Written exam in Statistical Inference, ST703A 7.5 ECTS credits

2010-03-24

All solutions should be well motivated. The total amount of points is 100. Grades are assigned as follows. **A** (100-90p), **B** (89-80p), **C** (79-70p), **D** (69-60p), **E** (59-50p), **Fx** (49-30p), **F** (29-0p).

Use the department's paper only. Begin the solution of each assignment on top of a new paper. Do not use red pencil.

You are allowed to use a pocket calculator, a table of common distributions from Casella, Berger book, and a reasonable table of mathematical formulas (e.g. TEFYMA).

Review of the exam will be held on Thursday 15th of April from 10.00 to 12.00 in B705.

1.

a) Describe the sufficiency principle as a method of data reduction. Explain how to find a sufficient statistics by inspection of the pdf or pmf of the sample. 3p

b) Let $X_1, \dots, X_n$ be a sequence of independent Bernoulli random variables with $P(X_i = 1) = \theta$. Verify that $T = \sum_{i=1}^n X_i$ is sufficient statistics for θ. 8p

c) Let $X_1, \dots, X_n$ be a random sample from a $n(\theta, 1)$ population. Find a minimal sufficient statistics for θ. 8p

2. Let $X_1 \dots, X_n$ be a random sample from a population with mean μ and variance σ^2.

a) Show that the estimator $\sum_{i=1}^n a_i X_i$ is an unbiased estimator of μ if $\sum_{i=1}^n a_i = 1$. 3p

b) Among the unbiased estimators of this form find the one with minimum variance, and calculate the variance.
Hint: Add and subtract the mean of a_is, $1/n$ in $\sum_{i=1}^n a_i^2$ when minimizing it subject to the constraint $\sum_{i=1}^n a_i = 1$. 5p

3. Let $X_1, \dots, X_n$ be a sample of iid observations from $n(\theta, \sigma^2)$ population, σ^2 is known. An LRT of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is a test that reject H_0 if $|\bar{X} - \theta_0|/(\sigma/\sqrt{n}) > c$.

a) Find an expression, in terms of standard normal probabilities, for the power function of this test. 7p

b) The experimenter desires a Type I Error probability of 0.05 and a Type II Error probability of 0.25 at $\theta = \theta_0 + \sigma$. Find values of n and c that will achieve this. 7p

4. Let $X_1, \dots, X_n$ be an iid sample from an exponential distribution with the density function

$$f(x|\tau) = \frac{1}{\tau} e^{-x/\tau}, \quad 0 \leq x < \infty.$$

a) Find the maximum likelihood estimator (mle) of τ. 3p

b) Show that the mle is unbiased, and find its exact variance. 5p

c) Use the central limit theorem to find a normal approximation to the sampling distribution of mle of τ. 5p

d) Is there any other unbiased estimate $\tilde{\tau}$ with smaller variance? 6p
Hint: Use Cramér-Rao inequality

$$\mathrm{Var}(\tilde{\tau}) \geq \frac{1}{I(\tau)} = \frac{1}{n\Big\{-\mathrm{E}\Big[\frac{\partial^2}{\partial \tau^2}\log f(X_i|\tau)\Big]\Big\}}.$$

e) Using the results of (a) and (b) find the form of an approximate confidence interval for τ. 6p

5. The independent random variables $X_1, \ldots, X_n$ have the common distribution

$$P(X_i \leq x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ (x/\beta)^\alpha & \text{if} \quad 0 < x < \beta, \\ 1 & \text{if} \quad x \geq \beta \end{cases}$$

If α is a known constant, $\alpha_0 > 0$ find an upper confidence limit for β with confidence level 0.95.
Hint: Use the MLE of β, $\hat{\beta}_{ML} = \max\{X_1, \ldots, X_n\}$ and observe that, since β is a scale parameter, $\max\{X_1, \ldots, X_n\}/\beta$ is a pivot. 7p

6. Let $X_1, \ldots, X_n$ be iid $n(\theta, \sigma^2)$. Consider testing

$$H_0 : \theta \leq \theta_0 \quad \text{vs} \quad H_1 : \theta > \theta_0.$$

a) If σ^2 is known, show that the test that rejects H_0 when $\bar{X} > \theta_0 + z_\alpha\sqrt{\sigma^2/n}$ is a test of size α. 4p

b) Show that the test in (a) can be derived as a likelihood ratio test. 7p

c) Show that the test in (a) is a uniformly most powerful (UMP) test. 6p
Hint. Use Karlin-Rubin Thm: Consider testing $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$. If T is *sufficient* statistics for θ and the family of pdfs $\{g(t|\theta) : \theta \in \Theta\}$ of T has a monotone likelihood ratio (nonincreasing or nondecreasing function of t), then for any t_0, the test that rejects H_0 if and only if $T > t_0$ is a UMP level α test, where $\alpha = P_{\theta_0}(T > t_0)$. Use the sufficiency of $\bar{X}$.

7.

a) Explain the principal difference between the classical confidence set and Bayes credible set, and for each approach suggest a technique for constructing the set. 5p

b) In the computer exercise we discussed the interval estimation based on the bootstrap resampling. Explain the idea behind this approach. 5p