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Testing Centrality in Random Graphs

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Abstract

Centrality is an important concept in social network analysis which involves identification of important or prominent actors. Three common definitions of centrality are degree centrality, closeness centrality and betwenness centrality. These definitions yield actor indices which can be aggregated across actors to obtain a single group-level index. In this paper we consider how eight of these group-level indices can be used for graph centrality tests. Two of the tests are based on degree, whereas the remaining six tests are based on closeness. Our null hypothesis model, showing no centrality structure, is the Bernoulli graph model which we test against a block model reflecting graph centrality. We perform a simulation study where the power of the tests are compared.

Keywords: Bernoulli graphs; Closeness centrality; Degree centrality; Power of centrality tests; Random graphs; Stochastic blockmodels.

1 Introduction

Data that involve relationships representing the interactions between members in a social network arise in a wide variety of settings, including economics, psychology, sociology and criminology. The development of stochastic models and statistical methods appropriate for analyzing this kind of data is therefore of considerable importance. The members and their relational structure are usually represented by graphs, where the members are referred to as vertices or actors and the relations are referred to as edges.

A substantial amount of research has been devoted to a concept usually referred to as blockmodels, which involves partitioning of actors with similar contact patterns and relational features into subsets called blocks. A definition of a deterministic approach to blockmodels was given by White, Boorman and Breiger (1976), which states that actors with the same adjacency structure are partitioned into the same block. Fienberg and Wasserman

(1981) and Holland, Laskey and Leinhardt (1983) generalized the deterministic blockmodels by using the concept *stochastic equivalence*; in a random directed graph model two actors are defined as stochastic equivalent if their probabilistic relational structures to the other actors in the graph are the same. Under an additional assumption of independent dyads and permutation invariance of actors, they called models with such probabilistic relational structures *stochastic blockmodels*.

Blockmodels may also be used to model the important concept of centrality by defining the block consisting of members with the largest probability to generate relations as central. A variety of indices measuring actor *centrality* has been proposed in the literature. By aggregating an actor centrality index over all actors, we obtain a graph centrality index, measuring how "centralized" the set of actors is as a whole. Three centrality definitions frequently discussed are those based on degree, closeness and betweenness. The definition of centrality was first developed by Bavelas (1948 , 1950). Many graph theoretic concepts are discussed in Hage and Harare (1983), and various aspects of centrality in graphs has been published by, for instance, Beauchamp (1965), Nieminen (1974), Höivik and Gleditsch (1975), Freeman (1977, 1979) and Snijders (1981a, 1981b). Recent publications include work by Hagberg (2000) and Frank (2002). The licentiate thesis Tallberg (2000a) included two papers, Tallberg (2000b, 2000c). In Tallberg (2000b), a likelihood ratio test procedure for testing centrality was considered, where centrality was defined by a stochastic blockmodel described above. The idea was abandoned because of the difficulties with obtaining the maximum likelihood estimators which includes identifying the blocks. In Tallberg (2000c) some preliminary comparisons between various centrality tests based on degree-centrality and closeness-centrality are made. Similar comparisons are continued here.

In this paper, we present test statistics based on well-known graph centrality indices for testing whether the observed data is likely to have come from a stochastic blockmodel, indicating centrality, or some non-central random model. By considering the Bernoulli model as our non-central random null hypothesis model, the performance of the tests are evaluated by comparing their power functions.

We now outline the content of this paper. In Section 2 basic notation is given that will be used throughout the remainder of this paper. In Section 3, the graph models considered are described, and in Section 4 the graph centrality tests are presented. A simulation study where the power functions are compared is covered in Section 5. In Section 6, we develop a model where critical values of the tests are expressed as functions of graph parameters. In Section 7, we illustrate the results on Kapferer's tailor shop data. Some concluding remarks are given in Section 8.

2 Notation

Consider an undirected graph of known order n on the vertex set $V = \{1, ..., n\}$, and let V^2 denote the set of all distinct ordered pairs of vertices (i, j) from V. By denoting the corresponding $n \times n$ adjacency matrix of the graph A, each entry $a_{ij} = a_{ji}, (i, j) \in V^2$ takes the value 1 if an edge is present between i and j. By convention, the diagonal entries of A are equal to 0. The number of edges, often referred to as the size of the graph, is here denoted by R and given by

$$R = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij}.$$

The degree of any vertex i is defined as the number of edges incident to vertex i and is denoted by a_i, thus

$$a_i = \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}.$$

The maximum degree is denoted by $\max_i a_i$ and the mean degree is denoted by $\bar{a}$ and given by

$$\bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}.$$

Finally the variance of the degrees is denoted by s_a^2 and given by

$$s_a^2 = \frac{1}{n} \sum_{i=1}^{n} (a_i - \bar{a})^2 .$$

A *walk* in a graph is an alternating sequence of vertices and edges, starting and ending with vertices, in which each edge is incident with the vertices following and preceding it in the sequence. Vertices and edges may be repeated. A *path* is a walk in which all vertices and edges are distinct. The *length* of a path is the number of edges used. The *geodesic* is the shortest path between two vertices. If any two vertices are connected by a walk the graph is said to be connected. The maximal connected subgraphs of a disconnected graph are called the connected components of the graph. Let D be an $n \times n$ distance matrix where the element d_{ij} is defined as the length of the geodesic between vertex i and vertex j. Then, the average distance in a connected graph is denoted by $\bar{d}$ and given by

$$\bar{d} = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij}.$$

If a geodesic does not exist between vertex i and vertex j, i.e. they are located in two different components, d_{ij} is set to infinity which only allows computation of the average distance in connected components. For a more detailed description of concepts in graph theory; see Palmer (1985) or the extensive work on social networks by Wasserman and Faust (1994).

3 Models

3.1 The Bernoulli model

As our stochastic null hypothesis model H_0, showing no centrality structure, we will use the Bernoulli graph model. That is, the edges are generated independently with an unknown common edge probability $p = \Pr(a_{ij} = 1)$, $i \neq j$. The probability of a graph G is given by

$$\Pr(G) = \begin{cases} p^r (1-p)^{\binom{n}{2}-r} & \text{if } G \text{ is of order } n \text{ with } r \text{ edges} \\ 0 & \text{otherwise.} \end{cases}$$

This model is commonly used to study random graph properties; see for instance Palmer (1985).

3.2 Blockmodels

To be able to estimate the powers of the tests, we need an alternative random graph model that captures centrality against which we can test the Bernoulli graph model. For this purpose, we define a block model in the following way:

Assume that we have a graph of fixed order n where V is partitioned into b mutually exclusive non-empty vertex blocks labeled $1, ..., b$. The dyad involving i and j is characterized by (a_{ij}, y_i, y_j), where y_i and y_j, $(i,j) \in N$, are block labels of actors i and j. Conditional on all $y_1, ..., y_n$, the elements a_{ij} for $i < j$ are independent random variables with probability

$$\begin{aligned} & \Pr(a_{ij} = 1 \,|y_1, ..., y_n) \\ = \; & \Pr(a_{ij} = 1 \,|y_i, y_j) = p(y_i, y_j). \end{aligned}$$

Set $p(k,l) = p_{k,l}$. The blocks are labeled so that $p_{kk} \geqslant p_{ll}$ for $k < l$. Usually we assume $p_{kk} > p_{ll}$ for $k < l$. If $p_{kk} = p_{ll}$, then we require $p_{kl} \neq p_{kk}$ in order to be able to separate the blocks k and $l \neq k$. The edge set between any vertex i in block k and any vertex j in block l is denoted by E_{kl}.

To simplify, we will impose constraints on the parameter space and carry out our analysis for the simplest case, $b = 2$. Then, the first block is regarded as the central block which we assume contain only one vertex, and the second block is regarded as the non-central block consisting of the remaining vertices. When the number of edge probabilities are reduced to two, p_{12} and p_{22}, such that

$$p_{12} - p_{22} > 0,$$

we conveniently reparametrize so that p_{22} is denoted by p. By introducing a new parameter explicitly denoted by Δ and given by

$$\Delta = p_{12} - p > 0,$$

the block model is reformulated with the two unknown parameters, p and Δ. Hence, the edge probability between b_1 and b_2 is now given by $p + \Delta$. We denote by H_Δ this blockmodel with parameters p and Δ, and for $\Delta = 0$, H_0 is the Bernoulli model.

4 Tests of centrality

4.1 Introduction

Three centrality definitions are usually mentioned in the literature, degree centrality, closeness centrality and betweenness centrality. All three are in different ways trying to capture the popularity, influence or importance of actors in a social network. The choice of an appropriate centrality definition and its associated measure depends on the context of the application. According to Freeman (1979), degree-based centrality measures the actors' communication activity, betweenness-based centrality measures the actors' control of communication, and closeness-based centrality measures the actors' interference or efficiency.

Centrality indices are evaluated for all n actors indicating the status of each actor according to popularity, influence etc.. To obtain a centrality index at group level the centrality indices are aggregated across all actors. There are different measures at group level such as the maximum, the average, or the variance of the actor level indices. According to many network researchers interested in centrality, for example Leavitt (1951), Nieminen (1974) and Freeman (1977), a group level index should have the property to increase if the difference between a single actor's centrality status and the remaining actors' centrality status increases. Thus, an appropriate group level index should measure the variability or heterogeneity of the actor centrality indices. We shall employ this view of graph centralization and consider eight graph centrality indices, measuring the variability of the actor centralities, as test statistics. Six of the tests are based on closeness and two are based on degree.

The numerical computation package Matlab was used to simulate properties of the tests for various values of the model parameters.

4.2 Tests based on degree.

The simplest and most prominent definition of actor centrality is that based on degree. It focuses only on direct or adjacent choices. Actors that have the most edges to other actors in the network are considered as central. As our test statistics, we will use two standard group level centrality measures which are frequently suggested in the literature. The first test is the difference of the maximum actor degree and the mean actor degree,

$$T_1 = \max_i \ a_i - \bar{a},$$

proposed by Freeman (1979). The second test is the variance of the actor degrees,

$$T_2 = s_a^2.$$

Properties of the distribution of T_2 have been investigated in detail by for example Snijders (1981a, 1981b) and Hagberg (2000) to mention a few.

4.3 Tests based on closeness

The second view of actor centrality is based on closeness or distance, which is more dependent on indirect ties compared to centrality based on degree which only involves direct ties. The idea is that an actor will be considered as central if he can interact directly or indirectly with many others, implying that centrality is inversely related to distance. Since a desirable property of an actor's centrality index should be that it decreases when its distance to another actor grows, Sabidussi (1966) proposed the following index

$$\left[\sum_{j=1}^{n} d_{ij}\right]^{-1},$$

where the length of the geodesic distances are inversely weighted. The disadvantage with Sabidussi's index is that it can only be evaluated for connected graphs, since the distance between two disconnected vertices is set to infinity. In this paper we present two modified versions of Sabidussi's index with weights, which allow us to compute centrality indices based on distances in disconnected graphs.

In the first version we simply evaluate Sabidussi's index in connected components. Let n_i be the number of vertices in the component of vertex i. The average distance from vertex i to its other connected vertices is denoted by $\bar{d}_i$. Since the minimum value of $\bar{d}_i$ is unity for $n_i > 1$, we set $\bar{d}_i = 1$ if vertex i is isolated. To obtain a centrality index that increases as the length of the geodesics decrease, we use the reciprocal of $\bar{d}_i$ rather than $\bar{d}_i$ itself as a centrality index on actor level. The range of $\bar{d}_i^{-1}$ is between zero and unity, and its maximum is obtained for any vertex i that is adjacent to all the other vertices in the same component. If the underlying model is a Bernoulli graph model generating edges with a small common edge probability p, the realized graphs will consist of a large fraction of small order components. The vertices in those components will obtain large centrality values and will consequently to a large extent dominate the group level centrality index. In the extreme case, for a sufficiently small p, a large amount of vertices will be isolated with the maximum centrality index value one. To revoke this excessive influence of small order components, the actor centrality index is weighted with a factor of size n_i/n. Thus, the centrality index for vertex i is given by

$$c_i = \frac{n_i}{n\bar{d}_i}, \text{ for } n_i \geqslant 1. \tag{1}$$

Note that if G is connected we arrive at Sabidussi's index multiplied by the constant $(n-1)$.

In the second version of Sabidussi's index, we set $d_{ij}^{-1} = 0$ if $d_{ij} = \infty$, which allows us to calculate an actor centrality index without considering connected components. Thus, a second centrality index based on closeness for any vertex i is given by

$$c_i' = \frac{1}{n-1} \sum_{j \neq i} \frac{1}{d_{ij}}.$$

This index also possesses the desirable property of a centrality index of having a range between zero and unity, where its maximum value is reached when the actor is adjacent to all other $n-1$ actors. Note that c_i and c_i' are closely related since they both are inverted means; c_i is the inverse of the arithmetic mean and c_i' is the inverse of the harmonic mean.

In order to define graph centralities from the actor centralities we use Freeman's (1979) index and the variance of the actor centralities like we did for degree based actor centralities. In addition, as an alternative to those two indices we also consider the maximum of the actor centralities as a graph centrality index. Thus, we consider six tests of graph centrality based on distances given by

- $T_3 = \max_i c_i$
- $T_4 = \max_i c_i - \bar{c}$, where $\bar{c} = \frac{1}{n}\sum_i^n c_i$
- $T_5 = s_c^2 = \frac{1}{n}\sum_i (c_i - \bar{c})^2$
- $T_6 = \max_i c_i'$
- $T_7 = \max_i c_i' - \bar{c}'$, where $\bar{c}' = \frac{1}{n}\sum_i^n c_i'$
- $T_8 = s_{c'}^2 = \frac{1}{n}\sum_i (c_i' - \bar{c}')^2$

Note that none of the eight tests are standardized with their theoretical maximum value, since such a standardization would not affect the following analysis.

5 Power against centrality

Hypothesis tests are evaluated and compared through their probabilities of making mistakes. There are two types of error probabilities, 1) $\Pr(\text{rejecting } H_0 \,|H_0 \text{ is true})$ and 2) $\Pr(\text{not rejecting } H_0 \,|H_0 \text{ is not true})$. Since it is impossible to make both types of error probabilities small, it is common to control the first probability at a specified level. Within a given class of tests, we then search for tests that have the second probability that is as small as possible. Thus, in this section we examine $1 - \Pr(\text{not rejecting } H_0 \,|H_\Delta \text{ is true})$, i.e. the power functions of the tests. Since the power functions will depend on n, p and Δ, they are compared for all combinations of the parameter values $n = 10, 20, 50$, $p = 0.1, 0.3, 0.5$ and $\Delta = 0.0, 0.2, 0.4, 0.6, 0.8$. Note that for the combinations $n = 20, 50$ and $p = 0.3, 0.5$, the probability that G is connected is large. The index given in (1) then equals Sabidussi's index.

As previously mentioned, a simulation study was conducted in order to compare the power functions. For each combination of n and p, the test statistics were computed in 20,000 generated Bernoulli graphs (when $\Delta = 0.0$) in order to estimate the distributions of the tests under the null hypothesis. Then, the critical values, C, were determined such that we could obtain significance levels of approximately the same size for all the tests. Since the test statistics are discrete we may, particularly in graphs of small order, find it difficult to obtain critical values at the desired significance level α.

We are now able to estimate the powers of the tests against centrality, using the blockmodel described in Section 3.2 as our centrality model. For each combination of n, p and $\Delta > 0$, we generated 20,000 block model graphs. We then estimated the tests' powers by computing the proportion of test statistics exceeding the critical values. Since the power is a function of n, p and Δ, we denote this explicitly by writing $K(n, p, \Delta)$ or simpler $K(\Delta)$ for fixed n and p.

All tests presented in this paper have been considered in the simulations, but we only present results for some of them. Figure **??** exhibits the simulated power functions for fixed $n = 10$ and $p = 0.3$, and various values of Δ. Not surprisingly, the power functions of T_3, T_4 and T_5 are almost equal to their closely related tests T_6, T_7 and T_8, respectively. The tests T_5 and T_8 perform very poorly; for sufficiently large Δ and small p, $K(\Delta)$ is even a decreasing function. The simulation results also demonstrate that both of the degree-based tests, T_1 and T_2, perform well and that T_3 and T_4 were consistently better than T_6 and T_7. Therefore, the remaining results are presented only for the tests T_1, T_2, T_3 and T_4.

By examining the simulated power functions in Figure **??** for each considered combination of n, p and Δ, we see that none of the tests are uniformly most powerful, although a general tendency is that T_1 and T_3 are more powerful than the other tests. A bit surprising is that the frequently proposed graph centrality index T_2, the variance of the actor centralities based on degree, shows relatively poor power, especially for large values of p. Notable also is that $K(\Delta)$ increases faster for small p and large n, and that the power functions of T_1, T_3 and T_4 are approximately equal for large values of n.

6 Critical values

Although some rather tight constraints are imposed on the parameter space for the block model, such as assuming two blocks and only one vertex in the central block, we will now provide some convenient results to increase the applicability of the test procedures given in the previous section. As in the previous section, we only present results for the tests T_1, T_2, T_3 and T_4.

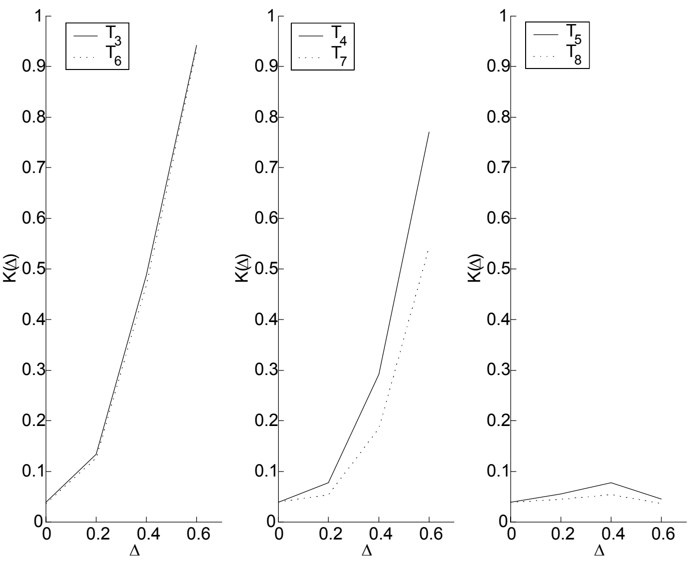

Figure 1: Power functions for some of the tests for $p = 0.3$ and $n = 10$.

We now provide the modeling and fitting details. Let G be generated by a Bernoulli graph model. Then, we assume that for any test T, $T - \mu$ is exponentially distributed with parameter θ, where the significance level of T is given by

$$\begin{aligned} \alpha &= \Pr\left(T > C\right) = \Pr\left(T - \mu > C - \mu\right) \\ &= \exp\left[-\left(C - \mu\right)/\theta\right],\ C > \mu,\ \theta > 0. \end{aligned}$$

so that

$$C = \mu - \theta \log \alpha. \tag{2}$$

Since T is not exactly exponentially distributed, an approximation is considered by assuming an ordinary least squares regression model with C considered as a continuous response taking values on the real line. Let C, besides depending on α, also depend on the graph parameters n and p through μ and θ in the following way

$$C\left(\alpha, n, p\right) = \mu\left(n, p\right) - \theta\left(n, p\right) \log \alpha = \boldsymbol{\gamma}_1' \mathbf{x} - \left(\log \alpha\right) \boldsymbol{\gamma}_2' \mathbf{x}, \tag{3}$$

where $\mathbf{x}' = (1, n, p, n^2, p^2, np)$ is a vector of known covariates, and $\boldsymbol{\gamma}_1' = (\gamma_{10}, \gamma_{11}, ..., \gamma_{15})$ and $\boldsymbol{\gamma}_2' = (\gamma_{20}, \gamma_{21}, ..., \gamma_{25})$ are two vectors of unknown regression coefficients.

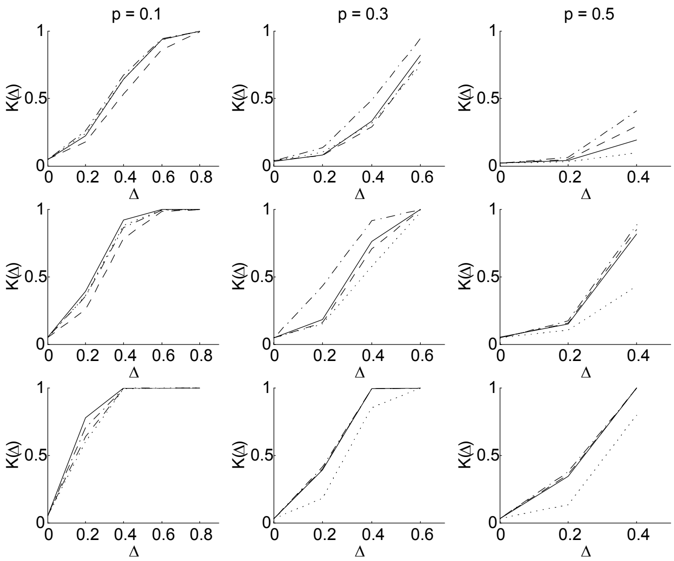

Figure 2: The simulated power functions of T_1 (solid line), T_2 (dotted line), T_3 (dashed dotted line) and T_4 (dashed line) for $n = 10$ (first row), $n = 20$ (second row) and $n = 50$ (bottom row), $p = 0.1$, 0.3, 0.5 and various Δ.

To evaluate the model a simulation study was conducted, where distributions of the tests were simulated for various combinations of n and p as described in previous section. Model (3) were fitted to the simulated data, and the vectors $\boldsymbol{\gamma}_1'$ and $\boldsymbol{\gamma}_2'$ estimated for each combination of n and p. Ordinary least squares estimates of $\boldsymbol{\gamma}_1'$ and $\boldsymbol{\gamma}_2'$ and their corresponding standard errors are given in Table 1.

To evaluate the fit of the models, we report adjusted R^2, the maximum absolute deviation and the mean absolute deviation of the residuals which seem reasonably adequate for all four fitted models according to Table 1. Furthermore, critical values obtained from the simulated distributions of the tests and critical values computed from (3) are plotted against α for various combinations of n and p in Figures **??** and **??**. Although the fit of (3) is not satisfactory for all combinations of n and p, the differences in general are not very large. Unfortunately, there is no obvious rule for which combinations of n, p and α, the critical values computed from (3) are almost equal to the simulated critical values.

	C_{T_1}	C_{T_2}	C_{T_3}	C_{T_4}
γ_{10}	-1.0 (0.3)	-10.7 (0.8)	0.44 (0.01)	0.13 (0.01)
γ_{11}	0.090 (0.006)	0.18 (0.01)	-80 (15)*	-80 (23)*
γ_{12}	15.6 (2.3)	57.4 (3.0)	0.95 (0.04)	-
γ_{13}	-	-	-	-
γ_{14}	-16.4 (2.6)	-56.9 (3.0)	-0.33 (0.04)	-
γ_{15}	-	-	-	-
γ_{20}	-	-	0.022 (0.002)	0.031 (0.004)
γ_{21}	0.011 (0.002)	0.026 (0.005)	-20.0 (5.1)*	-30.0 (7.7)*
γ_{22}	2.1 (0.5)	-	-	-0.019 (0.004)
γ_{23}	-	-10.0 (5.1)*	-	-
γ_{24}	-2.3 (0.6)	-	-	-
γ_{25}	-0.005 (0.002)	-	-	30.0 (7.7)*
R^2(adj)	96%	92%	94%	75%
$\text{mean}_{\lvert res \rvert}$	0.17	0.10	0.02	0.01
$\text{max}_{\lvert res \rvert}$	0.49	0.21	0.09	0.03

Table 1: OLS estimators of the regression parameters. Standard errors are given in parenthesis. A star implies that the coefficient and associated standard error should be multiplied by a factor 10^{-5}.

7 An empirical example

We apply the results discussed in the preceding sections to Kapferer's tailor shop data from a study concerning the observed interactions among workers in a tailor shop in Zambia. A description of the data, as well as the data itself, is presented in Kapferer (1973). The part of the data used here, consists of a symmetric binary matrix of order $n = 39$ and size $R = 446$ representing social relations among 39 of the families.

In the testing procedure proposed, it is assumed that p is known under the null hypothesis. Since this is not the case, we computed the maximum likelihood (M.L.) estimate, $\hat{p} = 0.602$, for the considered data set. In Table 2, computed values of C, are given together with the probabilities $\Pr(T > C \,|n = 39, p = 0.602)$ for the estimated model (3) as well as for the simulated distributions. The probability values computed with the model seem to agree with the probability values obtained from the simulated distributions for T_1 and T_2; they are less than 0.0001 for both the simulated and modeled critical values. For T_4, the modeled critical values are almost twice the simulated which is quite unsatisfactory. Although T_1, T_2 and T_4 yield

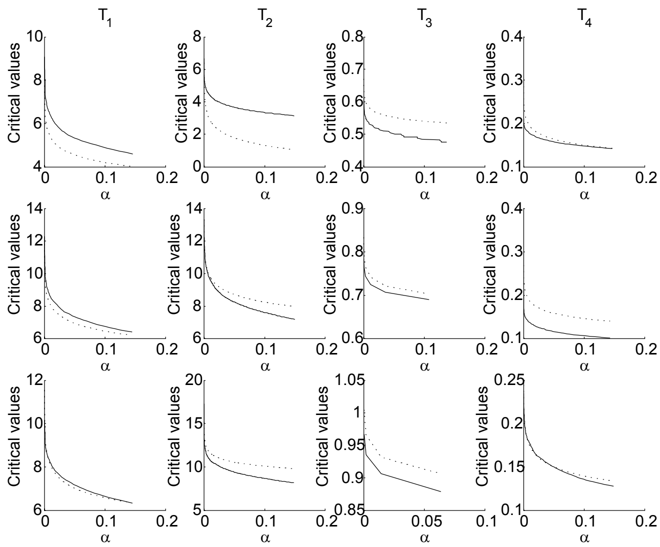

Figure 3: Simulated (solid curve) and approximated (dotted curve) critical values plotted against α for $n = 30$ and $p = 0.1$ (top row), $p = 0.3$ (second row) and $p = 0.6$ (bottom row), respectively.

different probability values, the decision taken is the same. That is, the null hypothesis is rejected and we assume graph centrality, whereas for T_3, the null hypothesis cannot be rejected and we assume that the observed network is non-central. This result indicates that either T_1, T_2 and T_4 or T_3 are not appropriate test statistics for testing graph centrality. Without any further investigation we assume that T_3 yields results that will lead to making the wrong decision, and therefore continue the analysis with T_1, T_2 and T_4.

Since p is unknown and estimated from data, there are several other plausible values of p apart from $\hat{p}$. Depending on which value of p is considered under the null hypothesis, different decisions can be made. It is easily verified that the first two moments of the M.L. estimator are given by

$$E\left(\hat{P}\right) = p$$
$$Var\left(\hat{P}\right) = \frac{2p(1-p)}{n\,(n-1)}.$$

If $n\,(n-1)\,p\,(1-p)$ is sufficiently large, $\hat{P}$ is approximately normally dis-

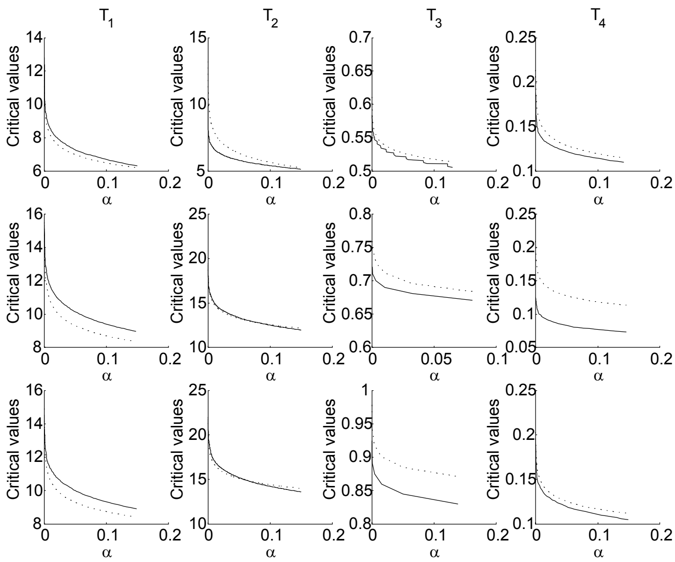

Figure 4: Simulated (solid curve) and approximated (dotted curve) critical values plotted against α for $n = 50$ and $p = 0.1$ (top row), $p = 0.3$ (second row) and $p = 0.6$ (bottom row), respectively.

tributed. Thus, an approximate 95% confidence interval for p is given by

$$\hat{p} \pm 2\sqrt{\frac{\hat{p}\,(1-\hat{p})}{\binom{n}{2}}}.$$

The computed approximate 95% confidence interval for p is $(0.584; 0.620)$. Which of these possible values should be considered as a natural candidate to $\hat{p}$? A joint property of the distributions for the centrality tests considered in this paper is that the probability $\Pr(T > C\,|n)$ increases if $p > 0.5$ decreases. Hence, because we reject the null hypothesis of no centrality when p is replaced with $\hat{p}$, it is sufficient to carry out the test procedure with p equal to the lower bound of the confidence interval, $\hat{p}_{lb}$. If we make the same decision, then the test is consistent for any other p in the confidence interval. However, if we cannot reject the null hypothesis, the test is inconclusive. In Table 2, we see by comparing the probability values for the simulated distributions of the tests, that the null hypothesis is still rejected for T_1, T_2 and T_4 at 5% significance level. Hence, by applying any of the three tests, we would conclude that there is strong evidence of graph centrality both when the critical values are simulated and computed with the proposed model. In Figure **??**, the simulated critical values and critical values obtained from (3) are plotted against α for this combination of n and p. The accuracy is rather poor for T_2 for sufficiently large α, whereas it is satisfactory for T_1 and T_4 for any of the α considered here.

	T_1	T_2	T_3	T_4
T (observed)	13.6	29.6	0.7451	0.1722
$P(T > C \mid n = 39, p = \hat{p} = 0.602)$ (s)	<0.0001	<0.0001	>0.15	<0.0027
$P(T > C \mid n = 39, p = \hat{p} = 0.602)$ (m)	<0.0001	<0.0001	>0.15	<0.0046
$P(T > C \mid n = 39, p = \hat{p}_{lb} = 0.584)$ (s)	<0.0001	<0.0001	-	<0.0026
$P(T > C \mid n = 39, p = \hat{p}_{lb} = 0.584)$ (m)	<0.0001	<0.0001	-	<0.0046

Table 2: Probability values of the four test statistics for Padgett's family data. The probability values are calculated from simulated distributions (s) as well as from distributions obtained by model (2) (m).

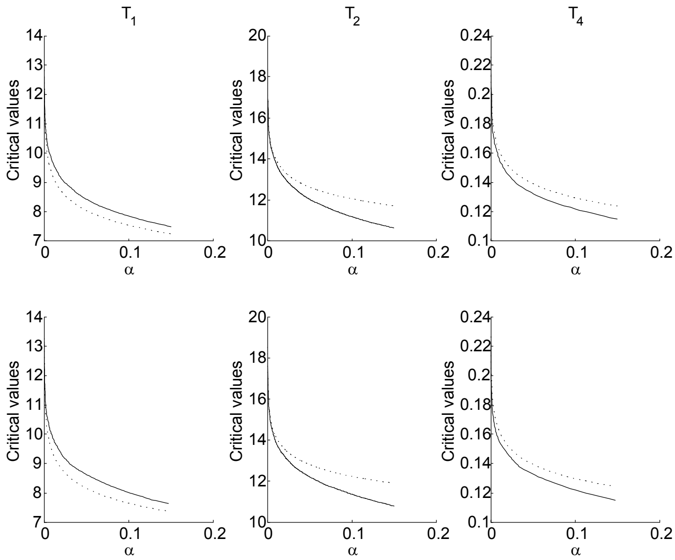

Figure 5: Simulated (solid curve) and approximated (dotted curve) critical values plotted against α for Kapferer's tailor shop data. In top row $p = \hat{p} = 0.602$, in bottom row $p = \hat{p}_{lb} = 0.584$.

8 Concluding remarks

Well-known graph centrality measures are used as tests of graph centrality. The Bernoulli graph model is considered as our null hypothesis model. As our alternative random graph model against which we can test centrality, we use a blockmodel. Some rather tight constraints are imposed on the parameter space for the blockmodel. The number of blocks are restricted to two, and allowing only one actor in the central block. By performing computer simulations, the power of the tests are compared in graphs of order 10, 20 and 50 for various values of model parameters. Although none of the tests is uniformly most powerful, Freeman's index when centrality is based on degree and the maximum of the actor centralities when centrality is based on closeness are more powerful in general. The variance of the actor centralities when centrality is based on degree performs rather well, whereas the variance of the actor level indices when centrality is based on closeness performs very poorly.

Besides comparing the power of the tests, we provide a regression model with critical values as functions of the order of the graph, the edge probability in the Bernoulli graph and significance level for four of the tests which show strong power. Although the predictions of critical values are quite similar to the simulated values, caution has to be taken when predicting for sufficiently large values of the edge probability for three of the tests.

Obvious extensions to the centrality blockmodels considered here, is to revoke the impose of restrictions on the number of the blocks and the order of the blocks.

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