Solutions to 2009-01-15.

① a) $f(x,y)=\begin{cases}\frac{1}{\pi}, & x^2+y^2\le 1\\ 0\end{cases}$, $f(x), f(y)$ - ?

□ $f(x)=\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{\pi}\,dy=\frac{2}{\pi}\sqrt{1-x^2}$ for $-1<x<1$ and

$f(x)=0$ for $|x|\ge 1$.

By symmetry $f(y)=\frac{2}{\pi}\sqrt{1-y^2}$, $-1<y<1$. ■

b) $f(x)\cdot f(y)\ne f(x,y)$

②. $X\sim N(0,1)$, $Y\sim N(0,1)$, $X+Y$ and $X-Y$ are indep. $N(0,2)$ distr.

□ $U=X+Y$, $V=X-Y$. Inversion yields

$X=\frac{U+V}{2}$, $Y=\frac{U-V}{2}$, which implies that

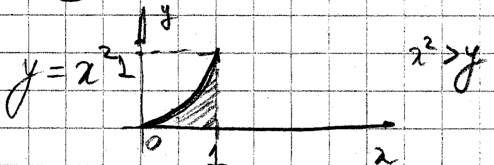
$J=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}=-\frac{1}{2}$. $f_{UV}(u,v)=f_{X,Y}(x,y)=f_{X,Y}\left(\frac{u+v}{2},\frac{u-v}{2}\right)\cdot|J|=$

$=f_X\left(\frac{u+v}{2}\right)\cdot f_Y\left(\frac{u-v}{2}\right)\cdot|J|=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(u+v)^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(u-v)^2}{2}}\cdot\frac{1}{2}=$

$=\frac{1}{\sqrt{2\pi\cdot 2}}e^{-\frac{1}{2}\frac{u^2}{2}}\cdot\frac{1}{\sqrt{2\pi\cdot 2}}e^{-\frac{1}{2}\frac{v^2}{2}}\Rightarrow$ U and V are indep.

$N(0,2)$ distributed r.v. ■

③ a) $P(X>\sqrt{Y})$, $f_{X,Y}(x,y)=x+y$, $0\le x\le 1, 0\le y\le 1$

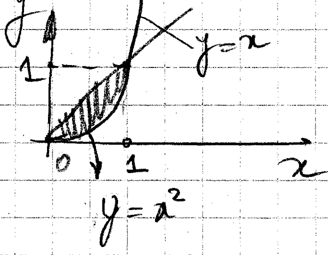


$P(X>\sqrt{Y})=\int_0^1\left[\int_{\sqrt{y}}^1(x+y)\,dx\right]dy=$

$=\int_0^1\left(xy+\frac{1}{2}x^2\right)\Big|_{x=\sqrt{y}}^1 dy=\ldots=\left[\frac{1}{4}y^2+\frac{1}{2}y\right]_0^1-$

$-\frac{2}{5}y^{5/2}\Big|_0^1=\ldots=\frac{15}{20}-\frac{8}{20}=7/20$ ■

6) $P(X^2 < Y < X)$, (X,Y), $f_{X,Y}(x,y) = 2x$, $0 \le x \le 1$, $0 \le y \le 1$



$$\int_0^1 \left[\int_{x^2}^{x} 2x\, dy\right] dx = \int_0^1 \left[2x \left[y\right]_{x^2}^{x}\right] dx =$$

$$\int_0^1 2x\left[x - x^2\right] dx = \int_0^1 2x^2 dx - \int_0^1 2x^3 dx = 2\left[\frac{x^3}{3}\right]_0^1 - 2\left[\frac{x^4}{4}\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$ $\blacksquare$

④ $P(X_T = x) = P(X = x / X > 0) = \dfrac{P(X=x)}{P(X>0)}$, $x = 1, 2, \dots$

□ a) $X \sim Po(\lambda)$, $P(X=x) = e^{-\lambda}\dfrac{\lambda^x}{x!}$, $x = 1, 2, \dots \Rightarrow$

$$P(X_T = x) = \frac{P(X=x)}{P(X>0)} = \frac{P(X=x)}{1 - P(X \le 0)} = \frac{P(X=x)}{1 - e^{-\lambda}} = \frac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})}, \quad x = 1, 2, \dots$$ $\blacksquare$

□ 6) $$E[X_T] \doteq \sum_{x=1}^{\infty} x \cdot P(X_T = x) = \sum_{x=1}^{\infty} x \cdot \frac{P(X=x)}{P(X>0)} =$$

$$= \frac{1}{P(X>0)} \cdot \sum_{x=1}^{\infty} x \cdot P(X=x) = \frac{1}{P(X>0)} \cdot \underbrace{\sum_{x=0}^{\infty} x P(X=x)}_{\lambda,\ \text{since } Po(\lambda)} = \frac{\lambda}{1 - e^{-\lambda}}.$$

$$E[X_T^2] = \sum_{x=1}^{\infty} x^2 \cdot P(X_T = x) = \dots = \frac{E[X^2]}{P(X>0)} = \frac{\lambda^2 + \lambda}{1 - e^{-\lambda}}$$

$$Var[X_T^2] = E[X_T^2] - \left(E[X_T]\right)^2 = \frac{\lambda^2+\lambda}{1-e^{-\lambda}} - \frac{\lambda^2}{(1-e^{-\lambda})^2}$$ $\blacksquare$

⑤ a) $T = \max(X_1, \dots, X_4)$, $X_i \sim Exp(\lambda)$ indep.

□ $F_{X_i}(x) = 1 - e^{-\lambda x} \Rightarrow F_T(x) = \prod_{i=1}^{4} F_{X_i}(x) = \left[1 - e^{-\lambda x}\right]^4.$ $\blacksquare$

6) □ $X \sim Exp(0.5)$, $Y \sim Exp(0.5) \Rightarrow F_X(z) = F_Y(z) = 1 - e^{-2z}$

$Z = \min(X,Y)$ and $F_Z(z) \doteq P(\min(X,Y) \le z) =$

$= 1 - P(X \ge z)P(Y \ge z) = 1 - [1 - F_X(z)][1 - F_Y(z)] =$

$= 1 - e^{-2z} \cdot e^{-2z} = 1 - e^{-4z}$, $z > 0$. So $\alpha = 4$. $\blacksquare$

6 a) $Y \sim Bin(n, X)$, $X \sim U[0,1]$, $E[Y]$-?, $V[Y]$-?

□ $E[Y] = E[E[Y/X]] = E[nX] = nE[X] = 0.5 \cdot n$

$Var[Y] \overset{\circ}{=} Var[E[Y/X]] + E[Var[Y/X]] =$

$= Var[nX] + E[n \cdot X(1-X)] = n^2 \cdot \frac{(b-a)^2}{12} + nE[X(1-X)] =$

$= \frac{n^2}{12} + n(\underbrace{E[X]}_{0.5} - E[X^2]) =$

$Var[X] = E[X^2] - (E[X])^2 \Rightarrow E[X^2] = \underbrace{Var[X]}_{\frac{1}{12}} + \underbrace{(E[X])^2}_{\frac{1}{4}} =$
$= \frac{1}{12} + \frac{3}{12} = \frac{1}{3}$

$= \frac{n^2}{12} + n\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{n^2}{12} + \frac{n}{6}$ ■

b) $f(x,y) = f_{Y/X}(y/x) \cdot f_X(x) = \binom{n}{y} x^y (1-x)^{n-y} \cdot 1$, $0 < x < 1$, $y = \overline{0,n}$.

c) $f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} dx =$

$= \binom{n}{y} \cdot \underbrace{\int_0^1 x^y (1-x)^{n-y} dx}_{\text{kernel of } \beta\text{-distr.}: f(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} x^{\alpha-1} \cdot (1-x)^{\beta-1}} =$

$= \binom{n}{y} \cdot \frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta)}$ where $\alpha = y+1$, $\beta = n-y+1$. ■

7 a) N-approx. $\overset{CLT}{,}$ indep. of observations.

b) $P\left(\sum_{i=1}^{100} X_i \geq 100.4\right) = P\left(\frac{1}{100}\sum_{i=1}^{100} X_i \geq 1.004\right) =$ (by CLT $\bar{X} \overset{Appr}{\sim} N(E[X_i], \frac{\sigma^2}{n})$) $= P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > \frac{1.004-\mu}{\sigma/\sqrt{n}}\right) \approx P\left(Z > \frac{0.004}{0.005}\right) =$

$= 1 - P(Z < 0.8) = 0.2119.$ ■

8 $X \sim t(p)$

a) □ $X = \dfrac{Z}{\sqrt{\frac{V}{p}}}$, $Z \sim N(0,1)$, $V \sim \chi^2(p)$ indep.

$$E[X] = (\text{by indep.}) = \underbrace{\frac{\overbrace{E[Z]}^{=0}}{E[..]}}_{\text{finite}} = 0.$$

To derive $Var[X]$ we first observe that $X^2 \sim F_{1,p}$.

$X^2 = \dfrac{Z^2}{V/p}$ and $Z^2 \sim \chi^2(1)$, $V \sim \chi^2(p)$ $\Rightarrow$ By the properies of F distr. we see that $Var[X] = E[X^2]$ for $E[X^2]$, $X^2 \sim F_{1,p}$. Therefore $Var[X] = E[X^2] = \dfrac{p}{p-2}$, $p > 2$. ■

b) See above, $X^2 = \dfrac{Z^2}{V/p}$, $Z^2 \sim \chi^2(1)$, $V \sim \chi^2(p)$ indep.

c) $X^2 \xrightarrow[p \to \infty]{D} \chi^2_1$.

□ Observe that for $X^2 \sim F_{1,p} = t_p^2$. It is known that $t_p \xrightarrow[p \to \infty]{D} n(0,1)$. Therefore $t_p^2 \xrightarrow[p \to \infty]{} n^2(0,1) \sim \chi^2(1)$. ■