Written exam in Probability Theory, ST701A 7.5 ECTS credits

All solutions should be well motivated. The total amount of points is 100. Grades are assigned as follows. **A** (100-90p), **B** (89-80p), **C** (79-70p), **D** (69-60p), **E** (59-50p), **Fx** (49-30p), **F** (29-0p).

Use the department's paper only. Begin the solution of each assignmet on top of a new paper. Do not use red pencil. Write your full name on each paper.

You are allowed to use a pocket calculator, a table of common distributions from Casella, Berger book, and a reasonable table of mathematical formulas (e.g. TEFYMA).

1.
Let (X, Y) be a point that is uniformly distributed on the unit disc; that is the joint density of X and Y is

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & \text{for} \quad x^2 + y^2 \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

a) Determine the marginal densities of X and Y. 10p

b) Are X and Y independent? Motivate yout answer. 5p

2.
Let X and Y be independent $N(0,1)$-distributed random variables. Show that $X+Y$ and $X-Y$ are independent $N(0,2)$-distributed random variables. 15p

3.

a) Find $P(X > \sqrt{Y})$ if X and Y are jointly distributed with pdf

$$f(x,y) = x + y, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

5p

b) Find $P(X^2 < Y < X)$ if X and Y are jointly distributed with pdf

$$f(x,y) = 2x, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

5p

4.

a) A paralel system is one that functions as long as at least one component of it functions. A particular parallel system is composed of four independent components, each of which has a lifelength with an exponential(λ) distribution. The lifetime of the system is the maximum of the individual lifelengths. What is the distribution of the lifetime of the system? 5p

b) Determine constant a so that the function

$$F(z) = 1 - e^{-az}, \quad z > 0$$

will be the distribution function for the minimum, $Z = \min(X, Y)$ of two independent exponetially distributed random variables, both with the expectation 0.5. 5p

5.

a) For the hierarchical model

$$Y \sim \text{binomial}(n, X) \quad \text{and} \quad X \sim \text{uniform}(0, 1)$$

find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. 5p

($Hint$: Use $\mathrm{E}(Y) = \mathrm{E}(\mathrm{E}(Y|X)), \mathrm{Var}(Y) = \mathrm{E}(\mathrm{Var}(Y|X)) + \mathrm{Var}(\mathrm{E}(Y|X))$)

b) Find the joint distribution of X and Y. 5p

c) Find the marginal distribution of Y. 10p

6.

A manufacturer of booklets packages them in boxes of 100. It is known that, on the average, the boklets weigh 1 ounce, with a standard deviation. 0f 0.05 ounce. The manufacturer is interested in calculating

$$P(100 \quad \text{booklets weigh more than 100.4 once}),$$

a number that would help detect whether too many booklets are being put in a box.

a) Determine the approximate value of this probability. 10p

b) Explaine the detailes of the approximation from part a), mention any relevant theorems or assumptions needed. 5p

7. Let X be a random variable with a Student's t distribution with p degrees of freedom.

a) Derive the mean and variance of X. 5p

b) Show that X^2 has an F distribution with 1 and p degrees of freedom. 5p

c) Show that, as $p \to \infty$, X^2 converges in distribution to χ^2_1 random variable. 5p