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Abstract

Equilibrium bid functions in common value auctions are in general complex and not easy to analyze. A handful closed form solutions have been derived, but only for quite unrealistic model assumptions. We derive closed form approximations of the bid function for two empirically important models within second price common value auctions. We treat both the case with a known number of bidders, and the case where bidders enter the auction stochastically. The approximated bid functions are of a very simple and easily interpretable form. Moreover, the approximate bid functions can be evaluated directly without time consuming numerical integration. This is crucial for speeding up likelihood/Bayesian estimations on auction data. Several interesting features are discerned from the bid approximation, e.g. the winner's curse effect is quantified analytically and explicit bidding strategies, as a weighted function between a bidder's private information and public information, are identified.

Keywords: closed form solution, equilibrium bidding strategies, bid approximation, normal valuations, winner's curse, likelihood estimation

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1 Introduction

The theory of auctions has developed extensively since Vickrey's (1961) seminal paper, particularly over the last decades, see e.g. Wolfstetter (1996), Klemperer (1999, 2004), and Milgrom (2004) for recent surveys and a general introduction. Most of the existing literature on auction theory analyzes either the private or the common value model[1]. Private value models are desirable in auctions with non-durable goods, where every bidder knows the value to himself and knowledge of other bidders' valuation would not affect his valuation. In a common value auction, the value of the object, V, is unknown but the same for all bidders. Each bidder uses his private information (the signal) of the object's value to estimate V. Common value auctions occur in markets where the market price is unknown at the time of bidding. For example, the sale of oil rights, timber tracts, and markets for Treasury bills.

In an influential paper, Milgrom and Weber (1982, Theorem 6) derive the equilibrium bid function for a second price common value auction. In practice it is difficult, however, to specify distributional assumptions that yield closed form solutions of the bid function. The lack of closed form solutions has two major drawbacks. First, it is hard to see how the bid function depends on various distributional components of the model, which makes it more difficult to bring out model characteristics. Second, to evaluate the bid function one has to make use of numerical integration which is very time demanding. This is a crucial step for econometric analysis of auction data (e.g. likelihood/Bayesian estimation) where the equilibrium bid function has to be evaluated over and over again (Bajari and Hortacsu, 2003). Some closed form solutions exists, though, for specific distributional assumptions.

Kagel and Levin (1986) obtain closed form solutions for values and signals following uniform distributions. Matthews (1984) find a closed form solution for signals as the highest order statistic of θ independent draws from a uniform distribution, and values drawn from a Pareto distribution. Other specifications are even more restrictive. Thiel (1988) imposes three restrictions which guarantee the existence of linear strategies. Engelbrecht-Wiggans and Weber (1979) and Wilson (1988) note their existence when the prior value distribution is assumed to be diffuse (restriction 1), Rothkopf (1980) and Winkler and Brooks (1980) derive linear bidding strategies when estimation errors are assumed to be independent of the object's true value (restriction 2), and Levin and Smith (1991) find closed form solutions under all three restrictions in a comment on Thiel (1988).

However, these restrictions rule out most realistic models of empirical interest. Models with a diffuse prior, for example, are not that realistic

[1] Almost every auction includes both a private and a common value element, but models with a mixture of both elements are often too complex to analyze. As such, these models are rare in the literature.

since there are no bounds on the object's public value, which is a natural ingredient in common value auction models. To decide between the private or the common value paradigm for a certain auction, Paarsch (1992) develop an empirical framework by using two models of bidding. One of the models is build upon the results in Levin and Smith (1991), and the other is a model by Smiley (1979) with restrictions on the signal and value distributions which yield bid functions that are proportional to the signal.

More recently, Gordy (1997) introduces two more realistic distributional assumptions and derives nearly closed form solutions of the bid function. A problem with his setup is that the inverse of the equilibrium bid function, an integral part of the likelihood, can not be calculated explicitly. Instead, the need of root finding algorithms implies time consuming evaluations of the inverse bid function and thus slow down likelihood estimation. Probably the most important model, at least from an empirical point of view, is the hierarchical normal model in Bajari and Hortacsu (2003). They assume normal priors for the unknown public value, as a part of an hierarchical normal valuation structure, to estimate an eBay auction model.

In this paper, we show how convenient closed form solutions can be obtained by approximating the equilibrium bid function for two realistic distributional assumptions. First, a linear bid approximation is derived for the Normal-Normal model, defined in Bajari and Hortacsu (2003), and then a non-linear approximation is obtained for the Gamma-Gamma model, as defined by Gordy (1997). The accuracy of both approximations is quite good, especially for the normal model, and yield straightforward and fast explicit solutions of the equilibrium bid functions that can be inverted analytically. Furthermore, we also derive a closed form approximation for the normal model with a stochastic number of bidders.

Section 2 presents the general equilibrium bid function for a second price common value auction together with the distributional assumptions of the normal and gamma model. In section 3 we derive a linear approximation for the normal model, document approximation accuracy, and present several interesting features that can be discerned from the approximated bid function in both an analytical and a graphical way. Using a similar technique in section 4, the approximation of the gamma case is derived and evaluated. Finally, section 5 concludes.

2 The Models

Following Milgrom and Weber (1982) we consider a second price common value auction in which risk-neutral bidders follow the same strategy and compete for a single object. The value of the object, V, is unknown and the same for each bidder, but a prior distribution for V is shared by the bidders. To estimate V, each bidder receives a private signal X drawn independently from the same distribution of $X|V$. We will consider two cases. First, the case with a known number of bidders, and further on a model with a stochastic number of bidders.

Since the auction involves symmetric bidders and a symmetric equilibrium we can focus on bidder i without loss of generality[2]. Let X_i be the signal for bidder i, and let Y_i be the highest signal among the other bidders' signals $X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$. The equilibrium bid is then given by (Milgrom and Weber, 1982, Theorem 6)

$$b(x) = v(x, x) = E[V|X_i = x, Y_i = x].$$

In words, bidder i submits a bid equal to the expected value of V given his own signal and the signal of his worst competitor, both equal to x. Let $f_V(v)$ denote the probability density function of V, $f_{X_i|V}(x|v)$ the conditional probability density function of $X_i|V$, and $F_{X_i|V}(x|v)$ the conditional cumulative distribution function of $X_i|V$. Using Bayes' theorem repeatedly we can rewrite the bid function easily as[3]

$$b(x) = \frac{\int_{-\infty}^{\infty} v \cdot f^2_{x_i|v}(x|v) \cdot F^{n-2}_{x_i|v}(x|v) \cdot f_v(v)\ dv}{\int_{-\infty}^{\infty} f^2_{x_i|v}(x|v) \cdot F^{n-2}_{x_i|v}(x|v) \cdot f_v(v) dv}. \tag{1}$$

The announcement of a *public reserve price (minimum bid)* is commonly used in second price auctions[4]. Therefore, assume that the seller sets a minimum bid $r \geq 0$. Then, the equilibrium bid function becomes

$$b(x) = \begin{cases} \frac{\int_{-\infty}^{\infty} v \cdot f^2_{x_i|v}(x|v) \cdot F^{n-2}_{x_i|v}(x|v) \cdot f_v(v)\ dv}{\int_{-\infty}^{\infty} f^2_{x_i|v}(x|v) \cdot F^{n-2}_{x_i|v}(x|v) \cdot f_v(v) dv}, \ if \quad x \geq x^\star \\ \\ 0, \ if \quad x < x^\star \ or \ if \ b(x) < 0, \end{cases} \tag{2}$$

where $x^\star$ is the cutoff signal, above which bidders participate in the auction with a positive bid. The cutoff signal is given in implicit form as (Milgrom and Weber, 1982)

$$x^\star(r) = \inf_x \ E[v|X_i = x, Y_i < x] \geq r,$$

and by using Bayes' theorem repeatedly as before gives

$$r(x^\star) = \frac{\int_{-\infty}^{\infty} v \cdot f_{x_i|v}(x^\star|v) \cdot F^{n-1}_{x_i|v}(x^\star|v) \cdot f_v(v)\ dv}{\int_{-\infty}^{\infty} f_{x_i|v}(x^\star|v) \cdot F^{n-1}_{x_i|v}(x^\star|v) \cdot f_v(v)\ dv}. \tag{3}$$

[2]Explanations of symmetric bidders/equilibrium are given by Krishna (2002).

[3]Consult Appendix A for details of derivation.

[4]A secret reserve price can also be set by the seller. Bajari and Hortacsu (2003) assume that the seller sets the secret reserve price by using the same bid function as the bidders, and therefore treat the seller as just another bidder.

Hence, the minimum bid is an explicit function of $x^\star$.

We now turn to the case with a stochastic number of bidders. Bajari and Hortacsu (2003) model an eBay auction as a second price auction where entry into the auction is stochastic. Specifically, N potential bidders view a particular listing for a common value object on eBay and participate only if they can bear a bid-preparation cost. The probability of bearing this cost is assumed to be identical for each bidder which form the stochastic feature of the auction. As in Levin and Smith (1994) and Bajari and Hortacsu (2003), assume that the unconditional distribution of bidders within an auction is binomial with the same probability for each bidder entering the auction. Thus, the model under consideration focus on the symmetric equilibrium of the endogenous-entry game. On eBay, however, the number of potential bidders is expected to be large compared to the actual bidder participation. As a consequence, we therefore use the Poisson approximation to the Binomial distribution.

Using the derivation of the equilibrium bid in Bajari and Hortacsu (2003) and rewriting by using Bayes' theorem repeatedly, gives the bid function for a stochastic number of bidders, with a minimum bid r, as

$$b(x,\lambda) = \begin{cases} \frac{\sum_{n=2}^{\infty}(n-1)\cdot p_n^i(\lambda)\cdot\int_{-\infty}^{\infty} v\cdot f_{x_i|v}^2(x|v)\cdot F_{x_i|v}^{n-2}(x|v)\cdot f_v(v)\ dv}{\sum_{n=2}^{\infty}(n-1)\cdot p_n^i(\lambda)\cdot\int_{-\infty}^{\infty} f_{x_i|v}^2(x|v)\cdot F_{x_i|v}^{n-2}(x|v)\cdot f_v(v)\ dv},\ if\ x \geq x^\star \\ \\ 0,\ if\ x < x^\star\ or\ if\ b(x,\lambda) < 0, \end{cases} \tag{4}$$

where $p_n^i(\lambda)$ is the poisson probability of $(n-1)$ bidders in the auction with λ as the expected value of the Poisson entry process[5]. The cutoff signal is now given in implicit form as (Milgrom and Weber, 1982)

$$x^\star(r,\lambda) = \inf_x\ E_n E[v|X_i = x, Y_i < x, n] \geq r,$$

which gives

$$r(x^\star,\lambda) = \sum_{n=1}^{\infty} p_n(\lambda)\cdot\frac{\int_{-\infty}^{\infty} v\cdot f_{x_i|v}(x^\star|v)\cdot F_{x_i|v}^{n-1}(x^\star|v)\cdot f_v(v)\ dv}{\int_{-\infty}^{\infty} f_{x_i|v}(x^\star|v)\cdot F_{x_i|v}^{n-1}(x^\star|v)\cdot f_v(v)\ dv}, \tag{5}$$

where $p_n(\lambda)$ is the poisson probability of n bidders in the auction with expected value λ. Hence, the minimum bid $r(x^\star,\lambda)$ is an explicit function of $x^\star$, and λ.

Following Bajari and Hortacsu (2003), we assume an hierarchical normal model for valuations as[6]

$$X_i|V \sim N(v,\kappa\sigma^2)$$

[5]The poisson probability notation, $p_n^i(\lambda)$, is used here as in Bajari and Hortacsu (2003), since the derivation of the equilibrium bid function is conditional on bidder i's presence.

[6]Negative valuations due to normal distributions are not unreasonable. First, in many auctions there exists administrative costs for a winning bidder. Second, unreasonable high negative signals (in absolute terms) are rare since the variance compared to the expected value can be assumed to be small, see for example estimation results in Bajari and Hortacsu (2003). If a signal gives a negative bid $b(x) < 0$, however, let $b(x) = 0$.

$$V \sim N(\mu, \sigma^2),$$

where κ scales the variance of the signal distribution relative to the variance of the common value V.

In order to account for other distributional settings, we also derive results for the Gamma-Gamma model in Gordy (1997). Following Gordy (1997) we define the prior distribution of V as a Gamma (α, β) distribution, where $\alpha > 0, \beta > 0$, and

$$f_V(v) = \frac{\beta^\alpha}{\Gamma(\alpha)} v^{\alpha-1} e^{-\beta v}.$$

To get the integrand of the bid function on a tractable form it is convenient to solve the model in terms of inverse signals $S = \frac{1}{X}$. Let S be conditionally independent given $V = v$ and identically Gamma $(\tau, \tau v)$ distributed, $\tau > 0, v > 0$, with probability density function

$$g_{S|V} = \frac{(\tau v)^\tau}{\Gamma(\tau)} s^{\tau-1} e^{-\tau v \cdot s}$$

and distribution function $G_{S|V}$.

3 Approximation of the equilibrium bid function for the Normal-Normal model

The derived equilibrium bid functions are quite complex and not easy to analyze. It is hard to see how the bid function depends on x for different distributional assumptions. In an influential paper, Bajari and Hortacsu (2003) perform Bayesian estimation of a second price common value auction with a stochastic number of bidders. For each bid b in every auction they find the corresponding signal x by numerical evaluations of integrals. That can be very time demanding, but by exploiting a linear property across auctions they reduce the computational complexity significantly. However, Bajari and Hortacsu (2003) argue that the convenient linear property also holds for cutoff signals $x^\star$, but it can be easily verified numerically that this is not true. As a consequence, one has to make use of a numerical routine like Newton-Rhapson to compute each $x^\star$ for each minimum bid, which is very time demanding.

Instead, we obtain convenient linear approximations for both the bid and the minimum bid functions of signals x and cutoff signals $x^\star$, respectively. The approximation is easily understood and includes several interesting features, for example: every signal (cutoff signal) connects to a bid (minimum bid) directly without numerical evaluations, quantifying the winner's curse is straightforward, and the bid function as a weighted function of the signal x and the expected value μ is identified.

3.1 Known number of bidders n

The derivation of the linear approximation can roughly be divided into the following steps[7]:

Step 1 : Substitute $t = \frac{x-v}{\sqrt{\kappa}\sigma}$ to get standard normal distributed signals. Then, the bid function becomes

$$b(x) = x - \sqrt{\kappa}\sigma \frac{\int_{-\infty}^{\infty} te^{-t^2}\Phi^{n-2}(t)e^{-\frac{1}{2\sigma^2}(x-\sqrt{\kappa}\sigma t-\mu)^2}\, dt}{\int_{-\infty}^{\infty} e^{-t^2}\Phi^{n-2}(t)e^{-\frac{1}{2\sigma^2}(x-\sqrt{\kappa}\sigma t-\mu)^2}\, dt},$$

where $\Phi(\cdot)$ is the standard normal $c.d.f.$

Step 2 : Approximate $\Phi(t)$ with the kernel of a normal probability density function over the interval $[-a, a]$ for $a = 2$.

Step 3 : Complete the squares in the exponents of the exponential functions to rewrite the integrands to one normal density. In the numerator we get the expected value of a normal density and in the denominator this normal density integrates to 1. Constants cancel out.

From this it follows that the bid approximation can be written as

$$b(x) \approx c + \omega\mu + (1-\omega)x, \tag{6}$$

where $c = -\frac{\sqrt{\kappa}\sigma\hat{\gamma}\hat{\theta}(n-2)}{\hat{\gamma}(n-2)+1+\frac{\kappa}{2}}$, and $\omega = \frac{\frac{\kappa}{2}}{\hat{\gamma}(n-2)+1+\frac{\kappa}{2}}$. Hence, the linear approximation is a weighted function of the signal x and the expected value μ. If $n = 2$, then $\Phi^{n-2}(t) = 1$, and the bid function in equation (1) can be computed exactly, in the same way as in step 3 above. The bid function then becomes

$$b(x) = \frac{\frac{\kappa}{2}}{1+\frac{\kappa}{2}}\mu \ + \ \frac{1}{1+\frac{\kappa}{2}}x,$$

which is exactly what the approximation also gives in this case.

Note, by using the same approximation technique as above, the minimum bid function in equation (3) can be approximated as

$$r(x^\star) \approx c_r + \omega_r\mu + (1-\omega_r)x, \tag{7}$$

where $c_r = -\frac{\sqrt{\kappa}\sigma\hat{\gamma}\hat{\theta}(n-1)}{\hat{\gamma}(n-1)+\frac{1}{2}+\frac{\kappa}{2}}$, and $\omega_r = \frac{\frac{\kappa}{2}}{\hat{\gamma}(n-1)+\frac{1}{2}+\frac{\kappa}{2}}$. In this case, the minimum bid function can be computed exactly for $n = 1$, which again gives the same result as the approximation.

Some interesting and valuable features should be noted from the approximation of the equilibrium bid function. A higher variance of the common value V implies a higher risk of drawing a large signal value and thus a higher risk of overestimating the true value of the object, why bidders should lower their bids. The constant term c of the approximated bid function captures

[7]See Appendix B for detailed explanations in each step.

this effect well, increasing the value of σ leads to lower bids. Less precision in signals, as κ increases, gives a bidder incentives to place more weight on public information, captured by parameter μ, and less weight on his own private signal x. In fact, the weight ω of the approximated bid function increases monotonically towards 1 as κ increases, and

$$b(x) \longrightarrow x \; if \; \kappa \longrightarrow 0, \quad \text{and} \quad b(x) \longrightarrow \mu \; if \; \kappa \longrightarrow \infty.$$

Intuitively, one could expect this result. The higher precision in signals the more the bidders trust their private information.

3.2 Stochastic number of bidders

The bid function for a stochastic number of bidders with a minimum bid r consists of $b(x, \lambda)$ in equation (4), and the corresponding minimum bid function $r(x^\star, \lambda)$ in equation (5). As for the case with a known number of bidders, the minimum bid function can be approximated and written as

$$r(x^\star, \lambda) \approx \sum_{n=2}^{\infty} p_n^i(\lambda) \left(c_r + \omega_r \mu + (1 - \omega_r) x^\star\right).$$

The same way to approximate the bid function, $b(x, \lambda)$, does not hold unfortunately. We could do the approximation term by term, but constants will not cancel out in this case since they are parts of the summation over n, and, more importantly, inverting the bid function is not possible. One obvious way to proceed, however, is to extract the information in λ, the mean of the Poisson process, by letting $n = \lambda$ and approximate the bid function for a stochastic number of bidders with the linear approximation in equation (6). This simple approximation turns out to be surprisingly good, see section 3.3.

3.3 Accuracy of approximations

The accuracy of the approximated bid function for a known number of bidders depends only on how well the standard normal distribution function is approximated. As we can see in Figure 1, the approximation with $a = 2$ is quite good within the approximation interval and seems to be the most suitable value for a by considering the whole graph. However, choosing the value of a is somewhat arbitrary. Other values than $a = 2$ could also work. For example, the shorter interval with $a = 1$ gives better approximations on $[-1, 1]$, but due to worse approximation outside this interval we prefer the approximation with $a = 2$.

The accuracy of the bid approximation is a function of the bidder's signal x, the parameters (μ, σ, κ), the number of bidders n (λ for the stochastic case), and the minimum bid r if the seller has set a public reserve price. In auctions with a minimum bid r, the bid functions are equal to zero whenever the signal x is below the cutoff signal $x^\star$, and identical to the bid functions without a minimum bid for $x \geq x^\star$. This fact is illustrated in Figures 2 to 9, where one also can note that the worst bid approximations occur for low values of x that gives a positive bid. Thus, the approximations of the bid functions with a minimum bid are remarkable better than the approximations without a minimum bid.

In Figures 2 to 3 for a known number of bidders, the approximation works very well for both a small and a large number of bidders[8], especially for $\kappa = 0.25$. The somewhat crude approximation for a stochastic number of bidders, by letting $n = \lambda$, works surprisingly good. In Figure 4 for $\kappa = 0.25$ there are only small differences between the bid functions, whereas there are some notable differences for a large number of expected bidders in Figure 5. However, these differences are especially for unusual signals, located almost 2 standard deviations from the expected value. By considering a very large number of bidders in Figures 6 to 9, the approximations still works very well. An increment in the number of bidders implies a shift of the bid function down to the right, and as such only gives worse bid function approximations for negative bids, which we never observe.

3.4 Illustrations of the winner's curse effect

The winner's curse is by far the most highlighted phenomena in common value auctions where bidders face effects from both information and competition perspectives[9]. More bidders leads to more competition which gives a bidder incentives to submit a higher bid (*competition effect*). However, a bidder must also account for the risk of overestimating the value if he wins, since his signal is then the highest signal among all bidders. As such, a bidder should also lower his bid when facing more bidders (*overestimation effect*). To illustrate how the approximated bid function captures the winner's curse effect as a mixture between the competition and the overestimation effect, we split the bid approximation in equation (6) into two parts as

$$(1-\omega)x = \frac{\hat{\gamma}(n-2)+1}{\hat{\gamma}(n-2)+1+\frac{\kappa}{2}} \cdot x \quad (I),$$

and

$$c + \omega\mu = \frac{\frac{\kappa\mu}{2} - \sqrt{\kappa}\sigma\hat{\gamma}\hat{\theta}(n-2)}{\hat{\gamma}(n-2)+1+\frac{\kappa}{2}} \quad (II).$$

Taking first-order derivatives in respect to n of parts (I, II) gives (I'_n, II'_n) as

$$\frac{\hat{\gamma}\kappa}{2\left[\hat{\gamma}(n-2)+1+\frac{\kappa}{2}\right]^2} x \ > \ 0 \ for \ x > 0 \quad (I'_n)$$

[8] Kagel and Levin (1986) mention that $3-4$ bidders can be considered as a small number of bidders, whereas a large number of bidders typically involves $7-8$ bidders.

[9] See Thaler (1988) for a careful discussion.

$$-\frac{\sqrt{\kappa}\sigma\hat{\gamma}\hat{\theta}(2+\kappa)+\hat{\gamma}\kappa\mu}{2\left[\hat{\gamma}(n-2)+1+\frac{\kappa}{2}\right]^2} < 0 \quad (II'_n).$$

Hence, increasing the number of bidders, n, increases bids for positive signals x in part (I), reflecting the economic effects of competition, and decreases bids for all signals in part (II), reflecting the overestimation effect of drawing the highest signal among bidders. In equilibrium, the effect of correcting for the winner's curse decreases bids (Krishna, 2002), see Figures 10 and 12 for an illustration, which implies that the overestimation effect decreases bids more than the competition effect increases bids in equilibrium. However, the effect of the winner's curse correction, for the bid function approximation in equation (6), does not always decrease bids, see Figures 11 and 13 for an illustration. This is not a big problem, though, since it only occurs for fairly high and unreasonable values of the variance scale parameter κ and signals x, see Appendix D for an analytical result.

3.5 Bidder's expected profit and expected seller revenue

Following Gordy (1997) one might expect that more competition, as the number of bidders n increases, drives expected seller revenue upwards. In addition, by intuition from the mechanism-design literature, one might also expect a bidder's expected profit to increase with the magnitude of his signal. Nevertheless, counter-examples are often available. For example, at low values of n, Matthews (1984) shows, in an example where signals follow a Pareto distribution, that seller revenue goes down by increasing the number of bidders n.

Gordy (1997) shows comparative statics for bidder profits and simulate expected seller revenue for the Gamma-Gamma model in Section 2, which resulted in no evidence of pathological behaviour. We now perform a similar analysis for the Normal-Normal model. Comparative statics for a bidder's expected profit and expected seller revenue are illustrated by using both the exact and the approximated bid function for a known number of bidders. To simulate expected seller revenue we utilize the same Monte Carlo techniques as in Gordy (1997).

In a second price common value auction, a bidder's expected profit for a given signal x is given by

$$\Pi(x) = \int_{-\infty}^{x} (v(x,y) - b(y))\, f_{Y_i|X_i}(y|x)\ dy. \tag{8}$$

This integral can be solved by using Gaussian quadrature methods. However, by using the bid approximation technique in section 3.1, we can derive an explicit approximative solution that yield a much faster computation of the bidder's expected profit. The approximation of the expected profit can be simplified to

$$\Pi(x) \approx \frac{1}{2\left(1+\hat{\gamma}(n-2)+\frac{\kappa}{2}\right)} \int_{-\infty}^{x} (x-y) f_{Y_i|X_i}(y|x)\ dy,$$

where $f_{Y_i|X_i}(y|x)$ is given by

$$f_{Y_i|X_i}(y|x) = (n-1)\frac{\int_{-\infty}^{\infty} f_{x_i|v}(x|v) f_{x_i|v}(y|v) F_{x_i|v}^{n-2}(y|v) f_v(v)\ dv}{\int_{-\infty}^{\infty} f_{x_i|v}(x|v) f_v(v)\ dv}.$$

Further simplifications requires some tedious algebra which results in a very messy and non-intuitive expression. Therefore, the interested reader is advised to consult Appendix E for a complete explicit approximative solution.

As we can see in Figures 22 and 23, the bidder's expected profit increases with signals x, and decreases with more competition as n increases. However, as we can see in Figure 23, the bidder's expected profits do not monotonically increase with higher precision in signals as κ decreases. By intuition this is an expected result. Gordy (1997) suggests that higher precision in signals only increase $\Pi(x)$ to a certain point and will eventually after this point decrease $\Pi(x)$ when signal precision becomes too high. In the limit, as $\kappa \longrightarrow 0$, signals become perfectly precise. Thus, the true unknown value of the object becomes common knowledge and the bidders face Bertrand competition, which results in zero expected profits. To estimate expected seller revenue we found that 100000 auctions were good enough for convergence. In Figures 24 and 25 we see that expected seller revenues increases with n and $E(V)$ as expected. Overall, we find no evidence of pathological behaviour whatsoever.

4 Approximation of the equilibrium bid function for the Gamma-Gamma model

In order to account for other distributional settings, we also show how the equilibrium bid function can be approximated for the Gamma-Gamma model [10]. Gordy (1997) obtains a nearly closed form solution for the Gamma-Gamma case (see $B_2(x)$ formula (7)), and mention that computations goes quite simple and fast. However, it is still time demanding since the inverse bid function has to be solved numerically for each bid, and it is still not easy to see how the bid function depends on signals x for different distributional asssumptions.

The bid function of signal X becomes

$$b(x) = \frac{\int_{-\infty}^{\infty} v \cdot (1-G_{S|V}(1/x|v))^{n-2} \cdot g_{S|V}^2(1/x|v) \cdot f_v(v)\ dv}{\int_{-\infty}^{\infty} (1-G_{S|V}(1/x|v))^{n-2} \cdot g_{S|V}^2(1/x|v) \cdot f_v(v)\ dv}. \tag{9}$$

[10] Similar illustrations as in Section 3.4 and 3.5 can be performed for the Gamma case too, but we do not give it here since it follows the same approach as in previous section.

The approximation goes over $(1-G_{S|V}(1/x|v))$ by using one unique Gamma probability density function with parameters $(\hat{\alpha}_\tau, \hat{\beta}_\tau)$ as the approximating function, see Appendix C for a complete discussion and derivation. Replacing $(1 - G_{S|V}(1/x|v))$ with $Gamma(\hat{\alpha}_\tau, \hat{\beta}_\tau)$ and simplifying, the approximated bid function can be written as

$$b(x) \approx \frac{[\alpha + 2\tau + (n-2)(\hat{\alpha}_\tau - 1)] \cdot x}{\beta x + 2\tau + (n-2)\hat{\beta}_\tau}. \tag{10}$$

The bid function in equation (9) can be computed exactly for $n = 2$, as for the normal case, and for $\tau = 1$, which yield the same results as what the approximation gives. The bid function for $n = 2$ is given by

$$b(x) = \frac{(\alpha + 2\tau) \cdot x}{\beta x + 2\tau}.$$

If $\tau = 1$, the conditional distribution of $S|V$ follows the exponential distribution with parameter v. This gives $(1-G_{S|V}(1/x|v)) = e^{-\frac{v}{x}}$, which is exactly the same result as the Gamma $p.d.f.$ with parameters $(\hat{\alpha}_\tau, \hat{\beta}_\tau) = (1, 1)$ used for approximation. The bid function is now equal to

$$b(x) = \frac{(\alpha + 2)x}{\beta x + n}.$$

4.1 Accuracy of approximations

It is not that informative to evaluate the approximation of $(1-G_{S|V}(1/x|v))$ for some values of τ. Instead, the approximated bid function, in equation (10), is compared to the exact bid function, in equation (9), for various sets of parameter values in Figures 14 to 21.

Considering Figures 14 and 15 for a known number of bidders, the approximation works pretty well when the number of bidders is small, regardless the other parameter values. Increasing the expected value μ tend to give worse approximations in absolute terms, but by comparing Figure 14 up left with Figure 14 up right and Figure 15 up left with Figure 15 up right, there seems to be no differences of how well the approximation performs in relative terms. The approximated bids, for signals equal to μ, are about 5 per cent higher than the exact bids in all these figures which is fairly low. Higher values of τ does not seem to change the accuracy of approximations for $n = 4$, but for $n = 8$ there is a small impairment in the approximation. In general, there are notably worse approximations for $n = 8$. Approximated bids are about 10 per cent higher than the exact bid function for signals equal to μ.

The crude approximation for the stochastic case works surprisingly good here too, as for the Normal-Normal model in the previous section. Figures 16 and 17 indicate that the approximations for the stochastic cases are almost as good as the approximations for a known number of bidders, and the bid functions change in a similar way for different sets of parameter values. In Figures 18 to 21 we allow for a very large number of bidders, where one get further indications that the bid approximations get worse as the number of bidders increases. Approximated bids are now about 20 per cent higher than the exact bid function for signals equal to μ, and $n = 16$.

5 Conclusions

In this paper, we derive approximative closed form solutions of the equilibrium bid function for two realistic models of empirical interest in second price common value auctions. The approximations bring out several interesting features which we divide into three major parts.

First, it is straightforward to measure how much the bid function depends on the signal for various distributional components. Second, we are able to compute the inverse of the equilibrium bid function (the signal) directly without time consuming numerical integration. This is a crucial step for Bayesian/likelihood estimation of auction data, where the inverse bid function has to be evaluated over and over again. Third, the magnitude of the winner's curse and the expected bidder profits can be quantified analytically.

We investigate the accuracy of the approximations and conclude that the approximation of the bid function in the normal model is highly accurate for all parameter values and number of bidders. The approximation for the gamma model is in general less accurate than the normal case, but it performs satisfactory unless the number of bidders is too large. A possible improvement of the approximation in the gamma case, especially when n is large, could be to approximate $(1-G_{S|V}(1/x|v))^{n-2}$ in equation (9) directly, rather than approximating $(1-G)$ and then taking the power as we have done here. The drawback is, however, that we would have a new approximation for every n, but polynomial interpolation could be used to handle this.

Finally, possible extensions could be to derive closed form bid approximations in auctions with both a private and a common value element of the object, multiunit objects, or auctions with risk-averse bidders.

Appendix A: Derivation of the equilibrium bid function by using Bayes' Theorem

The equilibrium bid function is given by

$$b(x) = v(x,x) = E[v|X_i = x, Y_i = y] = \int_{-\infty}^{\infty} v \cdot f_{v|x_i,y_i}(X_i = x, Y_i = y)dv.$$

Rewriting the density function in the integrand gives

$$f_{v|x_i,y_i}(X_i = x, Y_i = y) = \frac{f_{v,x_i,y_i}(v,x,x)}{f_{x_i,y_i}(x,x)} = \frac{f_{y_i|v,x_i}(x|v,x) \cdot f_{v,x_i}(v,x)}{\int_{-\infty}^{\infty} f_{x_i,y_i,v}(x,x,v)dv}$$

$$= \frac{f_{y_i|v,x_i}(x|v,x) \cdot f_{v,x_i}(v,x)}{\int_{-\infty}^{\infty} f_{y_i|v,x_i}(x|v,x) \cdot f_{v,x_i}(v,x)dv} = \frac{f_{y_i|v,x_i}(x|v,x) \cdot f_{x_i|v}(x|v)f_v(v)}{\int_{-\infty}^{\infty} f_{y_i|v,x_i}(x|v,x) \cdot f_{x_i|v}(x|v)f_v(v)dv}$$

$$= \Big[f_{y_i|v,x_i}(x|v,x) \text{ depends only on v}\Big] = \frac{f_{y_i|v}(x|v) \cdot f_{x_i|v}(x|v)f_v(v)}{\int_{-\infty}^{\infty} f_{y_i|v}(x|v) \cdot f_{x_i|v}(x|v)f_v(v)dv}.$$

Thus, $b(x)$ can be written as

$$b(x) = v(x,x) = \int_{-\infty}^{\infty} v \cdot \frac{f_{y_i|v}(x|v) \cdot f_{x_i|v}(x|v)f_v(v)}{\int_{-\infty}^{\infty} f_{y_i|v}(x|v) \cdot f_{x_i|v}(x|v)f_v(v)dv}\ dv$$

$$= \frac{\int_{-\infty}^{\infty} v \cdot f_{y_i|v}(x|v) \cdot f_{x_i|v}(x|v)f_v(v)\ dv}{\int_{-\infty}^{\infty} f_{y_i|v}(x|v) \cdot f_{x_i|v}(x|v)f_v(v)dv}$$

$$= \frac{\int_{-\infty}^{\infty} v \cdot (n-1) \cdot f_{x_i|v}(x|v) \cdot F_{x_i|v}^{n-2}(x|v) \cdot f_{x_i|v}(x|v)f_v(v)\ dv}{\int_{-\infty}^{\infty} (n-1) \cdot f_{x_i|v}(x|v) \cdot F_{x_i|v}^{n-2}(x|v) \cdot f_{x_i|v}(x|v)f_v(v)dv}$$

$$= \frac{\int_{-\infty}^{\infty} v \cdot f_{x_i|v}^2(x|v) \cdot F_{x_i|v}^{n-2}(x|v) \cdot f_v(v)\ dv}{\int_{-\infty}^{\infty} \cdot f_{x_i|v}^2(x|v) \cdot F_{x_i|v}^{n-2}(x|v) \cdot f_v(v)dv},$$

since the highest order statistic of $(n-1)$ competing signals, where competitors' (bidders') signals are i.i.d. with p.d.f. $f_{x_i|v}(x_i|v)$, has p.d.f. $f_{y_i|v}(x|v) = (n-1) \cdot f_{x_i|v}(x|v) \cdot F_{x_i|v}^{n-2}(x|v)$.

Appendix B: The linear approximation of the equilibrium bid function for the normal case

The derivation of the linear approximation is divided into four steps. Each step is here presentated carefully.

Step 1

Substitution gives the bid function on standard normal form as

$$b(x) = \frac{\int_{-\infty}^{\infty} v \cdot f_{x_i|v}^2(x|v) \cdot F_{x_i|v}^{n-2}(x|v) \cdot f_v(v)\ dv}{\int_{-\infty}^{\infty} f_{x_i|v}^2(x|v) \cdot F_{x_i|v}^{n-2}(x|v) \cdot f_v(v)\ dv} = \left[t = \frac{x-v}{\sqrt{\kappa}\sigma} \right]$$

$$= x - \sqrt{\kappa} \cdot \sigma \cdot \frac{\int_{-\infty}^{\infty} t \cdot e^{-t^2} \cdot F_t^{n-2}(t) \cdot e^{-\frac{1}{2\sigma^2}\cdot(x-\sqrt{\kappa}\sigma t-\mu)^2}\ dt}{\int_{-\infty}^{\infty} e^{-t^2} \cdot F_t^{n-2}(t) \cdot e^{-\frac{1}{2\sigma^2}\cdot(x-\sqrt{\kappa}\sigma t-\mu)^2}\ dt},$$

where $t \sim N(0,1)$.

Step 2

Let $h_t(t|\gamma,\theta) = e^{-\gamma\cdot(t-\theta)^2}$ be the approximating function to the standard normal distribution function $\Phi_t(t)$ on $[-a,a]$. The function $h_t(t|\gamma,\theta)$ can be considered as the "kernel" of a normal *p.d.f.* for t and is all we need for the approximation. Multiplicative constants, not depending on t, will cancel out in the numerator and the denumerator. Using a numerical routine[11] the best fitted pair of values $(\hat{\gamma},\hat{\theta})$ is obtained on $[-a,a]$ to approximate $\Phi_t(t)$ by a specified minimized function. To make things simple, we define a basic suitable minimized function, M_d, as

$$M_d(\hat{\gamma},\hat{\theta}) = \min_{\gamma,\theta}\left(\max_t |h_t(t|\gamma,\theta) - \Phi_t(t)|\right),$$

where $(\hat{\gamma},\hat{\theta})$ is the minimizing pair of the function.

As a result of this estimation procedure, we obtained the best fitted pair of values as

$$(\hat{\gamma},\hat{\theta}) = (0.1937, 1.9600).$$

The approximation works well, even if the normal *p.d.f.* is a bell-shaped density function, compared to the strict increasing *c.d.f.* Figures in Section 3.3 show that an acceptable approximation within $[-a,a]$, for $a = 2.2$, is good enough. Poor approximations for high values above a seems to give no considerable effect on the bid approximation.

[11] For example MatLab's build-in function "'fminsearch.m"'

Step 3

Replacing $\Phi_t(t)$ by $h_t(t|\hat{\gamma},\hat{\theta})$, the approximated bid function becomes

$$b(x) \approx x - \sqrt{\kappa}\cdot\sigma\cdot\frac{\int_{-\infty}^{\infty} t\cdot e^{-t^2}\cdot e^{-(n-2)\hat{\gamma}(t-\hat{\theta})^2}\cdot e^{-\frac{1}{2\sigma^2}\cdot(x-\sqrt{\kappa}\sigma t-\mu)^2}\ dt}{\int_{-\infty}^{\infty} e^{-t^2}\cdot e^{-(n-2)\hat{\gamma}(t-\hat{\theta})^2}\cdot e^{-\frac{1}{2\sigma^2}\cdot(x-\sqrt{\kappa}\sigma t-\mu)^2}\ dt}.$$

Expanding the exponent of the exponentialfunction gives

$$-\left[t^2+(n-2)\cdot\hat{\gamma}\cdot(t-\hat{\theta})^2+\frac{\kappa}{2}\cdot\left(t-\frac{x-\mu}{\sqrt{\kappa}\sigma}\right)^2\right]$$

$$\overset{t}{\propto} -\left[t^2\cdot\left(1+(n-2)\cdot\hat{\gamma}+\frac{\kappa}{2}\right)-2\cdot t\cdot\left((n-2)\cdot\hat{\gamma}\cdot\hat{\theta}+\frac{\sqrt{\kappa}\cdot(x-\mu)}{2\sigma}\right)\right]$$

$$\overset{t}{\propto} -\underbrace{\left(1+(n-2)\cdot\hat{\gamma}+\frac{\kappa}{2}\right)}_{m_3}\cdot\left(t-\overbrace{\frac{\left((n-2)\cdot\hat{\gamma}\cdot\hat{\theta}+\frac{\sqrt{\kappa}\cdot(x-\mu)}{2\sigma}\right)}{1+(n-2)\cdot\hat{\gamma}+\frac{\kappa}{2}}}^{m_4}\right)^2$$

$$= -m_3\cdot(t-m_4)^2.$$

Thus, the bid function $b(x)$ can be simplified to

$$b(x) \approx x - \sqrt{\kappa}\cdot\sigma\cdot\frac{\int_{-\infty}^{\infty} t\cdot e^{-m_3(t-m_4)^2}\ dt}{\int_{-\infty}^{\infty} e^{-m_3(t-m_4)^2}\ dt}.$$

$$b(x) \approx x - \sqrt{\kappa}\cdot\sigma\cdot\frac{\int_{-\infty}^{\infty} t\cdot e^{-m_3(t-m_4)^2}\ dt}{\int_{-\infty}^{\infty} e^{-m_3(t-m_4)^2}\ dt}$$

$$= x - \sqrt{\kappa}\cdot\sigma\cdot\frac{\mathrm{E}(t)}{1} = x - \sqrt{\kappa}\cdot\sigma\cdot m_4,$$

where $m_3 = 1+(n-2)\hat{\gamma}+\frac{\kappa}{2}$, and $m_4 = \frac{(n-2)\hat{\gamma}\hat{\theta}+\frac{\sqrt{\kappa}(x-\mu)}{2\sigma}}{1+(n-2)\hat{\gamma}+\frac{\kappa}{2}}$. Note, the constants of the normal kernel of the numerator and the denominator cancel out. By Substituting the expression for m_4, the bid approximation can be simplified to

$$b(x) \approx -\sqrt{\kappa}\cdot\sigma\cdot\frac{(n-2)\cdot\hat{\gamma}\cdot\hat{\theta}}{(n-2)\cdot\hat{\gamma}+1+\frac{\kappa}{2}} + \frac{\frac{\kappa}{2}}{(n-2)\cdot\hat{\gamma}+1+\frac{\kappa}{2}}\cdot\mu + \frac{(n-2)\cdot\hat{\gamma}+1}{(n-2)\cdot\hat{\gamma}+1+\frac{\kappa}{2}}\cdot x.$$

Hence, the linear approximation is a weighted function between the signal x and the expected value μ.

Appendix C: Gamma-Gamma approximation by using multivariate regression

The approximation goes over $(1-G_{S|V}(1/x|v))$ by using Gamma probability density functions. By substitution, the distribution function of $S|V$ can be written as

$$G_{S|V}(1/x|v) = \int_{-\infty}^{1/x} \frac{(\tau v)^\tau}{\Gamma(\tau)} l^{\tau-1} e^{-\tau v \cdot l}\, dl = \int_{-\infty}^{\tau\cdot\frac{v}{x}} \frac{1}{\Gamma(\tau)} t^{\tau-1} e^{-t}\, dt.$$

Hence, the distribution function $G_{S|V}$ depends on the parameter τ through the support $\frac{v}{x}$. Approximation of $(1 - G_{S|V}(1/x|v))$ with only one unique Gamma *p.d.f.* is therefore not possible. Tabulation of Gamma *p.d.f.* approximations for different values of τ is one way to tackle the non-uniqueness feature, but to get some structure we utilize multivariate regression.

Let the dependent variables be the two parameters, $(\alpha_{\tau,i}, \beta_{\tau,i})$, of the Gamma *p.d.f.* approximations for each value of τ_i, where $NAppr$ is the number of approximations and $i = 1, 2, \ldots, NAppr$. Further, define the independent variables as functions of τ_i, e.g. $\tau_i, \tau_i^2, log(\tau_i), 1/\tau_i$[12]. Then, by comparing adjusted R-square for each regression model we choose and estimate the best regression model $(\hat{\alpha}_\tau, \hat{\beta}_\tau)$ using all subsets of the independent variables as independent variables. This resulted in the following two best regression models, estimated as

$$\hat{\alpha}_\tau = 1.02 - 0.00488618 \cdot \tau + 0.00002205 \cdot \tau^2 + 0.125789 \cdot \log \tau,$$

and

$$\hat{\beta}_\tau = 0.448417 + 0.00095877 \cdot \tau + 0.496667 \cdot \frac{1}{\tau}$$

with adjusted R^2 equal to 98.6% and 99.5%, respectively.

Let $h_v(v|\hat{\alpha}_\tau, \hat{\beta}_\tau) = (\frac{v}{x})^{\hat{\alpha}_\tau - 1} \cdot e^{-\hat{\beta}_\tau \cdot \frac{v}{x}} \overset{v}{\propto} v^{\hat{\alpha}_\tau - 1} \cdot e^{-\hat{\beta}_\tau \cdot \frac{v}{x}}$ be the "Gamma kernel" approximating function to $(1 - G_{S|V}(1/x|v))$. Then, by replacing $(1-G_{S|V}(1/x|v))$ with $h_v(v|\hat{\alpha}_\tau, \hat{\beta}_\tau)$, the approximated bid function becomes

$$b(x) \approx \frac{\int_{-\infty}^{\infty} v \cdot v^{(n-2)(\hat{\alpha}_\tau - 1) + 2\tau + \alpha - 1} \cdot e^{-(\frac{1}{x}(\hat{\beta}_\tau(n-2)+2\tau)+\beta)\cdot v}\, dv}{\int_{-\infty}^{\infty} v^{(n-2)(\hat{\alpha}_\tau - 1) + 2\tau + \alpha - 1} \cdot e^{-(\frac{1}{x}(\hat{\beta}_\tau(n-2)+2\tau)+\beta)\cdot v}\, dv}.$$

Let $\alpha' = (n-2)(\hat{\alpha}_\tau - 1) + 2\tau + \alpha$, and let $\beta' = \frac{1}{x}(\hat{\beta}_\tau(n-2) + 2\tau) + \beta$. Then, we get the approximated bid function as

$$\begin{aligned} b(x) &\approx \frac{\int_{-\infty}^{\infty} Gamma(\alpha'+1, \beta')\, dv}{\int_{-\infty}^{\infty} Gamma(\alpha', \beta')\, dv} = \frac{\Gamma(\alpha'+1)}{\beta'^{\alpha'+1}} \cdot \frac{\beta'^{\alpha'}}{\Gamma(\alpha')} \\ &= \frac{\alpha'}{\beta'} = \frac{[\alpha + 2\tau + (n-2)(\hat{\alpha}_\tau - 1)] \cdot x}{\beta x + 2\tau + (n-2)\hat{\beta}_\tau}. \end{aligned}$$

[12]Several functions of τ_i were used but we do not list everyone here.

Appendix D: The winner's curse effect as a net bid shading effect of the approximated bid function

It is easily verified that the unconditional distribution of signals is given by $X \sim N(\mu, (\kappa+1)\sigma^2)$, and as such $x = \mu + d\sigma\sqrt{\kappa+1}$ represents signals that deviate from the expected value μ with d standard deviations. By replacing x with $\mu+d\sigma\sqrt{\kappa+1}$, the *net bid shading effect*, defined as the winner's curse effect by $NBS := (-II'_n - I'_n)$, can be written as

$$NBS = (-II'_n - I'_n) = \frac{\sqrt{\kappa}\sigma\hat{\gamma}\left[\hat{\theta}(2+\kappa) - d\sqrt{\kappa(\kappa+1)}\right]}{2\left[\hat{\gamma}(n-2)+1+\frac{\kappa}{2}\right]^2},$$

which is negative if $d > \frac{\hat{\theta}(2+\kappa)}{\sqrt{\kappa(\kappa+1)}}$. We could assume $\kappa = 0.25$ according to estimation results in Bajari and Hortacsu (2003), but in order to allow for larger values, let $\kappa = 1$. Then, the net bid shading effect only becomes negative for $d > 4.16$ (see Figure 11), corresponding to signals more than four standard deviations above their expected value, which is of course very unlikely.

Appendix E: Approximation of the bidder's expected profit

By using the approximation of the standard normal distribution function, the bidder's expected profit can be approximated and written as

$$\frac{(n-1)e^{-\hat{\gamma}(n-2)\hat{\theta}^2}\sqrt{c_2}}{3c_1\sqrt{2\pi\kappa c_1}} \cdot \left[\sqrt{\frac{\pi}{c_6}}\Phi[c_8(x)]\left(x - \frac{c_7(x)}{2c_6}\right) + \frac{1}{2c_6}e^{-\frac{c_8^2(x)}{2}}\right] \cdot$$
$$\exp\left[c_2c_4^2 - \frac{c_3c_5^2(x) + 8\sqrt{\kappa}\sigma c_2c_4c_5(x)}{8\kappa\sigma^2c_1c_2} + \frac{c_7(x)^2}{4c_6}\right],$$

where

$$c_1 = 1 + \hat{\gamma}(n-2) + \frac{k}{2}, \quad c_2 = \frac{k+1}{2}, \quad c_3 = 1 + 2\hat{\gamma}(n-2),$$

$$c_4 = \hat{\gamma}\hat{\theta}(n-2), \quad c_5(x) = x + \kappa\mu, \quad c_6 = \frac{c_3(k+1)^2}{8\kappa\sigma^2c_1c_2},$$

$$c_7(x) = \frac{8\sqrt{k}\sigma c_2c_4(k+1) + 2(k+1)c_3c_5(x)}{8\kappa\sigma^2c_1c_2}, \quad c_8(x) = \sqrt{2c_6}\left(x - \frac{c_7(x)}{2c_6}\right).$$

Figures

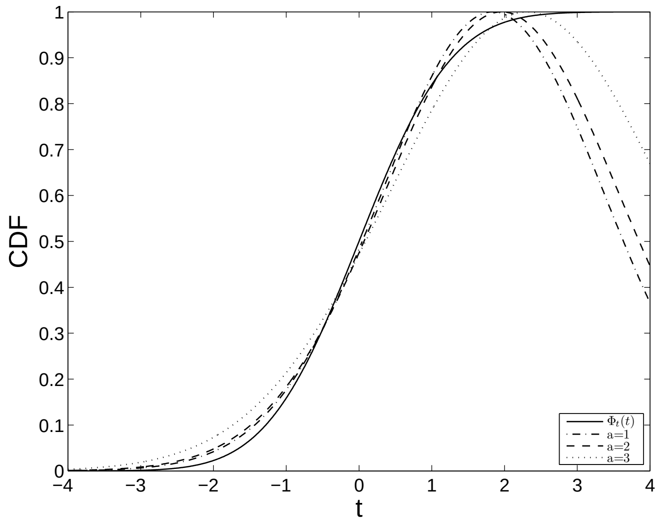

Figure 1: The approximation of the standard normal distribution function is compared to the exact function on $[-a, a]$, where $a = 1, 2$ or 3. The thick solid curve represents the standard normal distribution function $\Phi_t(t)$. Other curves are approximations of $\Phi_t(t)$ as kernels of normal $p.d.f.$s with different values of a.

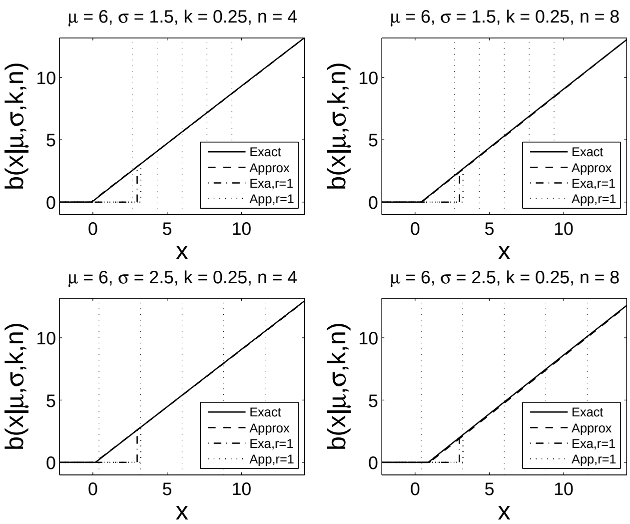

Figure 2: The exact versus the approximated bid function, with or without minimum bid r, for the Normal-Normal model with a known number of bidders. Dotted lines along the y-axis represent standard deviations from the expected value μ in the unconditional distribution of x, where the middle lines indicate the position of μ.

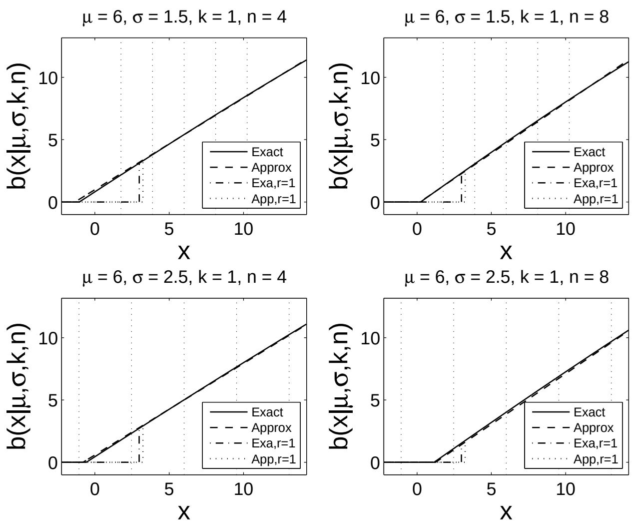

Figure 3: The exact versus the approximated bid function, with or without minimum bid r, for the Normal-Normal model with a known number of bidders. Dotted lines along the y-axis represent standard deviations from the expected value μ in the unconditional distribution of x, where the middle lines indicate the position of μ.

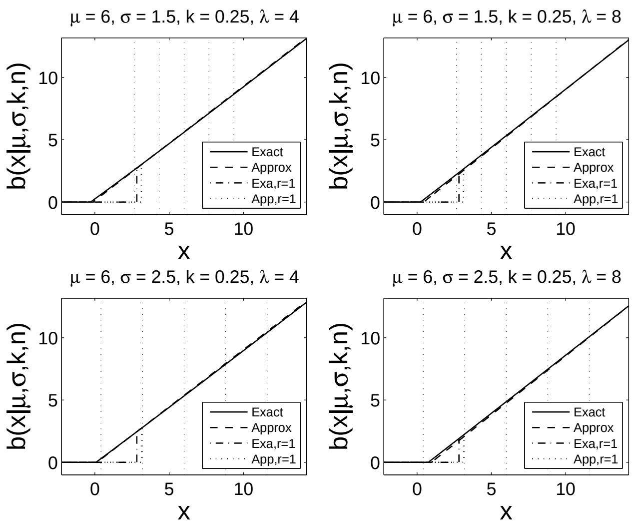

Figure 4: The exact versus the approximated bid function, with or without minimum bid r, for the Normal-Normal model with a stochastic number of bidders. Dotted lines along the y-axis represent standard deviations from the expected value μ in the unconditional distribution of x, where the middle lines indicate the position of μ.

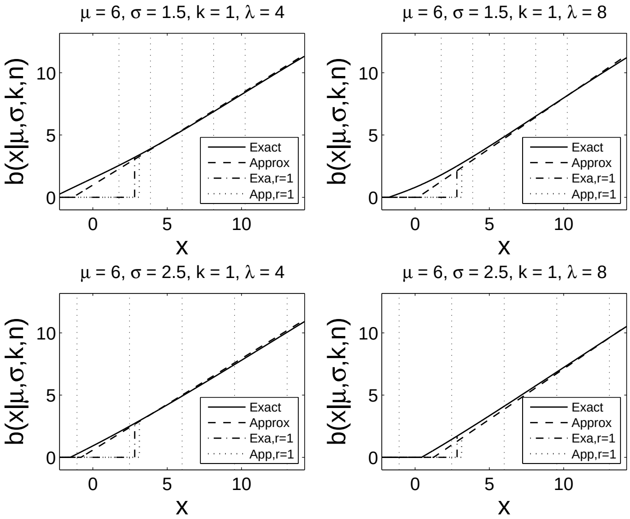

Figure 5: The exact versus the approximated bid function, with or without minimum bid r, for the Normal-Normal model with a stochastic number of bidders. Dotted lines along the y-axis represent standard deviations from the expected value μ in the unconditional distribution of x, where the middle lines indicate the position of μ.

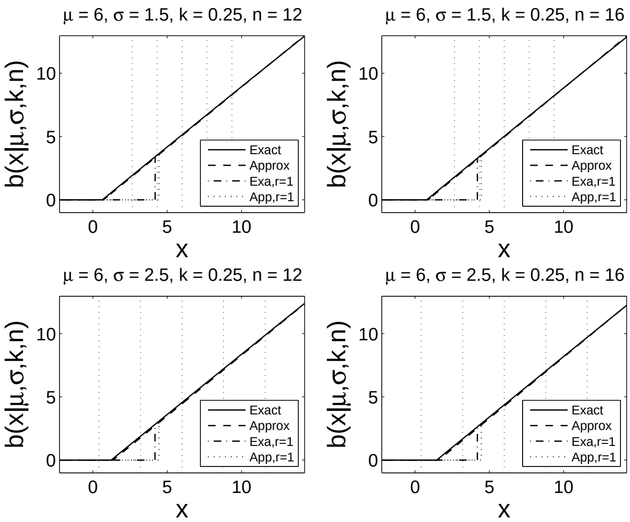

Figure 6: The exact versus the approximated bid function, with or without minimum bid r, for the Normal-Normal model with a known number of bidders. Dotted lines along the y-axis represent standard deviations from the expected value μ in the unconditional distribution of x, where the middle lines indicate the position of μ.

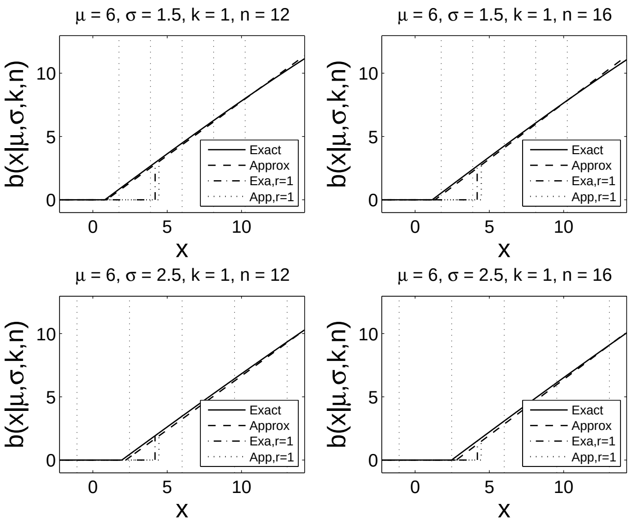

Figure 7: The exact versus the approximated bid function, with or without minimum bid r, for the Normal-Normal model with a known number of bidders. Dotted lines along the y-axis represent standard deviations from the expected value μ in the unconditional distribution of x, where the middle lines indicate the position of μ.

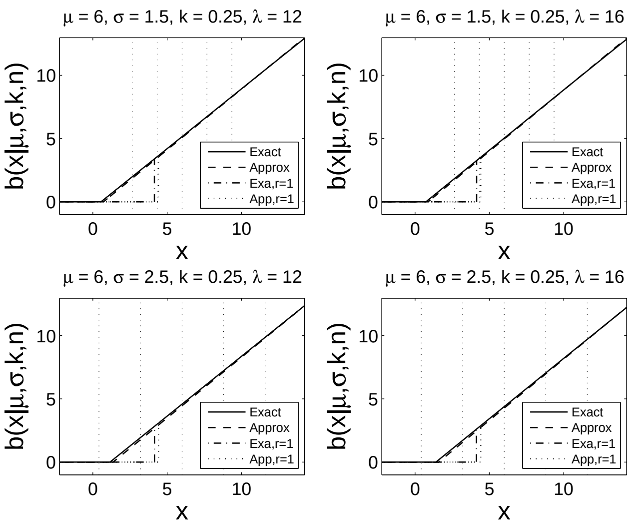

Figure 8: The exact versus the approximated bid function, with or without minimum bid r, for the Normal-Normal model with a stochastic number of bidders. Dotted lines along the y-axis represent standard deviations from the expected value μ in the unconditional distribution of x, where the middle lines indicate the position of μ.

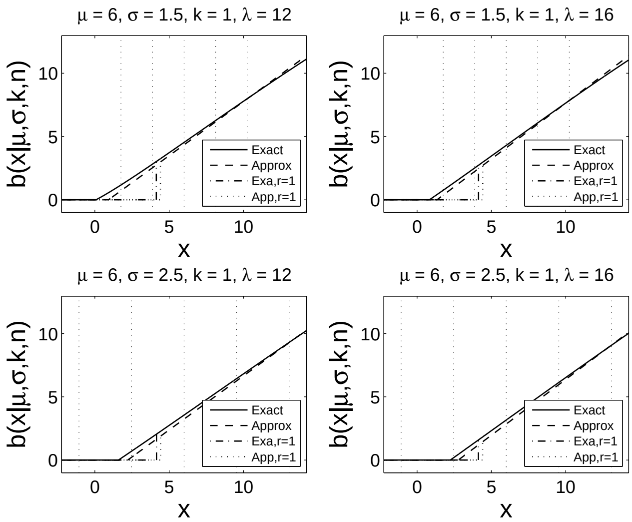

Figure 9: The exact versus the approximated bid function, with or without minimum bid r, for the Normal-Normal model with a stochastic number of bidders. Dotted lines along the y-axis represent standard deviations from the expected value μ in the unconditional distribution of x, where the middle lines indicate the position of μ.

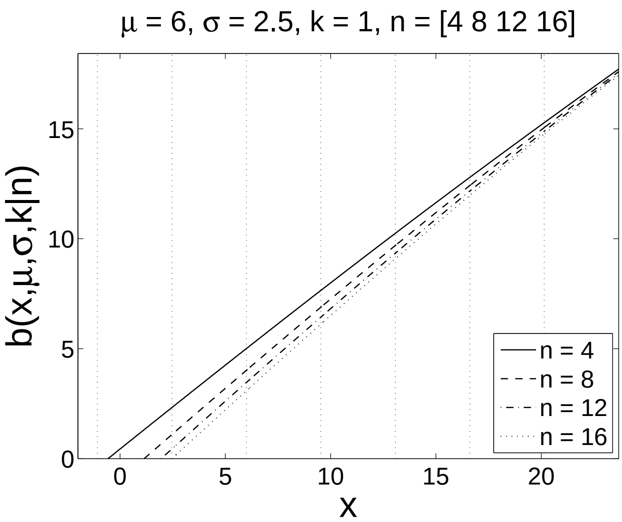

Figure 10: The Winner's curse effect of the exact bid function with a known number of bidders. Dotted lines along the $y-$axis represent the number of standard deviations from the expected value, starting from minus 4 to plus 4 standard deviations.

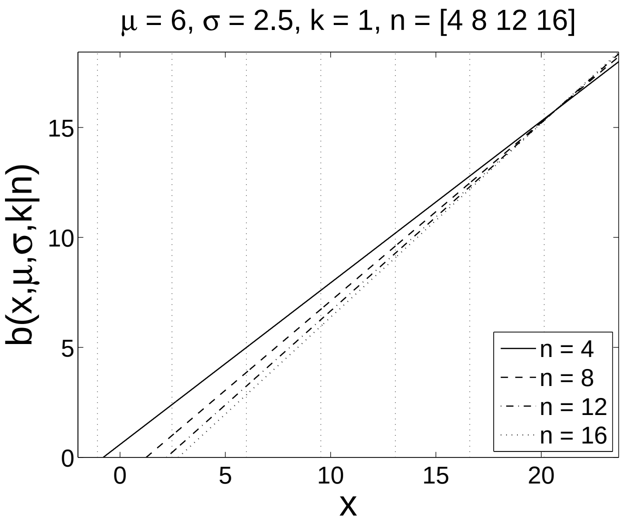

Figure 11: The Winner's curse effect of the approximated bid function with a known number of bidders. Despite a fairly large value of the variance scale parameter, $\kappa = 1$, large signals is needed to give a negative net bid shading (NBS) effect, since this only occur for signals larger than 4 standard deviations above the expected value μ. Dotted lines along the $y-$axis represent number of standard deviations from the expected value, starting from minus 2 to plus 4 standard deviations.

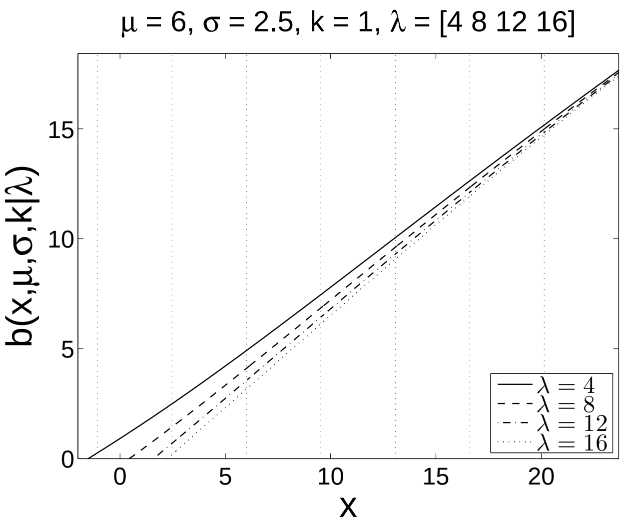

Figure 12: The Winner's curse effect of the exact bid function with a stochastic number of bidders. Dotted lines along the $y-$axis represent the number of standard deviations from the expected value, starting from minus 4 to plus 4 standard deviations.

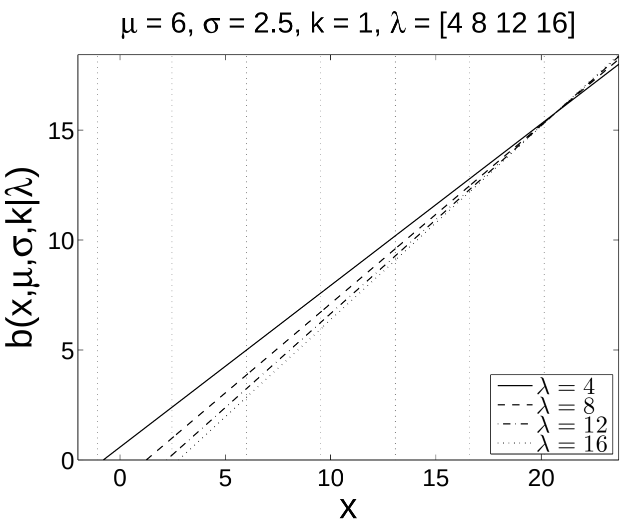

Figure 13: The Winner's curse effect of the approximated bid function with a stochastic number of bidders. Despite a fairly large value of the variance scale parameter, $\kappa = 1$, large signals is needed to give a negative net bid shading (NBS) effect, since this only occur for signals larger than 4 standard deviations above the expected value μ. Dotted lines along the $y-$axis represent number of standard deviations from the expected value, starting from minus 2 to plus 4 standard deviations.

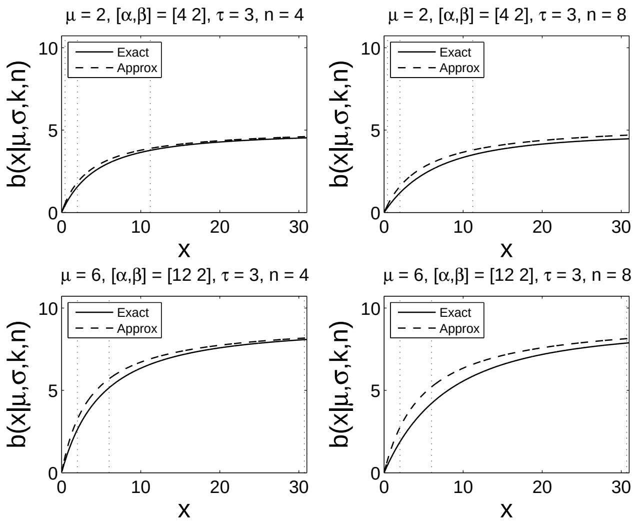

Figure 14: The exact versus the approximated bid function for the Gamma-Gamma model with a known number of bidders. Concerning the dotted lines along the y-axis, middle lines represent the position of μ, left lines the position of the $2.5-$percentile, and right lines the position of the $97.5-$percentile.

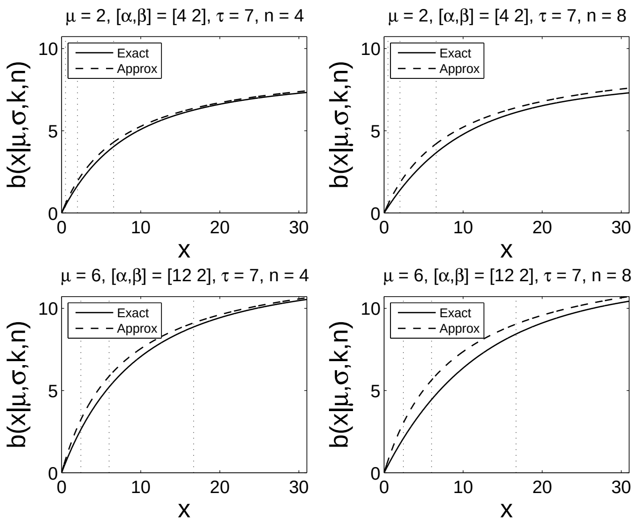

Figure 15: The exact versus the approximated bid function for the Gamma-Gamma model with a known number of bidders. Concerning the dotted lines along the y-axis, middle lines represent the position of μ, left lines the position of the $2.5-$percentile, and right lines the position of the $97.5-$percentile.

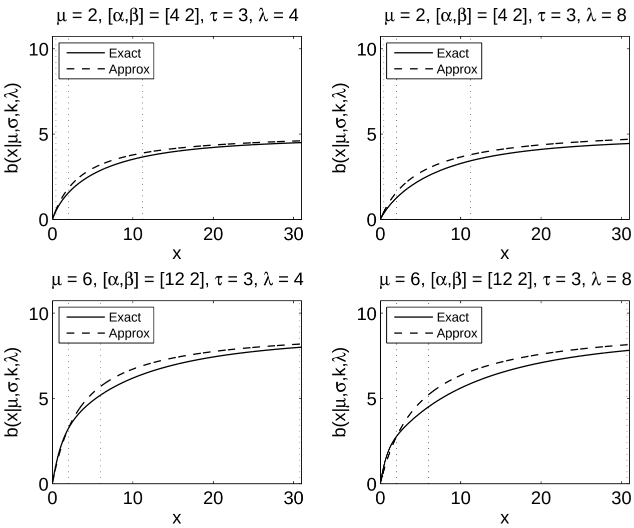

Figure 16: The exact versus the approximated bid function for the Gamma-Gamma model with a stochastic number of bidders. Concerning the dotted lines along the y-axis, middle lines represent the position of μ, left lines the position of the 2.5−percentile, and right lines the position of the 97.5−percentile.

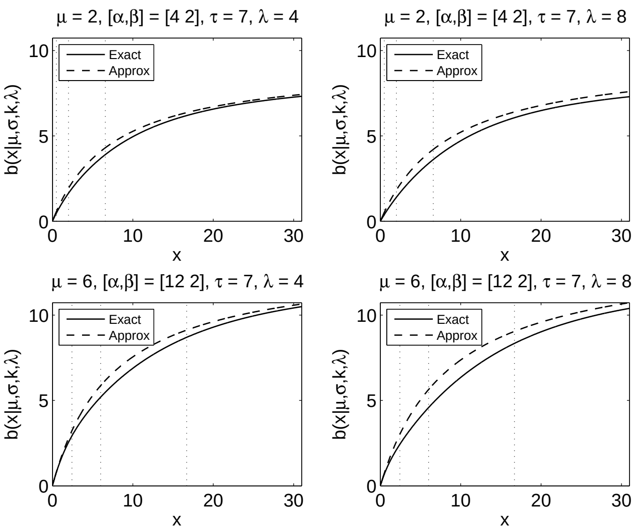

Figure 17: The exact versus the approximated bid function for the Gamma-Gamma model with a stochastic number of bidders. Concerning the dotted lines along the y-axis, middle lines represent the position of μ, left lines the position of the 2.5−percentile, and right lines the position of the 97.5−percentile.

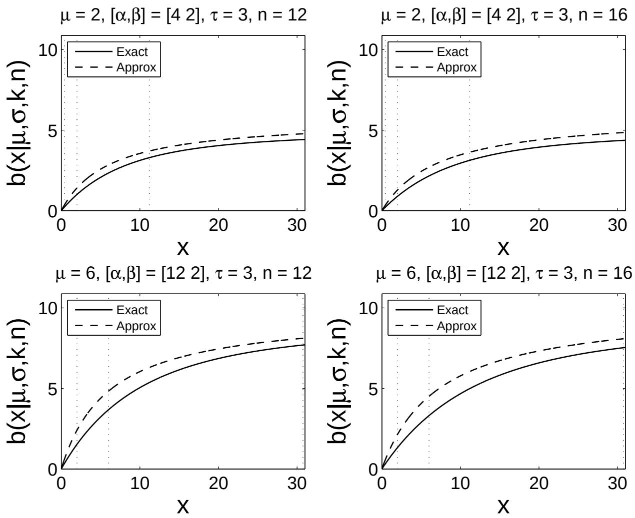

Figure 18: The exact versus the approximated bid function for the Gamma-Gamma model with a known number of bidders. Concerning the dotted lines along the y-axis, middle lines represent the position of μ, left lines the position of the 2.5−percentile, and right lines the position of the 97.5−percentile.

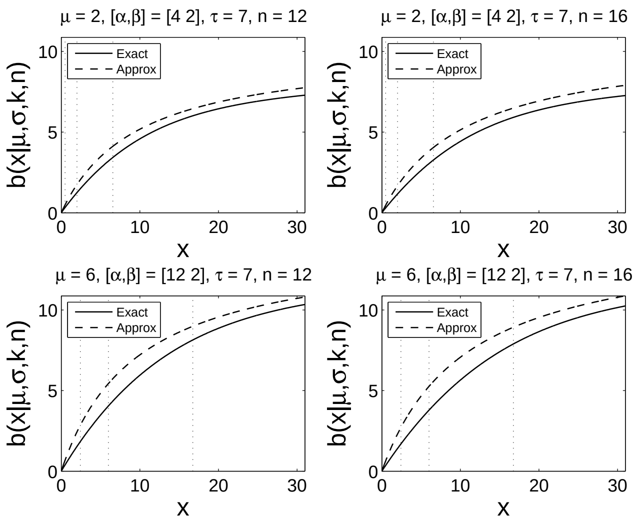

Figure 19: The exact versus the approximated bid function for the Gamma-Gamma model with a known number of bidders. Concerning the dotted lines along the y-axis, middle lines represent the position of μ, left lines the position of the 2.5−percentile, and right lines the position of the 97.5−percentile.

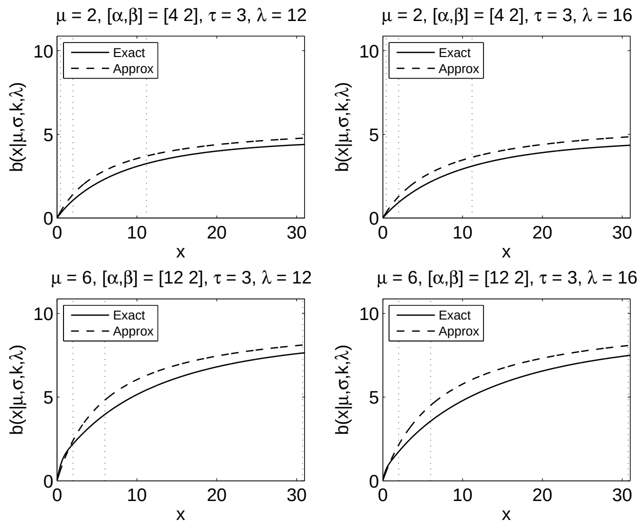

Figure 20: The exact versus the approximated bid function for the Gamma-Gamma model with a stochastic number of bidders. Concerning the dotted lines along the y-axis, middle lines represent the position of μ, left lines the position of the 2.5−percentile, and right lines the position of the 97.5−percentile.

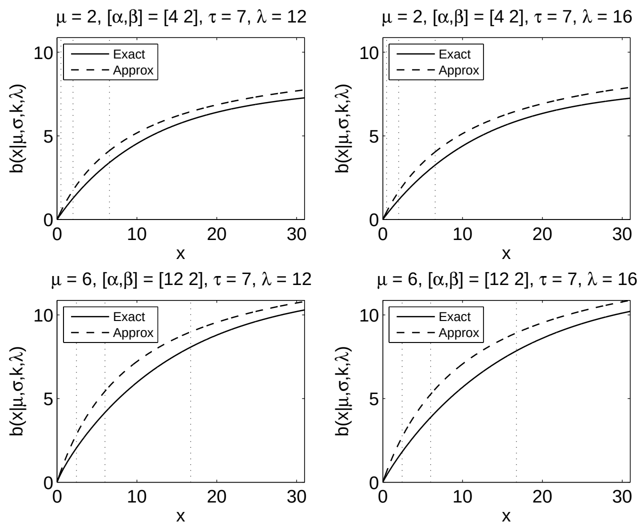

Figure 21: The exact versus the approximated bid function for the Gamma-Gamma model with a stochastic number of bidders. Concerning the dotted lines along the y-axis, middle lines represent the position of μ, left lines the position of the 2.5−percentile, and right lines the position of the 97.5−percentile.

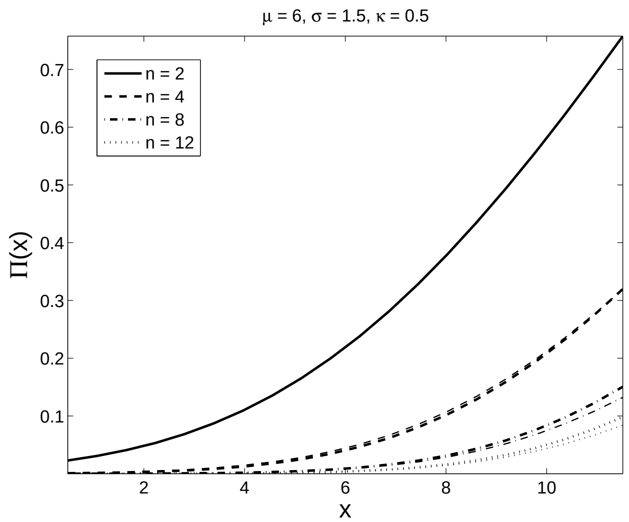

Figure 22: The bidder's expected profit for different number of bidders n. Thinner lines, located just below assigned thick lines for the exact cases, correspond to the approximated values of the bidder's expected profits.

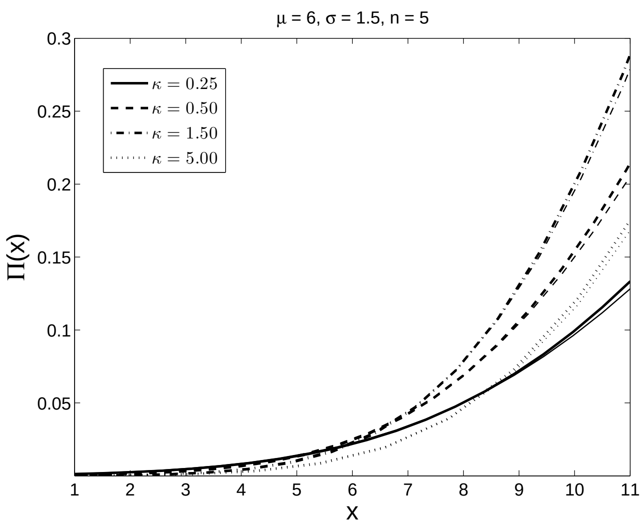

Figure 23: The bidder's expected profit for different values of κ, the variance scale parameter for signals. Thinner lines, located just below assigned thick lines for the exact cases, correspond to the approximated values of the bidder's expected profits. The precision in signals when $\kappa = 1.5$ was estimated to give the highest expected profits for sufficient high values of x. Lower and higher precision from this point results in a decline of $\Pi(x)$.

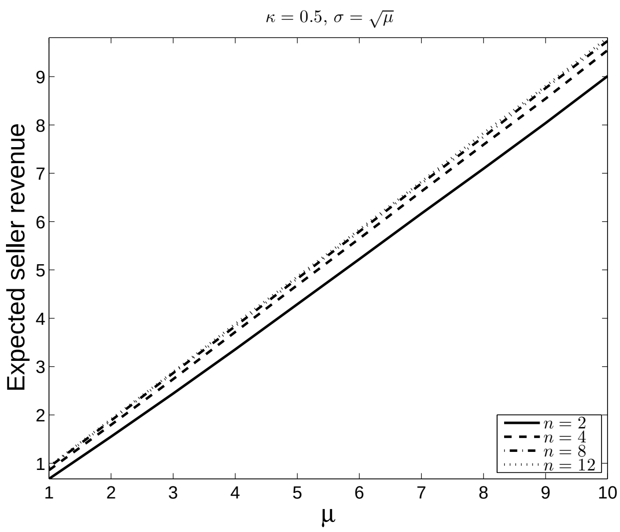

Figure 24: The expected seller revenue for different number of bidders n.

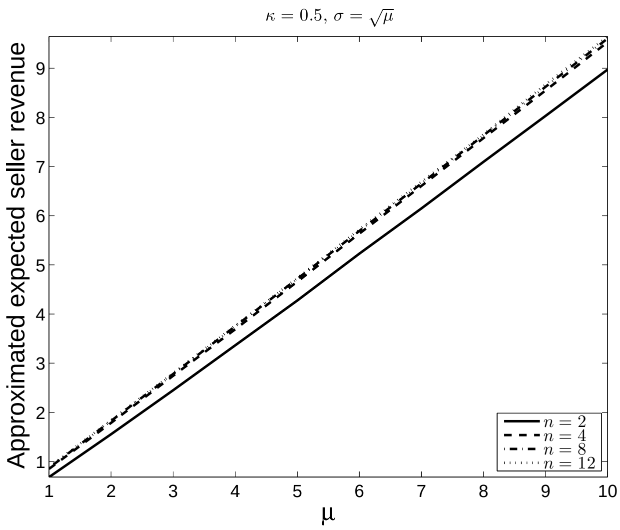

Figure 25: Approximated expected seller revenue for different number of bidders n. The approximations seem to work well compared to the exact cases above.

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