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# Block Circular Symmetry in Multilevel Models


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**Abstract**

Models that describe symmetries present in the error structure of observations have been widely used in different applications, with early examples from psychometric and medical research. The aim of this article is to study a multilevel model with a covariance structure that is block circular symmetric. Useful results are obtained for the spectra of these structured matrices.

*Keywords:* Covariance matrix, Circular block symmetry, Multilevel model, Symmetry model, Spectrum

## 1. Introduction

Real populations which are of interests in various research areas such as medicine, biology, social population studies, often exhibit hierarchical structures. For instance, in educational research, students are grouped within classes and classes are grouped within schools; in medical studies, patients are nested within doctors and doctors are nested within hospitals; in breeding studies, offsprings are grouped by sire and sires are grouped within some spatial factors (region); in political studies, voters are grouped within districts and districts are grouped within cities; in demographic studies, children are grouped within families and families are grouped within a macro-context such as neighborhoods or ethnic communities. It has been recognized that such grouping induces dependence between population units and, hence statistical models based upon independence assumption become invalid. Multilevel models (MM) is a widely accepted tool for analyzing data when observations are in a hierarchical form (for references see Hox and Kreft, 1994; Raudenbush, 1988; Goldstein, 2010).

In this article, we shall use the convention of calling single observations *cases* or *subjects*, and *units* will refer to clusters of observations. Thus, cases are observed within a unit in the same way as students may be observed within a class. When we would like to study the variations of the response variable, which arise at different levels, the following MM could be of use (e.g., see Goldstein, 2010):

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{E}, \\ \boldsymbol{E} &= \boldsymbol{Z}_s\boldsymbol{\gamma}_s + \ldots + \boldsymbol{Z}_2\boldsymbol{\gamma}_2 + \boldsymbol{Z}_1\boldsymbol{\gamma}_1, \end{aligned} \quad (1)$$

where $\boldsymbol{Y}$: $n \times 1$ is a response vector, $\boldsymbol{X}$: $n \times p$ is a known design matrix, $\boldsymbol{\beta} : p \times 1$ is a vector of fixed effects, $\boldsymbol{\gamma}_k : n_k \times 1$ is a random error at level $k$ and $\boldsymbol{Z}_k : n \times n_k$ is a known incidence matrix of random error, $k = 1, \ldots, s$. We assume that $\boldsymbol{\gamma}_k \sim N(0, \boldsymbol{\Sigma}_k)$ and $\boldsymbol{\gamma}_k$ is independent of $\boldsymbol{\gamma}_l$, $k \neq l$. Thus, $\boldsymbol{Y}$ is normally distributed with expectation $\boldsymbol{X\beta}$ and covariance matrix $\boldsymbol{\Sigma} = \sum_{k=1}^{s} \boldsymbol{Z}_k \boldsymbol{\Sigma}_k \boldsymbol{Z}_k'$.

Model (1) is also referred to as a hierarchical linear model (HLM) since it accounts for $\boldsymbol{Y}$ variation which comes from different $\boldsymbol{\gamma}_k$'s, reflecting the underlying hierarchical structure of the data. It may be noticed that any MM can be formulated as a linear mixed model (LMM), given by

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{Z\gamma} + \boldsymbol{\epsilon}, \tag{2}$$

where $\boldsymbol{Y}$, $\boldsymbol{X}$ and $\boldsymbol{\beta}$ have the same meaning as in (1), $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_s', \ldots, \boldsymbol{\gamma}_2')'$ is a vector of random-effects with a known incidence matrix $\boldsymbol{Z} = (\boldsymbol{Z}_s \vdots \ldots \vdots \boldsymbol{Z}_2)$, and $\boldsymbol{\epsilon} = \boldsymbol{Z}_1 \boldsymbol{\gamma}_1$ is an unknown random error vector whose elements are not required to be independent and homogeneous.

Though MM is commonly used to investigate multiple sources of variation in the context of an existence of an hierarchical structure, it can be adapted into the framework of LMM. For instance, in a medical study, patient 1 of doctor 1 is physically different from patient 1 of doctor 2 or on any other doctors. The same is true for any of the patients of any of the doctors. Similarly, doctor 1 in hospital 1 is different from doctor 1 in hospital 2, and so on. Therefore, understanding the input variables which are affected by the variation among those doctors, as well as understanding the variation across each doctor in a hospital, is important in statistical modelling. Furthermore, in the analysis, the factors which are located at the lower hierarchies (doctor and patient) are often treated as random, i.e., the levels of such factors have been randomly selected from a large population of levels, and the factor at the higher hierarchy is considered as fixed (hospital), i.e., the levels of this factor are the only ones of interest.

Let us now consider a balanced nested model, i.e., hierarchies have equal number of cases in all sub-hierarchies. Let $\boldsymbol{\gamma}_1 : n_2 \times 1$ and $\boldsymbol{\gamma}_2 : n_2 n_1 \times 1$ be two random nested effects and $\boldsymbol{\epsilon} : n_2 n_1 \times 1$ be the vector of random error. $\boldsymbol{\gamma}_i \sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_i)$ and $\boldsymbol{\epsilon} \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$ are assumed, $i = 1, 2$. Thus, the model in (1) reduces to:

$$\boldsymbol{Y} = (\boldsymbol{1}_{n_2} \otimes \boldsymbol{1}_{n_1})\mu + (\boldsymbol{I}_{n_2} \otimes \boldsymbol{1}_{n_1})\boldsymbol{\gamma}_1 + (\boldsymbol{I}_{n_2} \otimes \boldsymbol{I}_{n_1})\boldsymbol{\gamma}_2 + (\boldsymbol{I}_{n_2} \otimes \boldsymbol{I}_{n_1})\boldsymbol{\epsilon}, \tag{3}$$

where the scalar $\mu$ is the general mean, $\boldsymbol{1}_{n_i}$ is a column vector of size $n_i$ with all elements equal to 1, the symbol $\otimes$ denotes the Kronecker product and $\boldsymbol{I}_{n_i}$ is the identity matrix of order $n_i$. Thus,

$$\boldsymbol{Y} \sim N(\mu \boldsymbol{1}, \boldsymbol{\Sigma}), \text{ where } \boldsymbol{\Sigma} = \boldsymbol{Z}_1 \boldsymbol{\Sigma}_1 \boldsymbol{Z}_1' + \boldsymbol{\Sigma}_2 + \sigma^2 \boldsymbol{I},\ \boldsymbol{Z}_1 = \boldsymbol{I}_{n_2} \otimes \boldsymbol{1}_{n_1}.$$

As mentioned above, the presence of a hierarchical structure generally implies dependence within groups of observations. The dependence structure which is described via the covariance matrices can exhibit special patterns, for example an intraclass correlation pattern. Nowadays, the interest of studying various patterned covariance structures is increasing, e.g. see, Srivastava *et al.* (2008), Klein and Zezula (2009), Leiva and Roy (2010). The reason is that unstructured covariance matrices may not be suitable to model the error structure, in general. The number of unknown parameters in a $p \times p$ unstructured covariance matrix is $p(p+1)/2$. A parsimonious version of a covariance matrix may be both useful and meaningful when modelling data, especially for small sample sizes. E.g. in a $p \times p$ symmetric circular Toeplitz matrix, there are only $[p/2] + 1$ unknown parameters, the $[\bullet]$ stands for the integer part. Furthermore, in longitudinal studies, the number of covariance parameters to be estimated grows rapidly with the number of measured occasions and may approach or even become larger than the number of subjects enrolled in the study (Fitzmaurice *et al.*, 2004). In such situations it is common to impose some structures on the covariance matrix, e.g., autoregressive or banded structures. If we have a tenable prior knowledge about the true covariance structures of the random variables in the model, incorporation of this knowledge may increase the reliability of the estimation procedure. For example, Ohlson and von Rosen (2010) studied linearly structured covariance matrices in a classical growth curve model. Since the variance of the estimator of the mean parameter $\mu$ usually is a function of the covariance matrix, it is crucial to have a correct assumption about the covariance. Furthermore, an appropriate covariance structure also plays an important role in statistical diagnostics, such as outlier detection and influential observation identification, see Pan and Fang (2002), for example.

In this work we will study model (3) with a covariance structure that is block circular symmetric. Models that describe symmetries present in the error structure of observations have been widely used in different applications, with early examples from psychometric and medical research, such as Wilks (1946) and Votaw (1948).

The presence of symmetry in the data at one or several levels yields a patterned dependence structure within or between the corresponding levels in the model (Dawid, 1988). Symmetry here means, for example, that the units within given group are exchangeable (Draper *et al.*, 1993), i.e., dependence between neighboring units remains the same (invariant) after re-arrangement of units. Perlman (1987) discussed and summarized results related to group symmetry models. These are linear models for which the covariance structure of $\boldsymbol{Y}$ is assumed to satisfy certain symmetry restrictions, namely $D(\boldsymbol{Y}) = D(\boldsymbol{QY}) = \boldsymbol{Q}D(\boldsymbol{Y})\boldsymbol{Q}'$ for some orthogonal matrices, where $D(\bullet)$ stands for the covariance matrix. Properties of some patterned covariance matrices arising under different symmetry restrictions in balanced mixed models have been studied in Nahtman (2006), Nahtman and von Rosen (2008) and von Rosen (2011).

Circular symmetric model was considered by Olkin and Press (1969). They provided

MLEs for the parameters in such models. They also constructed different likelihood ratio tests (LRT) for testing different types of symmetry in the covariance matrix and tests concerning the mean structure. Olkin (1973) extended the circular symmetric model to the case where circular symmetry appeared in blocks, and blocks were unstructured. For this model, the covariance structure was studied and various LRTs were obtained.

The aim of this article is to extend models that are circular symmetric in blocks (Olkin, 1973), so-called dihedral block symmetry models. We prove that in case when both circular symmetry and exchangeability are present, these models have patterned blocks. We will follow up and combine in a certain sense the results obtained in Nahtman (2006), and Nahtman and von Rosen (2008) concerning the covariance structures in MM when a hierarchical data structure exists. We shall obtain expressions for the spectra of block circular symmetric covariance matrices which take into account the block structure.

The organization of the article is as follows. At the end of this section we give some examples concerning circular symmetry models. Section 2 states the preliminaries and presents some definitions and spectral properties of symmetric circular Toeplitz matrices; in Section 3 symmetry restrictions that yield the block circular symmetric covariance structure are obtained; in Section 4 the spectra of block circular symmetric matrices are considered; in Section 5 concluding remarks are presented.

### *1.1. Some examples of circular symmetry models*

Circular (block) symmetry models have been utilized in situations when there is a spatial circular layout on one factor and another factor satisfies the property of exchangeability.

**Example 1:** In a signal processing problem, Olkin and Press (1969) and Khattree and Naik (1994) studied a point source of a regular polygon of $n$ sides from which a signal received from a satellite is transmitted. The $n$ identical signal receivers with identical noise characteristics are placed at the $n$ vertices. Assuming that the signal strength is the same in all directions along the vertices of the polygon, one would expect a circular symmetric structure for the covariances between the messages received by the receivers placed at these vertices. Additionally, it might be possible to have a more general data structure, which contains another symmetric (with exchangeable categories) space factor (e.g., region), so that the data has the circulant property in the receiver (vertices) dimension and a symmetric pattern in the spatial dimension.

**Example 2:** In some public health studies (see Hartley and Naik, 2001), the disease incidence rates of (relatively homogeneous) city sectors placed around the city center may be circularly correlated. Additionally, if there are $n_2$ sectors within $n_1$ cities in the data, and $Y_{ij}$ denotes disease incidence rate in the $i$-th sector of the $j$-th city, then the covariance matrix of $Y_{ij}$ exhibits circular block symmetry when cities are exchangeable. Similarly, during an outbreak of a disease, the disease incidence rate in any sector around the initial ethological agent may be correlated with those adjacent sectors. With the existence of exchangeability of cities, this pattern of covariance structure is appropriate.

**Example 3:** Gotway and Cressie (1990) described a data set concerning soil-water-infiltration and it can be incorporated in our context by some modifications. As the location varies across the field, the ability of water to infiltrate soil will vary spatially so that locations nearby are more alike with regard to infiltration, than those far apart. Soil-water-infiltration measurements $Y_{ij}$ (uniresponse) or $Y_{ijk}$ (multiresponse) were made at $n_2$ locations contained by $n_1$ towns, which may be assumed to be exchangeable by our prior knowledge.

**Example 4:** Extending the studies of the standard "parent-sib" terminology (see Khattree and Naik, 1994; Hartley and Naik, 2001), we assume that there are $n_2$ siblings per parent (equal numbers of siblings in $n_1$ parents). It is reasonable to assume the between siblings covariance matrix has a circular structure, see Khattree and Naik (1994), and the dependence between families are equicorrelated. Let $Y_{ij}$ represent the score of the $i$-th child in the $j$-th family and we can formulate the model with a parsimonious covariance structure.

**Example 5:** Alternatively, in Example 2, if the factor "sector" is exchangeable but the factor "city" can be assumed to be circularly correlated, this "artificial" case will exhibit another combined shift and non-shift covariance structure of the disease incidence rate, which is different from the second example.

It can be found that the pattern of the covariance matrix in Example 1-4 is different from Example 5 since the circular correlation and the exchangeability are placed at different hierarchies of the model. For Example 1-4, the circular property occur at the lowest level while the exchangeability is at the highest level; in Example 5, the opposite was supposed.

## 2. Preliminaries

In this section, we will give some important definitions and provide useful results concerning certain patterned matrices which will be used in the subsequent. The concept of invariance is important throughout this work.

**Definition 2.1.** *The covariance matrix $D(\boldsymbol{\xi})$ of a random factor $\boldsymbol{\xi}$ is called invariant with respect to the transformation $\boldsymbol{Q}$ if $D(\boldsymbol{\xi}) = D(\boldsymbol{Q}\boldsymbol{\xi})$ which is the same as $D(\boldsymbol{\xi}) = \boldsymbol{Q}D(\boldsymbol{\xi})\boldsymbol{Q}'$, and $\boldsymbol{Q}$ is an orthogonal matrix.*

Next we will introduce specific matrices which are essential here and discuss their properties.

**Definition 2.2.** *A permutation matrix (P-matrix) is an orthogonal matrix whose columns can be obtained by permuting the columns of the identity matrix, e.g.*

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Definition 2.3.** *An orthogonal matrix $\boldsymbol{P} = (p_{ij}) : n \times n$ is a shift permutation matrix (SP-matrix) if*

$$p_{ij} = \begin{cases} 1, & \textit{if } j = i + 1 - n\mathbf{1}_{(i>n-1)}, \\ 0, & \textit{otherwise}, \end{cases}$$

*where $\mathbf{1}_{(.)}$ is the indicator function. For example, when $n = 3$ and $n = 4$, the SP-matrices are*

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

**Definition 2.4.** *A matrix $\boldsymbol{T} : n \times n$ of the form*

$$\boldsymbol{T} = \begin{pmatrix} t_0 & t_1 & t_2 & \cdots & t_1 \\ t_1 & t_0 & t_1 & \cdots & t_2 \\ t_2 & t_1 & t_0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ t_1 & t_2 & \cdots & t_1 & t_0 \end{pmatrix} = Toep(t_0, t_1, t_2, \ldots, t_1) \tag{4}$$

*is called a symmetric circular Toeplitz matrix (SC-Toeplitz matrix). The matrix $\boldsymbol{T} = (t_{ij})$ depends on $[n/2] + 1$ parameters, where $[.]$ stands for the integer part, and for $i, j = 1, \ldots, n$,*

$$t_{ij} = \begin{cases} t_{|j-i|}, & \textit{if } |j - i| \le [n/2], \\ t_{n-|j-i|}, & \textit{otherwise.} \end{cases}$$

An alternative way to define SC-Toeplitz matrix $\boldsymbol{T}$, see Olkin (1973), is given by

$$\boldsymbol{T} = \begin{pmatrix} t_1 & t_2 & t_3 & \cdots & t_n \\ t_n & t_1 & t_2 & \cdots & t_{n-1} \\ t_{n-1} & t_n & t_1 & \cdots & t_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \\ t_2 & t_3 & t_4 & \cdots & t_1 \end{pmatrix}, \text{ where } t_j = t_{n-j+2},\ j = 2, \ldots, n. \tag{5}$$

**Definition 2.5.** *A symmetric circular matrix $SC(n, k)$ is defined in the following way:*

$$SC(n, k) = Toep(\overbrace{\underbrace{0, \ldots, 0}_{k}, 1, 0, \ldots, 0, 1, \underbrace{0, \ldots, 0}_{k-1}}^{n})$$

*or equivalently*

$$(SC(n,k))_{ij} = \begin{cases} 1, & \textit{if } |i-j| = k \textit{ or } |i-j| = n-k, \\ 0, & \textit{otherwise,} \end{cases}$$

*where* $k \in \{1, \ldots, [n/2]\}$.
*For notational convenience denote* $SC(n,0) = \boldsymbol{I}_n$.

For example, when $n = 4$,

$$SC(4,1) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad SC(4,2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that

$$Toep(t_0, t_1, t_2, \ldots, t_1) = \sum_{k=0}^{[n/2]} t_k SC(n,k). \tag{6}$$

This way of representing SC-Toeplitz matrix can be useful when deriving MLEs for the model (3), see Olkin and Press (1969) and Olkin (1973).

The spectral properties of SC-Toeplitz matrices can, for example, be found in Basilevsky (1983). Nahtman and von Rosen (2008) gave some additional results concerning multiplicities of the eigenvalues of such matrices.

**Lemma 2.1.** *Let* $\boldsymbol{T} : n \times n$ *be a SC-Toeplitz matrix and let* $\lambda_h, h = 1, \ldots, n$, *be an eigenvalue of* $\boldsymbol{T}$.
(i) *If* $n$ *is odd, then*

$$\lambda_h = t_0 + 2 \sum_{j=1}^{[n/2]} t_j cos(2\pi hj/n). \tag{7}$$

*It follows that,* $\lambda_h = \lambda_{n-h}$, $h = 1, \ldots, n-1$, *there is only one eigenvalue,* $\lambda_n$, *which has multiplicity* 1, *and all other eigenvalues are of multiplicity* 2.
*If* $n$ *is even, then*

$$\lambda_h = t_0 + 2 \sum_{j=1}^{n/2-1} t_j cos(2\pi hj/n) + t_{n/2} cos(\pi h). \tag{8}$$

*It follows that, for* $h \neq n, n/2 : \lambda_h = \lambda_{n-h}$, *there are only two eigenvalues,* $\lambda_n$ *and* $\lambda_{n/2}$, *which have multiplicity* 1, *and all other eigenvalues are of multiplicity* 2.
(ii) *The number of distinct eigenvalues for SC-Toeplitz matrix is* $\left[\frac{n}{2}\right] + 1$.

(iii) *A set of eigenvectors* $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ *corresponding to the eigenvalues* $\lambda_1, \ldots, \lambda_n$*, is defined by*

$$v_{hi} = \frac{1}{\sqrt{n}} \left( cos(2\pi ih/n) + sin(2\pi ih/n) \right), \; i, h = 1, \ldots, n. \tag{9}$$

Furthermore, Lemma 2.1 provides immediately eigenvalues and eigenvectors for the matrix $SC(n, k)$ given in Definition 2.5. An important observation is that, the eigenvectors of a SC-Toeplitz matrix $\boldsymbol{T}$ in (9) do not depend on the elements of $\boldsymbol{T}$. A consequence of this result is the following.

**Theorem 2.2.** *Any pair of two SC-Toeplitz matrices of the same size commute.*

Another important result, given by Nahtman (2006), is presented in the next lemma. It will be used in Section 4. Let $\mathbf{1}_n$ denote a column vector of size $n$ with all elements equal to 1, and let $\boldsymbol{J}_n = \mathbf{1}_n \mathbf{1}_n'$.

**Lemma 2.3.** *The matrix* $\boldsymbol{\Sigma} = (a-b)\boldsymbol{I}_n + b\boldsymbol{J}_n$ *has two distinct eigenvalues,* $\lambda_0 = a - b$ *and* $\lambda_1 = a + (n-1)b$ *of multiplicities* $n-1$ *and* $1$*, respectively.*

See Nahtman (2006) for a proof.

## 3. Block circular symmetric covariance matrices

As mentioned above, the presence of symmetry in the data at one or several levels yields a patterned dependence structure within or between the corresponding levels (Dawid, 1988). In this section we shall obtain symmetry restrictions that keep the block circular symmetric covariance structures.

Let us consider model (3). We are specifically interested in the covariance matrices of the observation vector $\boldsymbol{Y} = (Y_{ij})$ and random factors in this model under circular symmetry.

A crucial assumption will be that if we permute or rotate the levels of one factor (i.e. permute or rotate the $i$th- or the $j$th-index in $Y_{ij}$), the others will not be affected. This leads to the concept of *marginal invariance*, see Nahtman (2006), i.e., each level within a factor can be permuted or shifted without any changes in the covariance structure of the model.

A symmetry model belongs to a family of models where the covariance matrix $\boldsymbol{\Sigma}$ remains invariant under a finite group $\mathcal{G}$ of orthogonal transformations (see Perlman, 1987). In the subsequent, we say that $\boldsymbol{\Sigma}$ is $\mathcal{G}$-invariant. More formally,

**Definition 3.1.** *A symmetry model determined by the group* $\mathcal{G}$ *is a family of models with covariance matrices*

$$S_{\mathcal{G}} \quad = \quad \{\boldsymbol{\Sigma} | \boldsymbol{G}\boldsymbol{\Sigma}\boldsymbol{G}' = \boldsymbol{\Sigma} \textit{ for all } \boldsymbol{G} \in \mathcal{G}\}. \tag{10}$$

The intraclass correlation model and the circular symmetry model are examples of symmetry models.

Let us define the following (finite) groups of orthogonal transformations:

$$\begin{aligned}
\mathcal{G}_0 &= \{\boldsymbol{P}^{(1)} \otimes \boldsymbol{P}^{(1)} | \boldsymbol{P}^{(1)} \text{ is a shift (rotation) matrix}\}, &(11)\\
\mathcal{G}_1 &= \{\boldsymbol{P}^{(2)} \otimes \boldsymbol{P}^{(2)} | \boldsymbol{P}^{(2)} \text{ is a permutation matrix}\}, &(12)\\
\mathcal{G}_2 &= \{\boldsymbol{P}_{12} | \boldsymbol{P}_{12} = \boldsymbol{P}^{(1)} \otimes \boldsymbol{P}^{(2)}\}, &(13)\\
\mathcal{G}_3 &= \{\boldsymbol{P}_{21} | \boldsymbol{P}_{21} = \boldsymbol{P}^{(2)} \otimes \boldsymbol{P}^{(1)}\}. &(14)
\end{aligned}$$

Thus, the following symmetry models can be considered.
(i) Symmetry model with complete block symmetry covariance matrices

$$S_{\mathcal{G}_1} = \{\boldsymbol{\Sigma} | \boldsymbol{G}\boldsymbol{\Sigma}\boldsymbol{G}' = \boldsymbol{\Sigma} \text{ for all } \boldsymbol{G} \in \mathcal{G}_1\} \tag{15}$$

implies that the covariance matrix $\boldsymbol{\Sigma}$ remains invariant under all permutations of the corresponding factor levels. Here, all the covariance matrices are of the form

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} & \cdots & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{A} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{B} \\ \boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{A} \end{pmatrix}. \tag{16}$$

(ii) Symmetry model with circular (dihedral) block symmetry covariance matrices

$$S_{\mathcal{G}_0} = \{\boldsymbol{\Sigma} | \boldsymbol{G}\boldsymbol{\Sigma}\boldsymbol{G}' = \boldsymbol{\Sigma} \text{ for all } \boldsymbol{G} \in \mathcal{G}_0\}. \tag{17}$$

Here, the covariance structure remains invariant under all rotations (and reflections) of the corresponding factor levels. All the covariance matrices are of the form, e.g.,

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{C} & \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{B} & \boldsymbol{A} \end{pmatrix}. \tag{18}$$

These models have been studied intensively during the last decades (see for example, Olkin and Press, 1969; Olkin, 1973; Marin and Dhorne, 2002, 2003).

The novelty of our work is the study of symmetry models determined by groups $\mathcal{G}_2$ and $\mathcal{G}_3$, i.e., models having the following covariance matrices,

$$\begin{aligned}
S_{\mathcal{G}_2} &= \{\boldsymbol{\Sigma} | \boldsymbol{G}\boldsymbol{\Sigma}\boldsymbol{G}' = \boldsymbol{\Sigma} \text{ for all } \boldsymbol{G} \in \mathcal{G}_2\}, &(19)\\
S_{\mathcal{G}_3} &= \{\boldsymbol{\Sigma} | \boldsymbol{G}\boldsymbol{\Sigma}\boldsymbol{G}' = \boldsymbol{\Sigma} \text{ for all } \boldsymbol{G} \in \mathcal{G}_3\}. &(20)
\end{aligned}$$

We shall show that a symmetry model determined by groups $\mathcal{G}_2$ and $\mathcal{G}_3$ is a special case of (i) or (ii) with an additional feature that blocks in the covariance matrix $\boldsymbol{\Sigma}$ are patterned. We also show how the symmetry models determined by groups $\mathcal{G}_2$ and $\mathcal{G}_3$ are related to each other.

The following should be especially noted: it is important to distinguish between full invariance and partial invariance. Full invariance concerns the covariance matrix $D(\boldsymbol{Y})$ of observation vector $\boldsymbol{Y}$ implying invariance for all factors in a model. Partial invariance concerns the covariance matrices of some (not all) factors in the model.

Nahtman (2006) and Nahtman and von Rosen (2008) gave the two following results, regarding the invariance of the main effect $\boldsymbol{\gamma}_1$ in model (3). Let $\boldsymbol{P}^{(1)}$ be a SP-matrix and $\boldsymbol{P}^{(2)}$ be a P-matrix.

**Theorem 3.1.** *Nahtman (2006) The covariance matrix $\boldsymbol{\Sigma}_1 : n_1 \times n_1$ of the factor $\boldsymbol{\gamma}_1$ is invariant with respect to all permutations $\boldsymbol{P}^{(2)}$, if and only if it has the following structure:*

$$\boldsymbol{\Sigma}_1 = \sum_{v_1=0}^{1} c_{v_1} \boldsymbol{J}_{n_1}^{v_1}, \tag{21}$$

*where $c_0$ and $c_1$ are constants, $v_1 \in \{0, 1\}$ and the matrices $\boldsymbol{J}_{n_1}^{v_1}$ are defined as follows:*

$$\boldsymbol{J}_{n_i}^{v_i} = \begin{cases} \boldsymbol{I}_{n_i}, & \text{if } v_i = 0, \\ \boldsymbol{J}_{n_i}, & \text{if } v_i = 1. \end{cases}$$

**Theorem 3.2.** *Nahtman and von Rosen (2008) The covariance matrix $\boldsymbol{\Sigma}_1 : n_1 \times n_1$ of the factor $\boldsymbol{\gamma}_1$ is shift invariant with respect to all shift permutations $\boldsymbol{P}^{(1)}$, if and only if it has the following structure:*

$$\boldsymbol{\Sigma}_1 = Toep(\tau_0, \tau_1, \tau_2, \ldots, \tau_1) = \sum_{k=0}^{[n_1/2]} \tau_k SC(n_1, k), \tag{22}$$

*where the matrices $SC(n_1, k)$, $k = 0, \ldots, [n_1/2]$, are given by* Definition 2.5, *and $\tau_k$, $k = 0, \ldots, [n_1/2]$, are constants.*

The results below reveal the structure of the covariance matrix of the factor representing the 2nd-order interaction effects $\boldsymbol{\gamma}_2$, which is invariant with respect to (19) or (20).

**Theorem 3.3.** *The matrix $D(\boldsymbol{\gamma}_2) = \boldsymbol{\Sigma}_{21} : n_2 n_1 \times n_2 n_1$ in model (3) is invariant with respect to $\mathcal{G}_3$, given in (14), if and only if it has the following structure:*

$$\begin{aligned} \boldsymbol{\Sigma}_{21} = \boldsymbol{I}_{n_2} \quad &\otimes \sum_{k_1=0}^{[n_1/2]} \tau_{k_1} SC(n_1, k_1) \\ &+ \quad (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \sum_{k_1=0}^{[n_1/2]} \tau_{k_1+[n_1/2]+1} SC(n_1, k_1), \end{aligned} \tag{23}$$

*where* $\tau_{k_1}$ *and* $\tau_{k_1+[n_1/2]+1}$ *are constants, and matrices* $SC(n_1,k_1)$ *are defined in* Definition (2.5)*,* $k_1 = 0,\ldots,[n_1/2]$.

**Proof.** Let $N = n_2 n_1$ and $\boldsymbol{P}_{21} \in \mathcal{G}_3$, given by (14). The matrix $\boldsymbol{\Sigma}_{21}$ can be written as

$$\boldsymbol{\Sigma}_{21} = \sum_{k=1}^{N}\sum_{l=1}^{N} \sigma_{kl}(\boldsymbol{e}_k \otimes \boldsymbol{e}_l'),$$

where $\boldsymbol{e}_k$, $\boldsymbol{e}_l$ are the $k$-th and the $l$-th columns of the identity matrix $\boldsymbol{I}_N$, respectively. We can define the element $\sigma_{kl}$ of $\boldsymbol{\Sigma}$ in a more informative way. Observe and one can write $\boldsymbol{e}_k = \boldsymbol{e}_{2,i_2} \otimes \boldsymbol{e}_{1,i_1}$ and $\boldsymbol{e}_l' = \boldsymbol{e}_{2,j_2}' \otimes \boldsymbol{e}_{1,j_1}'$, where $\boldsymbol{e}_{h,i_h}$ is the $i_h$-th column of the identity matrix $\boldsymbol{I}_{n_h}, h = 1,2$, and $\sigma_{kl} = \sigma_{(i_2,i_1)(j_2,j_1)} = Cov(\gamma_{2(i_2,i_1)}, \gamma_{2(j_2,j_1)})$, where $k = (i_2-1)n_1 + i_1$ and $l = (j_2-1)n_1 + j_1$.

Hence, we can express $\boldsymbol{\Sigma}_{21}$, using the following property of the Kronecker product,

$$(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D}) = \boldsymbol{AC}\otimes\boldsymbol{BD},$$

in the following way:

$$\begin{aligned}\boldsymbol{\Sigma}_{21} &= \sum_{i_2,j_2=1}^{n_2}\sum_{i_1,j_1=1}^{n_1} \sigma_{(i_2,i_1)(j_2,j_1)}(\boldsymbol{e}_{2,i_2}\otimes\boldsymbol{e}_{1,i_1})(\boldsymbol{e}_{2,j_2}'\otimes\boldsymbol{e}_{1,j_1}') \\ &= \sum_{i_2,j_2=1}^{n_2}\sum_{i_1,j_1=1}^{n_1} \sigma_{(i_2,i_1)(j_2,j_1)}(\boldsymbol{e}_{2,i_2}\boldsymbol{e}_{2,j_2}')\otimes(\boldsymbol{e}_{1,i_1}\boldsymbol{e}_{1,j_1}').\end{aligned}$$

The $\mathcal{G}_3$-invariance implies $\boldsymbol{P}_{21}\boldsymbol{\Sigma}_{21}\boldsymbol{P}_{21}' = \boldsymbol{\Sigma}_{21}$, for all $\boldsymbol{P}_{21} \in \mathcal{G}_3$. Therefore,

$$\begin{aligned}&\boldsymbol{P}_{21}\boldsymbol{\Sigma}_{21}\boldsymbol{P}_{21}' \\ &= \sum_{i_2,j_2=1}^{n_2}\sum_{i_1,j_1=1}^{n_1} \sigma_{(i_2,i_1)(j_2,j_1)}(\boldsymbol{P}^{(2)}\boldsymbol{e}_{2,i_2}\boldsymbol{e}_{2,j_2}'\boldsymbol{P}^{(2)'})\otimes(\boldsymbol{P}^{(1)}\boldsymbol{e}_{1,i_1}\boldsymbol{e}_{1,j_1}'\boldsymbol{P}^{(1)'}) \\ &= \sum_{i_2=j_2}\sum_{i_1,j_1=1}^{n_1} \sigma_{(i_2,i_1)(i_2,j_1)}(\boldsymbol{P}^{(2)}\boldsymbol{e}_{2,i_2}\boldsymbol{e}_{2,i_2}'\boldsymbol{P}^{(2)'})\otimes(\boldsymbol{P}^{(1)}\boldsymbol{e}_{1,i_1}\boldsymbol{e}_{1,j_1}'\boldsymbol{P}^{(1)'}) \\ &+ \sum_{i_2\neq j_2}\sum_{i_1,j_1=1}^{n_1} \sigma_{(i_2,i_1)(j_2,j_1)}(\boldsymbol{P}^{(2)}\boldsymbol{e}_{2,i_2}\boldsymbol{e}_{2,j_2}'\boldsymbol{P}^{(2)'})\otimes(\boldsymbol{P}^{(1)}\boldsymbol{e}_{1,i_1}\boldsymbol{e}_{1,j_1}'\boldsymbol{P}^{(1)'}).\end{aligned} \tag{24}$$

Since $\boldsymbol{P}^{(2)}$ is a P-matrix, it acts on the components of $\boldsymbol{\gamma}_2 = (\gamma_2)_{ij}$ via index $i$, which are associated with the corresponding factor levels of $\boldsymbol{\gamma}_1$, $i = 1,\ldots,n_2$, $j = 1,\ldots,n_1$. For the term $\boldsymbol{P}^{(2)}\boldsymbol{e}_{2,i_2}\boldsymbol{e}_{2,j_2}'\boldsymbol{P}^{(2)'}$, the invariance of $\boldsymbol{\Sigma}_{21}$ implies that in (24) we may define constants

$$\begin{aligned}\sigma_{1(i_1)(j_1)} &= \sigma_{(i_2,i_1)(i_2,j_1)}, \ \forall i_2 = j_2; \forall i_1, j_1, \\ \sigma_{2(i_1)(j_1)} &= \sigma_{(i_2,i_1)(j_2,j_1)}, \ \forall i_2 \neq j_2; \forall i_1, j_1,\end{aligned}$$

where $i_1, j_1 = 1, \ldots, n_1,\ i_2, j_2 = 1, \ldots, n_2$. Thus, (24) becomes

$$\begin{aligned}\boldsymbol{\Sigma}_{21} &= \sum_{i_1,j_1}^{n_1} \sigma_{1(i_1)(j_1)} \boldsymbol{I}_{n_2} \otimes (\boldsymbol{P}^{(1)} \boldsymbol{e}_{1,i_1} \boldsymbol{e}'_{1,j_1} \boldsymbol{P}^{(1)'}) \\ &+ \sum_{i_1,j_1}^{n_1} \sigma_{2(i_1)(j_1)} (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes (\boldsymbol{P}^{(1)} \boldsymbol{e}_{1,i_1} \boldsymbol{e}'_{1,j_1} \boldsymbol{P}^{(1)'}). \qquad (25)\end{aligned}$$

The SP-matrix $\boldsymbol{P}^{(1)}$ acts on the components of $\boldsymbol{\gamma}_2 = (\gamma_2)_{ij}$ via index $j$, which are nested within $\boldsymbol{\gamma}_1$ by assumption. We can express (25) in the following way:

$$\begin{aligned}\boldsymbol{\Sigma}_{21} &= \sum_{i_1=j_1} \sigma_{1(i_1)(i_1)} \boldsymbol{I}_{n_2} \otimes (\boldsymbol{P}^{(1)} \boldsymbol{e}_{1,i_1} \boldsymbol{e}'_{1,i_1} \boldsymbol{P}^{(1)'}) \\ &+ \sum_{k_1=1}^{[n_1/2]} \sum_{\substack{i_1,j_1 \\ |i_1-j_1|=k_1, \\ n_1-k_1}} \sigma_{1(i_1)(j_1)} \boldsymbol{I}_{n_2} \otimes (\boldsymbol{P}^{(1)} \boldsymbol{e}_{1,i_1} \boldsymbol{e}'_{1,j_1} \boldsymbol{P}^{(1)'}) \\ &+ \sum_{i_1=j_1} \sigma_{2(i_1)(i_1)} (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes (\boldsymbol{P}^{(1)} \boldsymbol{e}_{1,i_1} \boldsymbol{e}'_{1,i_1} \boldsymbol{P}^{(1)'}) \\ &+ \sum_{k_1=1}^{[n_1/2]} \sum_{\substack{i_1,j_1 \\ |i_1-j_1|=k_1, \\ n_1-k_1}} \sigma_{2(i_1)(j_1)} (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes (\boldsymbol{P}^{(1)} \boldsymbol{e}_{1,i_1} \boldsymbol{e}'_{1,j_1} \boldsymbol{P}^{(1)'}).\end{aligned}$$

By the invariance of $\boldsymbol{\Sigma}_{21}$ with respect to the term $\boldsymbol{P}^{(1)} \boldsymbol{e}_{1,i_1} \boldsymbol{e}'_{1,j_1} \boldsymbol{P}^{(1)'}$, we may define constants

$$\begin{aligned}&\tau_0 = \sigma_{1(i_1)(i_1)}, \forall i_1, \quad \tau_{k_1} = \sigma_{1(i_1)(j_1)}, \forall\ |i_1 - j_1| = k_1, n_1 - k_1, \\ &\tau_{[n_1/2]+1} = \sigma_{2(i_1)(i_1)}, \forall i_1, \quad \tau_{k_1+[n_1/2]+1} = \sigma_{2(i_1)(j_1)}, \forall\ |i_1 - j_1| = k_1, n_1 - k_1.\end{aligned}$$

Hence, we have the following structure for $\boldsymbol{\Sigma}_{21}$ :

$$\begin{aligned}&\boldsymbol{\Sigma}_{21} = \boldsymbol{I}_{n_2} \otimes \tau_0 \boldsymbol{I}_{n_1} + \boldsymbol{I}_{n_2} \otimes \sum_{k_1=1}^{[n_1/2]} \tau_{k_1} SC(n_1, k_1) \\ &+ (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \tau_{[n_1/2]+1} \boldsymbol{I}_{n_1} + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \sum_{k_1=1}^{[n_1/2]} \tau_{k_1+[n_1/2]+1} SC(n_1, k_1) \\ &= \boldsymbol{I}_{n_2} \otimes \sum_{k_1=0}^{[n_1/2]} \tau_{k_1} SC(n_1, k_1) + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \sum_{k_1=0}^{[n_1/2]} \tau_{k_1+[n_1/2]+1} SC(n_1, k_1).\end{aligned}$$

The structure in (23) is obtained, which implies that the "only if" part of the theorem is true. The "if" part is shown due to the structure of $\mathbf{\Sigma}_{21}$, since

$$\begin{aligned}
\boldsymbol{P}_{21}\mathbf{\Sigma}_{21}\boldsymbol{P}'_{21} &= (\boldsymbol{P}^{(2)} \otimes \boldsymbol{P}^{(1)}) \left[\boldsymbol{I}_{n_2} \otimes \sum_{k_1=0}^{[n_1/2]} \tau_{k_1} SC(n_1, k_1)\right] (\boldsymbol{P}^{(2)'} \otimes \boldsymbol{P}^{(1)'}) \\
&+ (\boldsymbol{P}^{(2)} \otimes \boldsymbol{P}^{(1)}) \left[(\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \sum_{k_1=0}^{[n_1/2]} \tau_{k_1+[n_1/2]+1} SC(n_1, k_1)\right] (\boldsymbol{P}^{(2)'} \otimes \boldsymbol{P}^{(1)'}) \\
&= \boldsymbol{I}_{n_2} \otimes \boldsymbol{P}^{(1)} \sum_{k_1=0}^{[n_1/2]} \tau_{k_1} SC(n_1, k_1) \boldsymbol{P}^{(1)'} \\
&\qquad + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{P}^{(1)} \sum_{k_1=0}^{[n_1/2]} \tau_{k_1+[n_1/2]+1} SC(n_1, k_1) \boldsymbol{P}^{(1)'},
\end{aligned}$$

followed by Theorem 3.2,

$$\boldsymbol{P}^{(1)} \sum_{k_1=0}^{[n_1/2]} \tau_{k_1} SC(n_1, k_1) \boldsymbol{P}^{(1)'} = \sum_{k_1=0}^{[n_1/2]} \tau_{k_1} SC(n_1, k_1)$$

and

$$\boldsymbol{P}^{(1)} \sum_{k_1=0}^{[n_1/2]} \tau_{k_1+[n_1/2]+1} SC(n_1, k_1) \boldsymbol{P}^{(1)'} = \sum_{k_1=0}^{[n_1/2]} \tau_{k_1+[n_1/2]+1} SC(n_1, k_1).$$

Hence, the proof is completed. ■

In order to emphasize the block-symmetric structure of $\mathbf{\Sigma}_{21}$ given in (23), the following result can be established.

**Corollary 3.4.** *The $\mathcal{G}_3$-invariant matrix $\mathbf{\Sigma}_{21}$ given in (23) has the following block structure:*

$$\begin{aligned}
\mathbf{\Sigma}_{21} &= \begin{pmatrix} \mathbf{\Sigma}^{(1)} & \mathbf{\Sigma}^{(2)} & \mathbf{\Sigma}^{(2)} & \cdots & \mathbf{\Sigma}^{(2)} \\ \mathbf{\Sigma}^{(2)} & \mathbf{\Sigma}^{(1)} & \mathbf{\Sigma}^{(2)} & \cdots & \mathbf{\Sigma}^{(2)} \\ \mathbf{\Sigma}^{(2)} & \mathbf{\Sigma}^{(2)} & \mathbf{\Sigma}^{(1)} & \cdots & \mathbf{\Sigma}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Sigma}^{(2)} & \mathbf{\Sigma}^{(2)} & \mathbf{\Sigma}^{(2)} & \cdots & \mathbf{\Sigma}^{(1)} \end{pmatrix} \\
&= \boldsymbol{I}_{n_2} \otimes \mathbf{\Sigma}^{(1)} + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \mathbf{\Sigma}^{(2)},
\end{aligned} \tag{26}$$

*where $\mathbf{\Sigma}^{(1)} = \sum_{k_1=0}^{[n_1/2]} \tau_{k_1} SC(n_1, k_1)$, $\mathbf{\Sigma}^{(2)} = \sum_{k_1=0}^{[n_1/2]} \tau_{k_1+[n_1/2]+1} SC(n_1, k_1)$. The number of distinct elements given in (26) is $2([n_1/2] + 1)$.*

The next example illustrates a $\mathcal{G}_3$-invariant covariance matrices when $n_2 = 4$ and $n_1 = 4$.

**Example 3.1.** If $n_2 = 4, n_1 = 4$, then

$$\boldsymbol{\Sigma}_{21} = \left(\begin{array}{cccc|cccc|cccc|cccc}
\tau_0 & \tau_1 & \tau_2 & \tau_1 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 \\
\tau_1 & \tau_0 & \tau_1 & \tau_2 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 \\
\tau_2 & \tau_1 & \tau_0 & \tau_1 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 \\
\tau_1 & \tau_2 & \tau_1 & \tau_0 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 \\
\hline
\tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_0 & \tau_1 & \tau_2 & \tau_1 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 \\
\tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_1 & \tau_0 & \tau_1 & \tau_2 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 \\
\tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_2 & \tau_1 & \tau_0 & \tau_1 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 \\
\tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_1 & \tau_2 & \tau_1 & \tau_0 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 \\
\hline
\tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_0 & \tau_1 & \tau_2 & \tau_1 & \tau_3 & \tau_4 & \tau_5 & \tau_4 \\
\tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_1 & \tau_0 & \tau_1 & \tau_2 & \tau_4 & \tau_3 & \tau_4 & \tau_5 \\
\tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_2 & \tau_1 & \tau_0 & \tau_1 & \tau_5 & \tau_4 & \tau_3 & \tau_4 \\
\tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_1 & \tau_2 & \tau_1 & \tau_0 & \tau_4 & \tau_5 & \tau_4 & \tau_3 \\
\hline
\tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_0 & \tau_1 & \tau_2 & \tau_1 \\
\tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_1 & \tau_0 & \tau_1 & \tau_2 \\
\tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_2 & \tau_1 & \tau_0 & \tau_1 \\
\tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_1 & \tau_2 & \tau_1 & \tau_0
\end{array}\right). \tag{27}$$

The next step is to derive the structure of the covariance matrix which is $\mathcal{G}_2$-invariant.

**Theorem 3.5.** *The matrix $D(\boldsymbol{\gamma}_2) = \boldsymbol{\Sigma}_{12} : n_2 n_1 \times n_2 n_1$ is $\mathcal{G}_2$-invariant if and only if it has the following structure:*

$$\boldsymbol{\Sigma}_{12} = \sum_{k_2=0}^{[n_2/2]} \left[ SC(n_2, k_2) \otimes \boldsymbol{\Sigma}^{(k_2)} \right], \tag{28}$$

*where $\boldsymbol{\Sigma}^{(k_2)} = \tau_{k_2} \boldsymbol{I}_{n_1} + \tau_{k_2+[n_2/2]+1} (\boldsymbol{J}_{n_1} - \boldsymbol{I}_{n_1}), \tau_{k_2}$ and $\tau_{k_2+[n_2/2]+1}$ are constants. $SC(n_2, k_2)$ is a SC-matrix, given in* Definition 2.5.

**Proof.** We use the same technique as in Theorem 3.3. Under the condition $\boldsymbol{P}_{12} \boldsymbol{\Sigma}_{12} \boldsymbol{P}'_{12} = \boldsymbol{\Sigma}_{12}$, for all $\boldsymbol{P}_{12} \in \mathcal{G}_2$, after the same presentation of $\boldsymbol{\Sigma}_{21}$ as used in Theorem 3.3 for $\boldsymbol{\Sigma}_{12}$, we have

$$\begin{aligned}
\boldsymbol{\Sigma}_{12} &= \sum_{i_2, j_2 = 1}^{n_2} \sum_{i_1, j_1 = 1}^{n_1} \sigma_{(i_2, i_1)(j_2, j_1)} (\boldsymbol{P}^{(1)} \boldsymbol{e}_{2,i_2} \boldsymbol{e}'_{2,j_2} \boldsymbol{P}^{(1)'}) \otimes (\boldsymbol{P}^{(2)} \boldsymbol{e}_{1,i_1} \boldsymbol{e}'_{1,j_1} \boldsymbol{P}^{(2)'}) \\
&= \sum_{i_2, j_2 = 1}^{n_2} \sum_{i_1 = j_1} \sigma_{(i_2, i_1)(j_2, i_1)} (\boldsymbol{P}^{(1)} \boldsymbol{e}_{2,i_2} \boldsymbol{e}'_{2,j_2} \boldsymbol{P}^{(1)'}) \otimes (\boldsymbol{P}^{(2)} \boldsymbol{e}_{1,i_1} \boldsymbol{e}'_{1,i_1} \boldsymbol{P}^{(2)'}) \\
&+ \sum_{i_2, j_2 = 1}^{n_2} \sum_{i_1 \neq j_1} \sigma_{(i_2, i_1)(j_2, j_1)} (\boldsymbol{P}^{(1)} \boldsymbol{e}_{2,i_2} \boldsymbol{e}'_{2,j_2} \boldsymbol{P}^{(1)'}) \otimes (\boldsymbol{P}^{(2)} \boldsymbol{e}_{1,i_1} \boldsymbol{e}'_{1,j_1} \boldsymbol{P}^{(2)'}).
\end{aligned}$$

Denoting $\sigma_{1(i_2)(j_2)} = \sigma_{(i_2,i_1),(j_2,i_1)}$ for $\forall i_1 = j_1$; $\forall i_2, j_2$ and $\sigma_{2(i_2)(j_2)} = \sigma_{(i_2,i_1),(j_2,j_1)}$ for $\forall i_1 \neq j_1$; $\forall i_2, j_2$, we have

$$\begin{aligned}
\boldsymbol{\Sigma}_{12} &= \sum_{i_2,j_2=1}^{n_2} \sigma_{1(i_2)(j_2)} (\boldsymbol{P}^{(1)} \boldsymbol{e}_{2,i_2} \boldsymbol{e}'_{2,j_2} \boldsymbol{P}^{(1)'}) \otimes \boldsymbol{I}_{n_1} \\
&+ \sum_{i_2,j_2=1}^{n_2} \sigma_{2(i_2)(j_2)} (\boldsymbol{P}^{(1)} \boldsymbol{e}_{2,i_2} \boldsymbol{e}'_{2,j_2} \boldsymbol{P}^{(1)'}) \otimes (\boldsymbol{J}_{n_1} - \boldsymbol{I}_{n_1}) \\
&= \sum_{k_2=0}^{[n_2/2]} \sum_{\substack{i_2,j_2 \\ |i_2-j_2|=k_2, \\ n_2-k_2}} \sigma_{1(i_2)(j_2)} (\boldsymbol{P}^{(1)} \boldsymbol{e}_{2,i_2} \boldsymbol{e}'_{2,j_2} \boldsymbol{P}^{(1)'}) \otimes \boldsymbol{I}_{n_1} \\
&+ \sum_{k_2=0}^{[n_2/2]} \sum_{\substack{i_2,j_2 \\ |i_2-j_2|=k_2, \\ n_2-k_2}} \sigma_{2(i_2)(j_2)} (\boldsymbol{P}^{(1)} \boldsymbol{e}_{2,i_2} \boldsymbol{e}'_{2,j_2} \boldsymbol{P}^{(1)'}) \otimes (\boldsymbol{J}_{n_1} - \boldsymbol{I}_{n_1}). \qquad (29)
\end{aligned}$$

Let us now define $\tau_{k_2} = \sigma_{1(i_2)(j_2)}, \forall |i_2 - j_2| = k_2, n_2 - k_2; \forall i_1 = j_1$, and $\tau_{k_2+[n_2/2]+1} = \sigma_{2(i_2)(j_2)}, \forall |i_2 - j_2| = k_2, n_2 - k_2; \forall i_1 \neq j_1$. Thus, (29) becomes

$$\boldsymbol{\Sigma}_{12} = \sum_{k_2=0}^{[n_2/2]} SC(n_2, k_2) \otimes \left[\tau_{k_2} \boldsymbol{I}_{n_1} + \tau_{k_2+[n_2/2]+1} (\boldsymbol{J}_{n_1} - \boldsymbol{I}_{n_1})\right],$$

and (28) is obtained. Due to the structure of $\boldsymbol{\Sigma}_{12}$, it is straightforward to show that $\boldsymbol{P}_{12} \boldsymbol{\Sigma}_{12} \boldsymbol{P}'_{12} = \boldsymbol{\Sigma}_{12}$. ■

The following corollary illustrates the block structure of $\boldsymbol{\Sigma}_{12}$ which is $\mathcal{G}_2$-invariant.

**Corollary 3.6.** *A $\mathcal{G}_2$-invariant covariance matrix $\boldsymbol{\Sigma}_{12}$ has the following block structure:*

$$\begin{aligned}
\boldsymbol{\Sigma}_{12} &= \begin{pmatrix}
\boldsymbol{\Sigma}^{(0)} & \boldsymbol{\Sigma}^{(1)} & \boldsymbol{\Sigma}^{(2)} & \dots & \boldsymbol{\Sigma}^{(2)} & \boldsymbol{\Sigma}^{(1)} \\
\boldsymbol{\Sigma}^{(1)} & \boldsymbol{\Sigma}^{(0)} & \boldsymbol{\Sigma}^{(1)} & \dots & \boldsymbol{\Sigma}^{(3)} & \boldsymbol{\Sigma}^{(2)} \\
\boldsymbol{\Sigma}^{(2)} & \boldsymbol{\Sigma}^{(1)} & \boldsymbol{\Sigma}^{(0)} & \dots & \boldsymbol{\Sigma}^{(4)} & \boldsymbol{\Sigma}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\boldsymbol{\Sigma}^{(2)} & \boldsymbol{\Sigma}^{(3)} & \boldsymbol{\Sigma}^{(4)} & \dots & \boldsymbol{\Sigma}^{(0)} & \boldsymbol{\Sigma}^{(1)} \\
\boldsymbol{\Sigma}^{(1)} & \boldsymbol{\Sigma}^{(2)} & \boldsymbol{\Sigma}^{(3)} & \dots & \boldsymbol{\Sigma}^{(1)} & \boldsymbol{\Sigma}^{(0)}
\end{pmatrix} \qquad (30) \\
&= \sum_{k_2=0}^{[n_2/2]} \left[SC(n_2, k_2) \otimes \boldsymbol{\Sigma}^{(k_2)}\right].
\end{aligned}$$

*where $\boldsymbol{\Sigma}^{(k_2)} = \tau_{k_2} \boldsymbol{I}_{n_1} + \tau_{k_2+[n_2/2]+1} (\boldsymbol{J}_{n_1} - \boldsymbol{I}_{n_1})$. The number of distinct elements of $\boldsymbol{\Sigma}_{12}$ is $2([n_2/2] + 1)$.*

In the next example $\mathcal{G}_2$-invariant $\mathbf{\Sigma}_{12}$ will be presented when $n_2 = 4$ and $n_1 = 4$.

**Example 3.2.** If $n_2 = 4, n_1 = 4$, then

$$\mathbf{\Sigma}_{12} = \left(\begin{array}{cccc|cccc|cccc|cccc}
\tau_0 & \tau_3 & \tau_3 & \tau_3 & \tau_1 & \tau_4 & \tau_4 & \tau_4 & \tau_2 & \tau_5 & \tau_5 & \tau_5 & \tau_1 & \tau_4 & \tau_4 & \tau_4 \\
\tau_3 & \tau_0 & \tau_3 & \tau_3 & \tau_4 & \tau_1 & \tau_4 & \tau_4 & \tau_5 & \tau_2 & \tau_5 & \tau_5 & \tau_4 & \tau_1 & \tau_4 & \tau_4 \\
\tau_3 & \tau_3 & \tau_0 & \tau_3 & \tau_4 & \tau_4 & \tau_1 & \tau_4 & \tau_5 & \tau_5 & \tau_2 & \tau_5 & \tau_4 & \tau_4 & \tau_1 & \tau_4 \\
\tau_3 & \tau_3 & \tau_3 & \tau_0 & \tau_4 & \tau_4 & \tau_4 & \tau_1 & \tau_5 & \tau_5 & \tau_5 & \tau_2 & \tau_4 & \tau_4 & \tau_4 & \tau_1 \\
\hline
\tau_1 & \tau_4 & \tau_4 & \tau_4 & \tau_0 & \tau_3 & \tau_3 & \tau_3 & \tau_1 & \tau_4 & \tau_4 & \tau_4 & \tau_2 & \tau_5 & \tau_5 & \tau_5 \\
\tau_4 & \tau_1 & \tau_4 & \tau_4 & \tau_3 & \tau_0 & \tau_3 & \tau_3 & \tau_4 & \tau_1 & \tau_4 & \tau_4 & \tau_5 & \tau_2 & \tau_5 & \tau_5 \\
\tau_4 & \tau_4 & \tau_1 & \tau_4 & \tau_3 & \tau_3 & \tau_0 & \tau_3 & \tau_4 & \tau_4 & \tau_1 & \tau_4 & \tau_5 & \tau_5 & \tau_2 & \tau_5 \\
\tau_4 & \tau_4 & \tau_4 & \tau_1 & \tau_3 & \tau_3 & \tau_3 & \tau_0 & \tau_4 & \tau_4 & \tau_4 & \tau_1 & \tau_5 & \tau_5 & \tau_5 & \tau_2 \\
\hline
\tau_2 & \tau_5 & \tau_5 & \tau_5 & \tau_1 & \tau_4 & \tau_4 & \tau_4 & \tau_0 & \tau_3 & \tau_3 & \tau_3 & \tau_1 & \tau_4 & \tau_4 & \tau_4 \\
\tau_5 & \tau_2 & \tau_5 & \tau_5 & \tau_4 & \tau_1 & \tau_4 & \tau_4 & \tau_3 & \tau_0 & \tau_3 & \tau_3 & \tau_4 & \tau_1 & \tau_4 & \tau_4 \\
\tau_5 & \tau_5 & \tau_2 & \tau_5 & \tau_4 & \tau_4 & \tau_1 & \tau_4 & \tau_3 & \tau_3 & \tau_0 & \tau_3 & \tau_4 & \tau_4 & \tau_1 & \tau_4 \\
\tau_5 & \tau_5 & \tau_5 & \tau_2 & \tau_4 & \tau_4 & \tau_4 & \tau_1 & \tau_3 & \tau_3 & \tau_3 & \tau_0 & \tau_4 & \tau_4 & \tau_4 & \tau_1 \\
\hline
\tau_1 & \tau_4 & \tau_4 & \tau_4 & \tau_2 & \tau_5 & \tau_5 & \tau_5 & \tau_1 & \tau_4 & \tau_4 & \tau_4 & \tau_0 & \tau_3 & \tau_3 & \tau_3 \\
\tau_4 & \tau_1 & \tau_4 & \tau_4 & \tau_5 & \tau_2 & \tau_5 & \tau_5 & \tau_4 & \tau_1 & \tau_4 & \tau_4 & \tau_3 & \tau_0 & \tau_3 & \tau_3 \\
\tau_4 & \tau_4 & \tau_1 & \tau_4 & \tau_5 & \tau_5 & \tau_2 & \tau_5 & \tau_4 & \tau_4 & \tau_1 & \tau_4 & \tau_3 & \tau_3 & \tau_0 & \tau_3 \\
\tau_4 & \tau_4 & \tau_4 & \tau_1 & \tau_5 & \tau_5 & \tau_5 & \tau_2 & \tau_4 & \tau_4 & \tau_4 & \tau_1 & \tau_3 & \tau_3 & \tau_3 & \tau_0
\end{array}\right). \tag{31}$$

It is interesting to observe that the $\mathcal{G}_3$-invariant matrix $\mathbf{\Sigma}_{21} : 16 \times 16$ in (27) has a different structure from the $\mathcal{G}_2$-invariant matrix $\mathbf{\Sigma}_{12} : 16 \times 16$. One is block compound symmetry with SC-Toeplitz blocks, another is block SC-Toeplitz with compound symmetric blocks. Transformation $\boldsymbol{P}_{12}$ and $\boldsymbol{P}_{21}$ only affect indices of a response vector $\boldsymbol{Y} = (y_{ij})$, and the question is whether the labeling of $y_{ij}$ (observations) affects the covariance structure of the model. The answer is negative. The relationship between the two covariance structures, obtained in Theorem 3.3 and 3.5, respectively, can be described in the theorem below.

The following matrix is needed. Let $\boldsymbol{e}_i$ be the $i$-th column vector of $\boldsymbol{I}_{n_1}$ and $\boldsymbol{d}_j$ be the $j$-th column vector of $\boldsymbol{I}_{n_2}$. Then the commutation matrix $\boldsymbol{K}_{n_1,n_2}$ is defined as

$$\boldsymbol{K}_{n_1,n_2} = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(\boldsymbol{e}_i\boldsymbol{d}_j') \otimes (\boldsymbol{d}_j\boldsymbol{e}_i'). \tag{32}$$

**Theorem 3.7.** *With rearrangement of the observations in the response vector* $\boldsymbol{Y}$ *in model (3), the covariance matrix* $\mathbf{\Sigma_{21}}$ *given in (23), can be transformed into the covariance matrix* $\mathbf{\Sigma_{12}}$ *given in (28), i.e.* $\mathbf{\Sigma_{12}} = \boldsymbol{K}_{n_1,n_2}\mathbf{\Sigma_{21}}\boldsymbol{K}'_{n_1,n_2}$, *where* $\boldsymbol{K}_{n_1,n_2} : n_2n_1 \times n_2n_1$ *is the commutation matrix given in (32).*

**Proof.** In Theorem 3.3,

$$\mathbf{\Sigma}_{21} = \boldsymbol{I}_{n_2} \otimes \sum_{k_1=0}^{[n_1/2]} \tau_{k_1} SC(n_1,k_1) + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \sum_{k_1=0}^{[n_1/2]} \tau_{k_1+[n_1/2]+1} SC(n_1,k_1)$$

$$
\begin{aligned}
&= \boldsymbol{I}_{n_2} \otimes \tau_0 SC(n_1, 0) + \ldots + \boldsymbol{I}_{n_2} \otimes \tau_{[n_1/2]} SC(n_1, [n_1/2]) \\
&+ (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \tau_{[n_1/2]+1} SC(n_1, 0) + \ldots \\
&\qquad\qquad + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \tau_{2[n_1/2]+1} SC(n_1, [n_1/2]).
\end{aligned}
$$

Using the following property of the Kronecker product $(c\boldsymbol{A}) \otimes \boldsymbol{B} = \boldsymbol{A} \otimes (c\boldsymbol{B})$, where $c$ is an arbitrary scalar, we have

$$
\begin{aligned}
\boldsymbol{\Sigma}_{21} &= \tau_0 \boldsymbol{I}_{n_2} \otimes SC(n_1, 0) + \ldots + \tau_{[n_1/2]} \boldsymbol{I}_{n_2} \otimes SC(n_1, [n_1/2]) \\
&+ \tau_{[n_1/2]+1} (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes SC(n_1, 0) + \ldots \\
&+ \tau_{2[n_1/2]+1} (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes SC(n_1, [n_1/2]) \\
&= \underbrace{\left[\tau_0 \boldsymbol{I}_{n_2} + \tau_{[n_1/2]+1} (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2})\right]}_{\boldsymbol{\Sigma}^{(0)}} \otimes SC(n_1, 0) + \ldots \\
&+ \underbrace{\left[\tau_{[n_1/2]} \boldsymbol{I}_{n_2} + \tau_{2[n_1/2]+1} (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2})\right]}_{\boldsymbol{\Sigma}^{([n_1/2])}} \otimes SC(n_1, [n_1/2]) \\
&= \sum_{k_1=0}^{[n_1/2]} \left[\boldsymbol{\Sigma}^{(k_1)} \otimes SC(n_1, k_1)\right],
\end{aligned}
$$

where $\boldsymbol{\Sigma}^{(k_1)} = \tau_{k_1} \boldsymbol{I}_{n_2} + \tau_{k_1+[n_1/2]+1} (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}), k_1 = 0, \ldots, [n_1/2]$. Moreover, let $\boldsymbol{Y} = (y_{11}, y_{12}, \ldots, y_{1n_1}, \ldots, y_{n_2 1}, y_{n_2 2}, \ldots, y_{n_2 n_1})'$. Applying $\boldsymbol{K}_{n_1,n_2}$ to $\boldsymbol{Y}$ yields,

$$
(\boldsymbol{K}_{n_1,n_2} \boldsymbol{Y}) = (y_{11}, y_{21}, \ldots, y_{n_1 1}, \ldots, y_{1n_2}, y_{2n_2}, \ldots, y_{n_1 n_2})',
$$

the labeling is changed.

With the help of the commutation matrix, we can interchange the elements of the Kronecker product, namely,

$$
\boldsymbol{K}_{n_1,n_2} \sum_{k_1=0}^{[n_1/2]} \left[\boldsymbol{\Sigma}^{(k_1)} \otimes SC(n_1, k_1)\right] \boldsymbol{K}'_{n_1,n_2} = \sum_{k_1=0}^{[n_1/2]} \left[SC(n_1, k_1) \otimes \boldsymbol{\Sigma}^{(k_1)}\right],
$$

and the structure of $\boldsymbol{\Sigma}_{12}$ in Theorem 3.5 is obtained.

If the covariance matrix has the structure $\boldsymbol{\Sigma}_{12}$, using the commutation matrix $\boldsymbol{K}_{n_2,n_1}$, we obtain the same structure as in Theorem 3.3, i.e.,

$$
\begin{aligned}
&\boldsymbol{\Sigma}_{21} = \boldsymbol{K}_{n_2,n_1} \boldsymbol{\Sigma}_{12} \boldsymbol{K}'_{n_2,n_1} \\
= \quad &\boldsymbol{I}_{n_1} \otimes \sum_{k_2=0}^{[n_1/2]} \tau_{k_2} SC(n_2, k_2) + (\boldsymbol{J}_{n_1} - \boldsymbol{I}_{n_1}) \otimes \sum_{k_2=0}^{(n_2/2)} \tau_{k_2+[n_2/2]+1} SC(n_2, k_2).
\end{aligned}
$$

■

We use a simple example to demonstrate the statement of Theorem 3.7.

**Example 3.3.** In the case of Theorem 3.3, when $n_2 = 3,\ n_1 = 4$, let

$$\boldsymbol{Y} = (y_{11}, y_{12}, y_{13}, y_{14}, y_{21}, y_{22}, y_{23}, y_{24}, y_{31}, y_{32}, y_{33}, y_{34})'$$

and $\boldsymbol{\gamma}_2$ has the covariance matrix $\boldsymbol{\Sigma}_{21}$ given in (27). According to Theorem 3.7, there exists the following commutation matrix

$$\boldsymbol{K}_{4,3} = \left(\begin{array}{cccc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

such that

$$\boldsymbol{K_{4,3}Y} = (y_{11}, y_{21}, y_{31}, y_{12}, y_{22}, y_{32}, y_{13}, y_{23}, y_{33}, y_{14}, y_{24}, y_{34})'$$

and $\boldsymbol{\Sigma}_{12} = \boldsymbol{K}_{4,3}\boldsymbol{\Sigma}_{21}\boldsymbol{K}'_{4,3}$. The example shows that, (27) and (31) reflect the dependence structure of the same data, which however, arise from different labeling of factor levels.

## 4. Spectra of $\mathcal{G}_2$ and $\mathcal{G}_3$-invariant matrices

In this section we study the spectra of the covariance matrices $\boldsymbol{\Sigma}_{21}$ and $\boldsymbol{\Sigma}_{12}$, given in Theorem 3.3 and Theorem 3.5, respectively. The novelty of our results is that we use the eigenvalues of the blocks which constitute corresponding matrices (26) and (30), instead of direct calculation of the eigenvalues using the elements of $\boldsymbol{\Sigma}_{21}$ and $\boldsymbol{\Sigma}_{12}$. Here the concept of *commutativity* is important since if two normal matrices commute then they have a joint eigenspace and can be diagonalized simultaneously, for example see Kollo and von Rosen (2005, Chapter 1). The multiplicities of the eigenvalues and the number of distinct eigenvalues will also be given.

**Theorem 4.1.** *Let the covariance matrix $\boldsymbol{\Sigma}_{21} : n_2n_1 \times n_2n_1$ be $\mathcal{G}_3$-invariant and have a structure obtained in (26). Let $\lambda_h^{(i)}$ be the eigenvalue of $\boldsymbol{\Sigma}^{(i)} : n_1 \times n_1$ with multiplicity $m_h,\ i, = 1, 2, h = 1, \ldots, [n_1/2] + 1$. The spectrum of $\boldsymbol{\Sigma}_{21}$ consists of the eigenvalues $\lambda_h^{(1)} - \lambda_h^{(2)}$, each of multiplicity $(n_2 - 1)m_h$, and $\lambda_h^{(1)} + (n_2 - 1)\lambda_h^{(2)}$, each of multiplicity $m_h$. The number of distinct eigenvalues is $2([n_1/2] + 1)$.*

**Proof.** The SC-matrices $SC(n_i, k_i)$, $k_i = 0, \dots, [n_i/2]$ commute. So $\boldsymbol{\Sigma}^{(1)}$ and $\boldsymbol{\Sigma}^{(2)}$ commute as well, and they have a joint eigenspace. Hence, there exists an orthogonal matrix $\boldsymbol{V}_2$, such that $\boldsymbol{V}_2'\boldsymbol{\Sigma}^{(1)}\boldsymbol{V}_2 = \boldsymbol{\Lambda}^{(1)}$ and $\boldsymbol{V}_2'\boldsymbol{\Sigma}^{(2)}\boldsymbol{V}_2 = \boldsymbol{\Lambda}^{(2)}$, where $\boldsymbol{\Lambda}^{(i)} = diag(\lambda_1^{(i)}, \dots, \lambda_{n_1}^{(i)})$, $i = 1, 2$. Furthermore, $\boldsymbol{I}_{n_2} \otimes \boldsymbol{\Sigma}^{(1)}$ and $(\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{\Sigma}^{(2)}$ also commute. Define the orthogonal matrix $\boldsymbol{V}_1 = (n_2^{-1/2}\boldsymbol{1}_{n_2} \vdots \boldsymbol{H})$, where $\boldsymbol{H}$ has the size of $n_2 \times (n_2 - 1)$, satisfying both $\boldsymbol{H}'\boldsymbol{1}_{n_2} = \boldsymbol{0}$ and $\boldsymbol{H}'\boldsymbol{H} = \boldsymbol{I}_{n_2-1}$. Then $\boldsymbol{V}_1'\boldsymbol{J}_{n_2}\boldsymbol{V}_1 = diag\{n_2, \boldsymbol{0}_{n_2-1}\}$. Let $\boldsymbol{V} = \boldsymbol{V}_1 \otimes \boldsymbol{V}_2$, then using the property of the Kronecker product $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = \boldsymbol{AC} \otimes \boldsymbol{BD}$, we have

$$\begin{aligned}
\boldsymbol{V}'\boldsymbol{\Sigma}_2\boldsymbol{V} &= (\boldsymbol{V}_1' \otimes \boldsymbol{V}_2')(\boldsymbol{I}_{n_2} \otimes \boldsymbol{\Sigma}^{(1)})(\boldsymbol{V}_1 \otimes \boldsymbol{V}_2) \\
&\quad + (\boldsymbol{V}_1' \otimes \boldsymbol{V}_2')\left[(\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{\Sigma}^{(2)}\right](\boldsymbol{V}_1 \otimes \boldsymbol{V}_2) \\
&= (\boldsymbol{V}_1'\boldsymbol{V}_1) \otimes (\boldsymbol{V}_2'\boldsymbol{\Sigma}^{(1)}\boldsymbol{V}_2) + (\boldsymbol{V}_1'(\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2})\boldsymbol{V}_1) \otimes (\boldsymbol{V}_2'\boldsymbol{\Sigma}^{(2)}\boldsymbol{V}_2) \\
&= \boldsymbol{I}_{n_2} \otimes \boldsymbol{\Lambda}^{(1)} + diag\{n_2 - 1, -\boldsymbol{I}_{n_2-1}\} \otimes \boldsymbol{\Lambda}^{(2)}. \qquad (33)
\end{aligned}$$

The obtained matrix in (33) is a diagonal matrix and the elements in $\boldsymbol{\Lambda}^{(1)}$ and $\boldsymbol{\Lambda}^{(2)}$ are obtained from Lemma 2.1, as well as their multiplicities. We know that there are $\left[\frac{n_1}{2}\right] + 1$ distinct eigenvalues in $\boldsymbol{\Lambda}^{(i)}$, $i = 1, 2$. From the diagonal matrix (33), the number of distinct eigenvalues in $\boldsymbol{\Sigma}_{21}$ is obtained. ■

Now we illustrate the results obtained in Theorem 4.1 on two examples.

**Example 4.1.**
Let $\boldsymbol{\Sigma}_{21} = \boldsymbol{I}_3 \otimes \boldsymbol{\Sigma}^{(1)} + (\boldsymbol{J}_3 - \boldsymbol{I}_3) \otimes \boldsymbol{\Sigma}^{(2)}$, where $\boldsymbol{\Sigma}^{(1)} = \sum_{k_1=0}^{2} \tau_{k_1} SC(4, k_1)$ and $\boldsymbol{\Sigma}^{(2)} = \sum_{k_1=0}^{2} \tau_{k_1+3} SC(4, k_1)$.

The block $\boldsymbol{\Sigma}^{(1)} : 4 \times 4$, is a SC-Toeplitz matrix with three distinct eigenvalues:

$$\begin{aligned}
\lambda_1^{(1)} &= \tau_0 - \tau_2, \\
\lambda_2^{(1)} &= \tau_0 - 2\tau_1 + \tau_2, \\
\lambda_3^{(1)} &= \tau_0 + 2\tau_1 + \tau_2,
\end{aligned}$$

with multiplicities $m_1 = 2$, $m_2 = 1$ and $m_3 = 1$, respectively.

Similarly, the block $\boldsymbol{\Sigma}^{(2)} : 4 \times 4$, is a SC-Toeplitz matrix with three distinct eigenvalues:

$$\begin{aligned}
\lambda_1^{(2)} &= \tau_3 - \tau_5, \\
\lambda_2^{(2)} &= \tau_3 - 2\tau_4 + \tau_5, \\
\lambda_3^{(2)} &= \tau_3 + 2\tau_4 + \tau_5,
\end{aligned}$$

with the same multiplicities $m_h$, $h = 1, \dots, 3$, as in $\boldsymbol{\Sigma}^{(1)}$.

The distinct eigenvalues of $\boldsymbol{\Sigma}_{21} : 12 \times 12$ are:

$$\begin{aligned}
\lambda_1 &= \lambda_1^{(1)} - \lambda_1^{(2)} = \tau_0 - \tau_2 - \tau_3 + \tau_5 \text{ with multiplicity } (3-1)m_1 = 4,\\
\lambda_2 &= \lambda_2^{(1)} - \lambda_2^{(2)} = \tau_0 - 2\tau_1 + \tau_2 - \tau_3 + 2\tau_4 - \tau_5\\
&\quad \text{with multiplicity } (3-1)m_2 = 2,\\
\lambda_3 &= \lambda_3^{(1)} - \lambda_3^{(2)} = \tau_0 + 2\tau_1 + \tau_2 - \tau_3 - 2\tau_4 - \tau_5\\
&\quad \text{with multiplicity } (3-1)m_3 = 2,\\
\lambda_4 &= \lambda_1^{(1)} + (n_2-1)\lambda_1^{(2)} = \tau_0 - \tau_2 + 2(\tau_3 - \tau_5) \text{ with multiplicity } m_1 = 2,\\
\lambda_5 &= \lambda_2^{(1)} + (n_2-1)\lambda_2^{(2)} = \tau_0 - 2\tau_1 + \tau_2 + 2(\tau_3 - 2\tau_4 + \tau_5)\\
&\quad \text{with multiplicity } m_2 = 1,\\
\lambda_6 &= \lambda_3^{(1)} + (n_2-1)\lambda_3^{(2)} = \tau_0 + 2\tau_1 + \tau_2 + 2(\tau_3 + 2\tau_4 + \tau_5)\\
&\quad \text{with multiplicity } m_3 = 1.
\end{aligned}$$

**Example 4.2.**
Let $\boldsymbol{\Sigma}_{21} = \boldsymbol{I}_3 \otimes \boldsymbol{\Sigma}^{(1)} + (\boldsymbol{J}_3 - \boldsymbol{I}_3) \otimes \boldsymbol{\Sigma}^{(2)}$, where $\boldsymbol{\Sigma}^{(1)} = \sum_{k_1=0}^{1} \tau_{k_1} SC(3, k_1)$ and $\boldsymbol{\Sigma}^{(2)} = \sum_{k_1=0}^{1} \tau_{k_1+2} SC(3, k_1)$.

Both blocks $\boldsymbol{\Sigma}^{(1)} : 3 \times 3$ and $\boldsymbol{\Sigma}^{(2)} : 3 \times 3$ are SC-Toeplitz matrices. The distinct eigenvalues are:

$$\begin{aligned}
\lambda_1^{(1)} &= \tau_0 - \tau_1,\ m_1 = 2;\ \lambda_2^{(1)} = \tau_0 + 2\tau_1,\ m_2 = 1,\\
\lambda_1^{(2)} &= \tau_2 - \tau_3,\ m_1 = 2;\ \lambda_2^{(2)} = \tau_2 + 2\tau_3,\ m_2 = 1.
\end{aligned}$$

The distinct eigenvalues of $\boldsymbol{\Sigma}_{21} : 9 \times 9$ are:

$$\begin{aligned}
\lambda_1 &= \lambda_1^{(1)} - \lambda_1^{(2)} = \tau_0 - \tau_1 - \tau_2 + \tau_3 \text{ with multiplicity } 4,\\
\lambda_2 &= \lambda_2^{(1)} - \lambda_2^{(2)} = \tau_0 + 2\tau_1 - \tau_2 - 2\tau_3 \text{ with multiplicity } 2,\\
\lambda_3 &= \lambda_1^{(1)} + (n_2-1)\lambda_1^{(2)} = \tau_0 - \tau_1 + 2(\tau_2 - \tau_3) \text{ with multiplicity } 2,\\
\lambda_4 &= \lambda_2^{(1)} + (n_2-1)\lambda_2^{(2)} = \tau_0 + 2\tau_1 + 2(\tau_2 + 2\tau_3) \text{ with multiplicity } 1.
\end{aligned}$$

**Note.** The spectrum of $\boldsymbol{\Sigma}_{12}$, given in (30), is the same as $\boldsymbol{\Sigma}_{21}$ in (26). Since it also can be found from Theorem 3.7, $\boldsymbol{\Sigma}_{12}$ and $\boldsymbol{\Sigma}_{21}$ are similar matrices, i.e, $\boldsymbol{\Sigma}_{12} = \boldsymbol{K}_{n_1,n_2}\boldsymbol{\Sigma}_{21}\boldsymbol{K}'_{n_1,n_2}$, where $\boldsymbol{K}_{n_1,n_2}$ is an orthogonal matrix. Essentially, it is an orthogonal matrix transformation and will not change the spectrum, i.e., the characteristic equation is given by the following determinant,

$$|\boldsymbol{\Sigma}_{12} - \lambda\boldsymbol{I}| = |\boldsymbol{K}_{n_1,n_2}\boldsymbol{\Sigma}_{21}\boldsymbol{K}'_{n_1,n_2} - \lambda\boldsymbol{I}| = |\boldsymbol{K}_{n_1,n_2}(\boldsymbol{\Sigma}_{21} - \lambda\boldsymbol{I})\boldsymbol{K}'_{n_1,n_2}| = |\boldsymbol{\Sigma}_{21} - \lambda\boldsymbol{I}|.$$

## 5. Concluding remarks

In practice, a symmetry study starts with a data set in which symmetry relations can be identified (Viana, 2008). We have derived the covariance structures under invariance related to two groups of orthogonal transformations (permutations and rotations). In MM, particular patterns of the covariance matrices reflect how the data share common characteristics in different hierarchies. This is important when performing estimation and testing. When we estimate the fixed effects, usually the imposed structure can improve the precision of the fixed effects estimator. Furthermore, there might exist a risk of misspecification of the covariance structure that could result in misleading inference of the fixed effects. Thus, it is also necessary to discuss different hypotheses of the covariance structures to verify the model (Jensen, 1988). In addition, the existence of explicit MLEs for such symmetry models should be studied, for example, Szatrowski (1980) and Ohlson and von Rosen (2010) provided the explicit MLEs of some patterned covariance structures. Our study of the spectral properties can be used to obtain explicit MLEs of a covariance matrix which has block circular symmetric structure and discuss concerning the existence of explicit MLEs.

In this article, we only considered model with two random factors and it could be of interest to study MM cases with more factors. In such cases, the higher order interactions will be involved. For example, when we investigate MM with $s$ random factors, the potential structured data might be possibly identified by considering different groups of symmetry transformations, i.e., when different symmetry patterns are observed in different hierarchies.

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