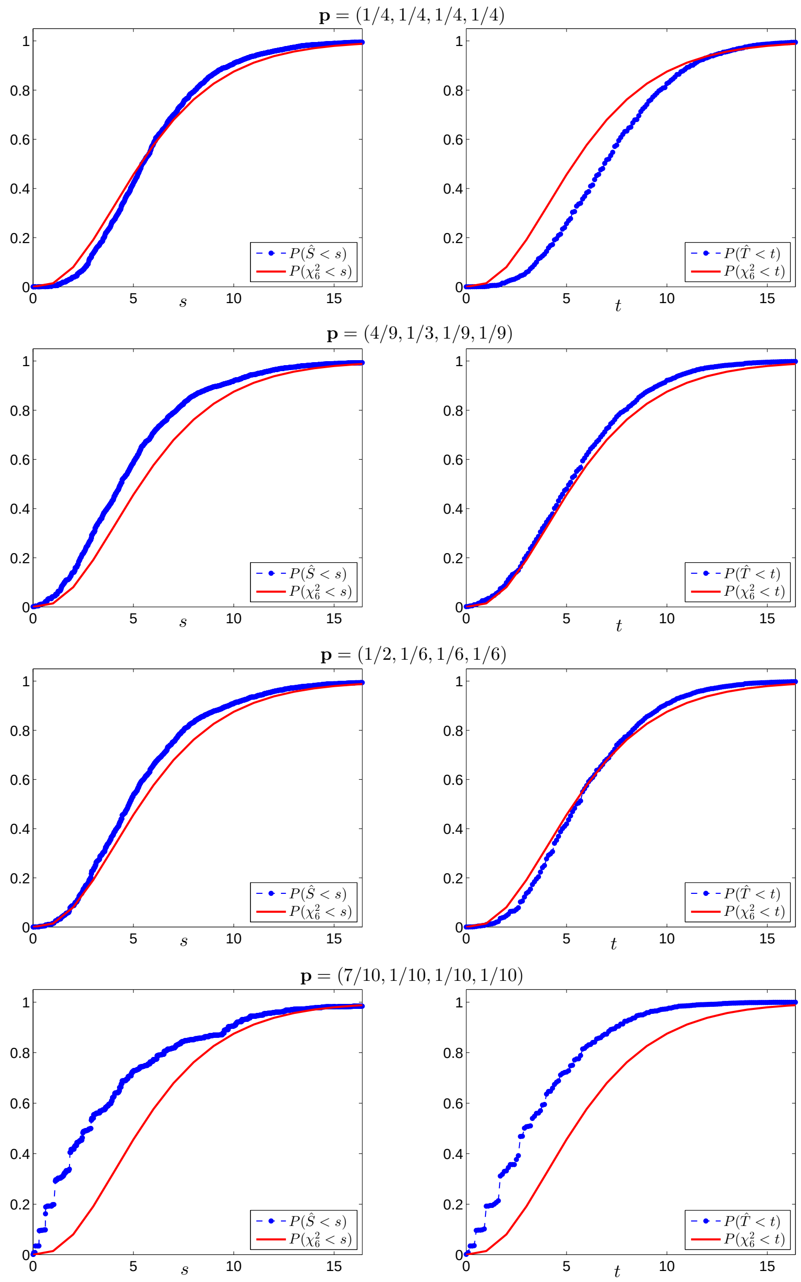

Figure 27: Distributions of $\hat{S}$, $\hat{T}$ and χ^2_6 for some ISA($\mathbf{p}$) models and composite ISA hypothesis with $n = 4$ and $m = 10$.

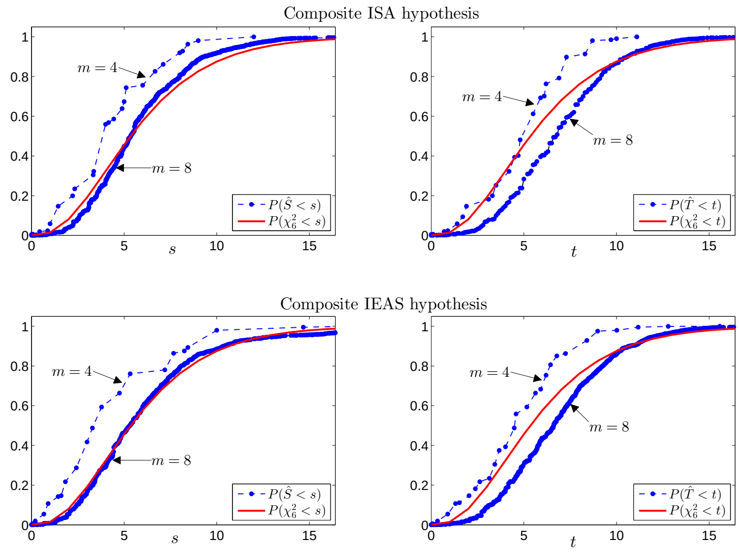

Figure 28: Null and non-null distributions of $\hat{S}$ and $\hat{T}$ for some ISA($\mathbf{p}$) models with flat $\mathbf{p}$ =(1/4, 1/4, 1/4, 1/4) and composite ISA and IEAS hypotheses when m increases.

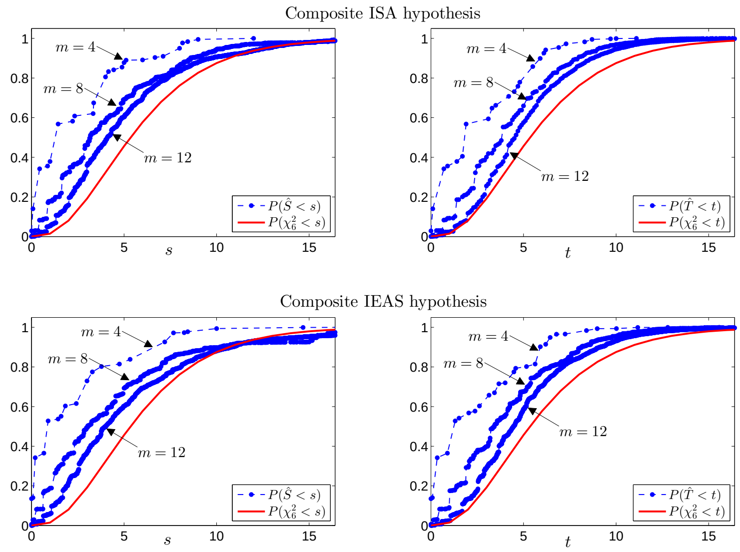

Figure 29: Null and non-null distributions of $\hat{S}$ and $\hat{T}$ for some ISA($\mathbf{p}$) models with skew $\mathbf{p}$ =(5/8, 1/8, 1/8, 1/8) and composite ISA and IEAS hypotheses when m increases.

5.4 Test Illustrations for RSM Models

In this section we illustrate some of the consequences of using previously described tests of composite hypotheses against a false IEA model when the true model is RSM. Here, both IEAS and ISA hypotheses are tested against RSM($\mathbf{d}$) models. We start by considering multigraphs with 4 vertices and 12 edges. The poor powers according to $\hat{S}$ and $\hat{T}$ of rejecting IEAS and ISA when RSM is true are presented in Table 13. To illustrate the fit of the distributions of the statistics $\hat{S}$ and $\hat{T}$ to that of χ^2_6, their cumulative distribution functions for some selected cases are shown in Figure 30. For all cases, there is reasonably good fit to χ^2_6 for this rather small m.

The expected values and variances of all the versions of the test statistics are presented in Table 14. For the majority of cases, the variances of $\hat{S}$ are roughly twice their expected values which are equal to 6. This indicates a good fit to the χ^2_6-distribution in terms of the first two moments. Note that $E(\hat{S})$ under IEAS hypotheses are not dependent on the values in the degree sequence, as mentioned in Section 5.1. For very skew $\mathbf{d}$, $\hat{T}^{''}$ is preferred for almost all cases and for the rest of the skew cases and for flat cases, $\hat{T}^{'}$ is preferred.

The poor powers of rejecting IEAS and ISA when RSM is true for multigraphs with 3 vertices and 45 edges are shown in Table 15. We see that $\alpha_{\hat{S}}$ is close to $\alpha = 0.04$ for all cases shown, while $\hat{T}$ is equal, less or greater than α for both skew and flat $\mathbf{d}$. The fit of the non-null distributions of $\hat{S}$ and $\hat{T}$ to that of χ^2_3 for some selected cases are shown in Figure 31 where we for all cases illustrated see a good fit. The expected values and variances of all the versions of the test statistics are presented in Table 16. For all cases with IEAS hypotheses, and almost all cases with ISA hypotheses, we note a good fit to the χ^2_3-distribution since the variances of both test statistics are roughly twice their expected values which are equal to 3. This indicates that the approximations are mostly unnecessary for large m.

In Figures 32 and 33 the effects of increasing m on the non-null distributions of $\hat{S}$ and $\hat{T}$ for some RSM($\mathbf{d}$) models with flat and skew $\mathbf{d}$ are illustrated. For all cases illustrated we see that these distributions are very close to the asymptotic null distribution. Further, the effect from increasing m on the non-null distributions is small. Thus it can be concluded that no matter the size of m, it is difficult to detect a false composite hypothesis under an RSM model, just as it is difficult to detect a false composite hypothesis under IEA models as demonstrated in Figures 25-26 for IEAS models, and in Figures 28-29 for ISA models.

Table 13: Power according to $\hat{S}$ and $\hat{T}$ when model is RSM($\mathbf{d}$) and a composite IEAS or ISA hypothesis is tested for $n = 4$ and $m = 12$. $\alpha = 0.04$.

	$\mathbf{d}$	(18, 2, 2, 2)	(16, 3, 3, 2)	(13, 5, 4, 2)	(8, 8, 4, 4)	(7, 7, 7, 3)	(7, 6, 6, 5)	(6, 6, 6, 6)
IEAS	$\hat{S}$	0.14	0.09	0.06	0.03	0.04	0.04	0.03
	$\hat{T}$	0.02	0.02	0.03	0.05	0.06	0.08	0.07
ISA	$\hat{S}$	0.04	0.08	0.06	0.03	0.03	0.03	0.02
	$\hat{T}$	0.02	0.01	0.02	0.06	0.06	0.09	0.06

Table 14: Moments of $\hat{S}$, $\hat{S}'$, $\hat{S}''$, $\hat{T}$, $\hat{T}'$ and $\hat{T}''$ when model is RSM($\mathbf{d}$) and a composite IEAS or ISA hypothesis is tested for $n = 4$ and $m = 12$. The unshaded rows correspond to the best approximations to the test statistics.

		Composite IEAS hypothesis						
	$\mathbf{d}$	(18, 2, 2, 2)	(16, 3, 3, 2)	(13, 5, 4, 2)	(8, 8, 4, 4)	(7, 7, 7, 3)	(7, 6, 6, 5)	(6, 6, 6, 6)
$\hat{S}$	Mean	6.29	6.29	6.29	6.29	6.29	6.29	6.29
	Variance	66.26	32.41	26.50	10.18	11.83	9.64	9.63
$\hat{S}'$	Mean	6.29	6.29	6.29	6.29	6.29	6.29	6.29
	Variance	13.18	13.17	13.17	13.17	13.17	13.17	13.17
$\hat{S}''$	Mean	6.29	6.29	6.29	6.29	6.29	6.29	6.29
	Variance	13.18	13.17	13.17	13.17	13.17	13.17	13.17
$\hat{T}$	Mean	3.80	5.08	6.26	7.34	7.39	7.83	7.90
	Variance	9.06	7.25	7.57	11.43	11.77	11.54	11.56
$\hat{T}'$	Mean	3.80	5.08	6.26	7.34	7.39	7.83	7.90
	Variance	9.63	10.32	13.04	15.40	15.62	17.51	17.82
$\hat{T}''$	Mean	3.80	5.08	6.26	7.34	7.39	7.83	7.90
	Variance	4.81	8.60	13.04	17.97	18.23	20.43	20.80
		Composite ISA hypothesis						
	$\mathbf{d}$	(18, 2, 2, 2)	(16, 3, 3, 2)	(13, 5, 4, 2)	(8, 8, 4, 4)	(7, 7, 7, 3)	(7, 6, 6, 5)	(6, 6, 6, 6)
$\hat{S}$	Mean	5.12	5.52	5.76	6.09	6.07	6.18	6.19
	Variance	23.58	13.19	10.87	8.41	9.07	8.70	8.78
$\hat{S}'$	Mean	5.12	5.52	5.76	6.09	6.07	6.18	6.19
	Variance	10.50	12.17	13.26	12.36	12.29	12.71	12.76
$\hat{S}''$	Mean	5.12	5.52	5.76	6.09	6.07	6.18	6.19
	Variance	8.75	10.14	11.05	12.36	12.29	12.71	12.76
$\hat{T}$	Mean	3.86	5.16	6.33	7.42	7.46	7.90	7.98
	Variance	6.78	5.36	6.40	10.83	11.44	11.33	11.37
$\hat{T}'$	Mean	3.86	5.16	6.33	7.42	7.46	7.90	7.98
	Variance	9.92	10.66	13.34	15.73	15.91	17.82	18.18
$\hat{T}''$	Mean	3.86	5.16	6.33	7.42	7.46	7.90	7.98
	Variance	4.96	8.88	13.34	18.35	18.56	20.79	21.21

Table 15: Power according to $\hat{S}$ and $\hat{T}$ when model is RSM($\mathbf{d}$) and a composite IEAS or ISA hypothesis is tested for $n = 3$ and $m = 45$. $\alpha = 0.04$.

	$\mathbf{d}$	(70, 10, 10)	(65, 15, 10)	(50, 20, 20)	(45, 35, 10)	(40, 30, 20)	(35, 30, 25)	(30, 30, 30)
IEAS	$\hat{S}$	0.04	0.04	0.04	0.04	0.05	0.05	0.05
	$\hat{T}$	0.02	0.04	0.07	0.05	0.06	0.06	0.07
ISA	$\hat{S}$	0.03	0.04	0.04	0.04	0.04	0.05	0.05
	$\hat{T}$	0.02	0.04	0.06	0.05	0.06	0.06	0.06

Table 16: Moments of $\hat{S}$, $\hat{S}'$, $\hat{S}''$, $\hat{T}$, $\hat{T}'$ and $\hat{T}''$ when model is RSM($\mathbf{d}$) and a composite IEAS or ISA hypothesis is tested for $n = 3$ and $m = 45$. The unshaded rows correspond to the best approximations to the test statistics.

				Composite IEAS hypothesis				
	$\mathbf{d}$	(70, 10, 10)	(65, 15, 10)	(50, 20, 20)	(45, 35, 10)	(40, 30, 20)	(35, 30, 25)	(30, 30, 30)
$\hat{S}$	Mean	3.03	3.03	3.03	3.03	3.03	3.03	3.03
	Variance	6.62	6.12	5.71	6.11	5.78	5.81	5.83
$\hat{S}'$	Mean	3.03	3.03	3.03	3.03	3.03	3.03	3.03
	Variance	6.14	6.14	6.14	6.14	6.14	6.14	6.14
$\hat{S}''$	Mean	3.03	3.03	3.03	3.03	3.03	3.03	3.03
	Variance	6.14	6.14	6.14	6.14	6.14	6.14	6.14
$\hat{T}$	Mean	3.43	3.48	3.31	3.22	3.21	3.15	3.14
	Variance	3.66	5.28	7.35	5.48	6.94	6.77	6.66
$\hat{T}'$	Mean	3.43	3.48	3.31	3.22	3.21	3.15	3.14
	Variance	7.82	8.09	7.30	6.92	6.88	6.63	6.57
$\hat{T}''$	Mean	3.43	3.48	3.31	3.22	3.21	3.15	3.14
	Variance	7.82	8.09	7.30	6.92	6.88	6.63	6.57
				Composite ISA hypothesis				
	$\mathbf{d}$	(70, 10, 10)	(65, 15, 10)	(50, 20, 20)	(45, 35, 10)	(40, 30, 20)	(35, 30, 25)	(30, 30, 30)
$\hat{S}$	Mean	2.91	2.95	3.01	2.98	3.03	3.03	3.03
	Variance	5.55	5.36	5.50	5.60	5.66	5.75	5.78
$\hat{S}'$	Mean	2.91	2.95	3.01	2.98	3.03	3.03	3.03
	Variance	8.49	8.69	6.05	8.88	6.10	6.13	6.14
$\hat{S}''$	Mean	2.91	2.95	3.01	2.98	3.03	3.03	3.03
	Variance	5.66	5.79	6.05	5.92	6.10	6.13	6.14
$\hat{T}$	Mean	3.45	3.50	3.32	3.23	3.23	3.17	3.15
	Variance	3.47	5.23	7.45	5.42	7.02	6.82	6.73
$\hat{T}'$	Mean	3.45	3.50	3.32	3.23	3.23	3.17	3.15
	Variance	7.92	8.16	7.35	6.98	6.95	6.71	6.60
$\hat{T}''$	Mean	3.45	3.50	3.32	3.23	3.23	3.17	3.15
	Variance	7.92	8.16	7.35	6.98	6.95	6.71	6.60

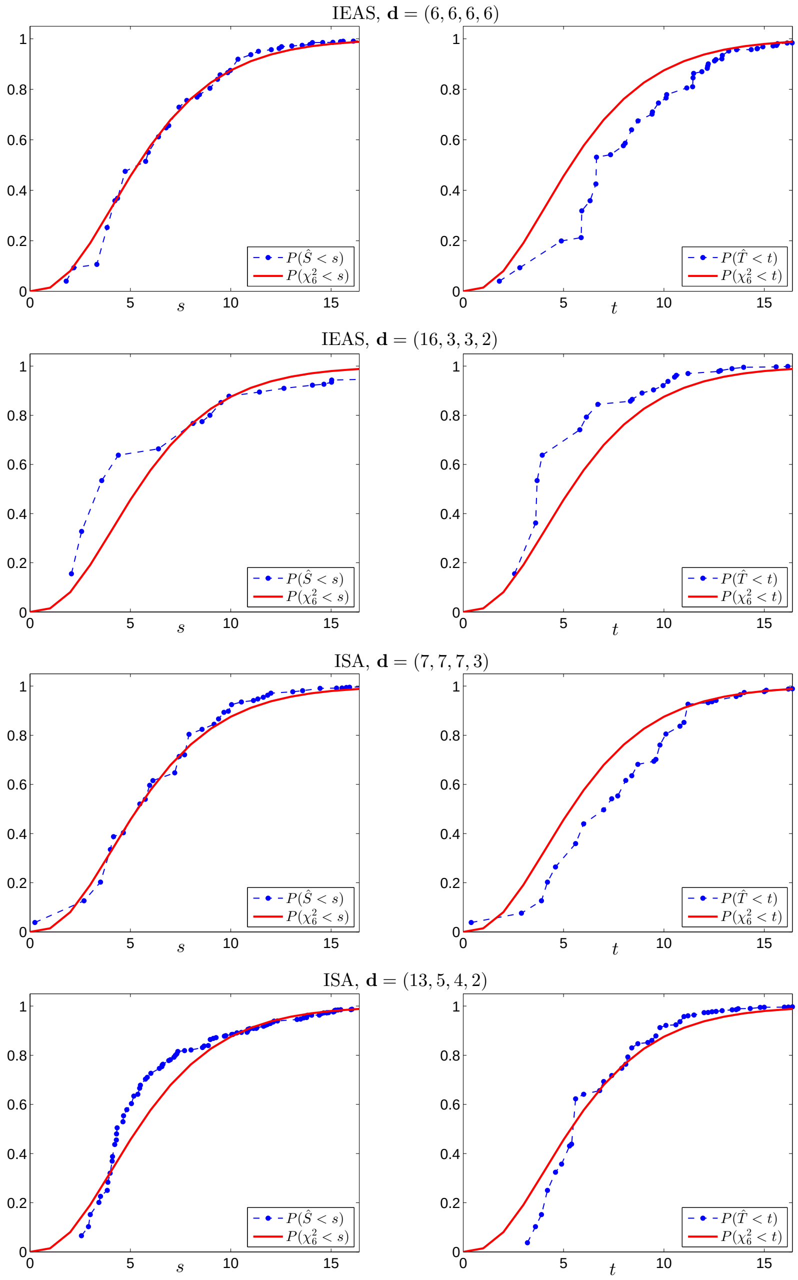

Figure 30: Distributions of $\hat{S}$, $\hat{T}$ and χ^2_6 for some RSM($\mathbf{d}$) models and composite IEAS or ISA hypotheses with $n = 4$ and $m = 12$.

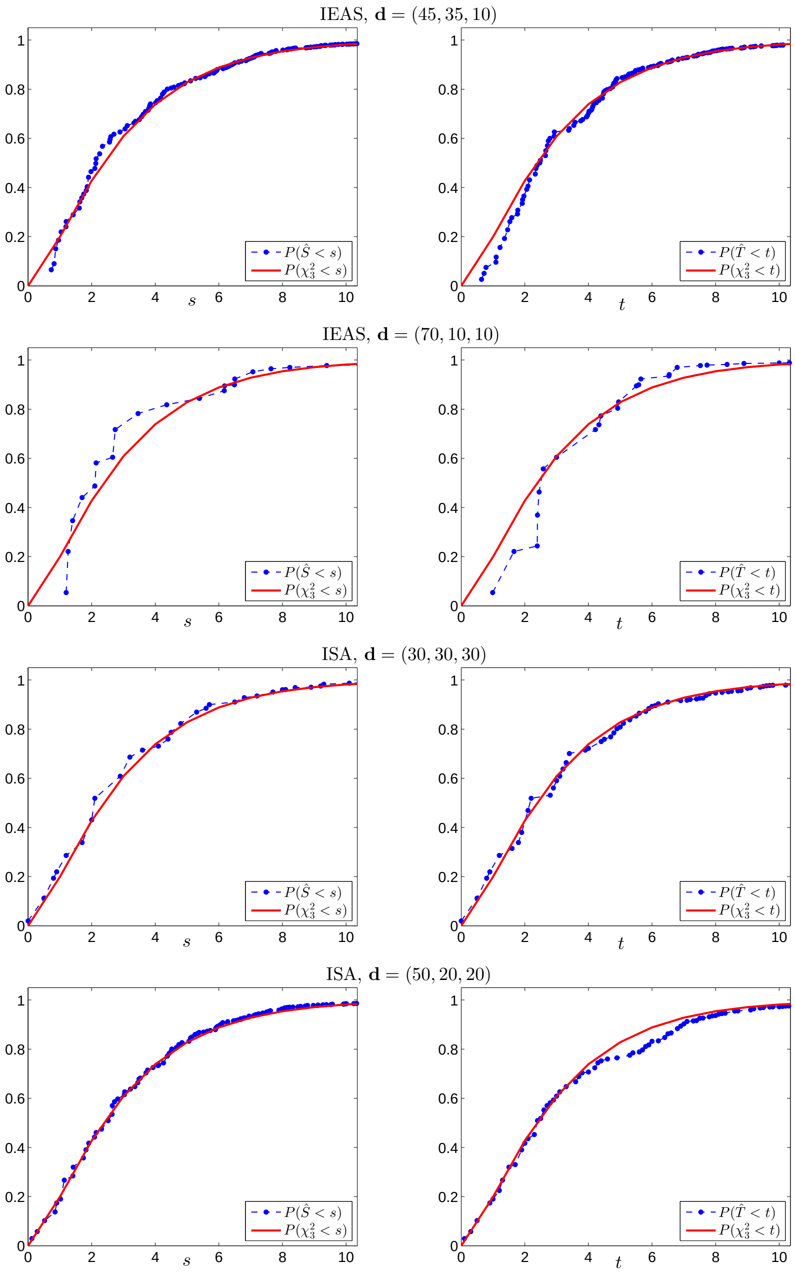

Figure 31: Distributions of $\hat{S}$, $\hat{T}$ and χ_6^2 for some RSM($\mathbf{d}$) models and composite IEAS or ISA hypotheses with $n = 3$ and $m = 45$.

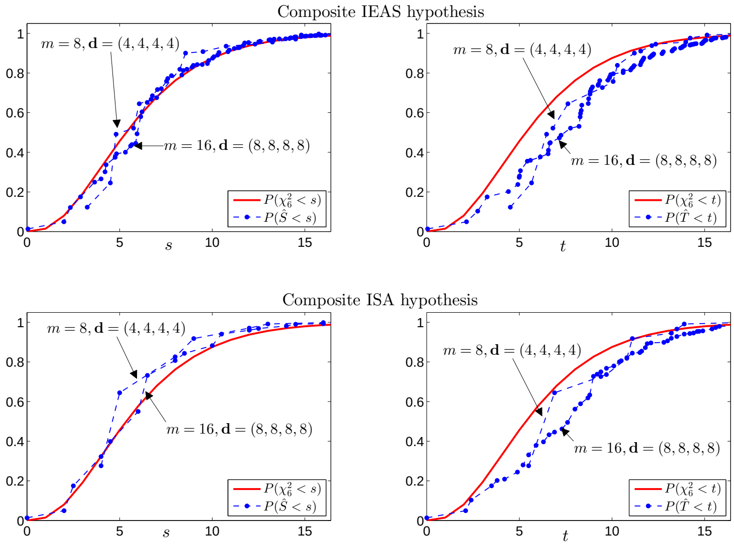

Figure 32: Non-null distributions of $\hat{S}$ and $\hat{T}$ for some RSM($\mathbf{d}$) models with flat $\mathbf{d}$ and composite IEAS and ISA hypotheses when m increases.

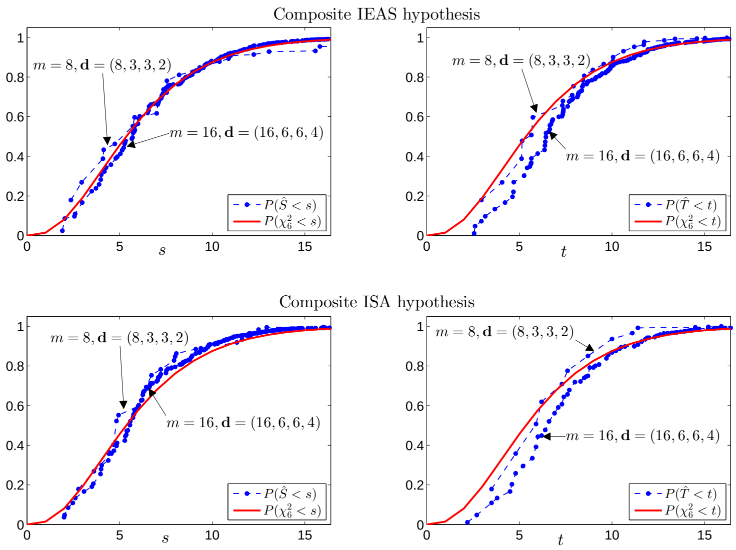

Figure 33: Non-null distributions of $\hat{S}$ and $\hat{T}$ for some RSM($\mathbf{d}$) models with skew $\mathbf{d}$ and composite IEAS and ISA hypotheses when m increases.

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