Lec 7: Nested Designs and Split-plot Designs

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November 28, 2011

Two-stage nested design

An arrangement of experiment with the levels of factor B under the levels of factor A.

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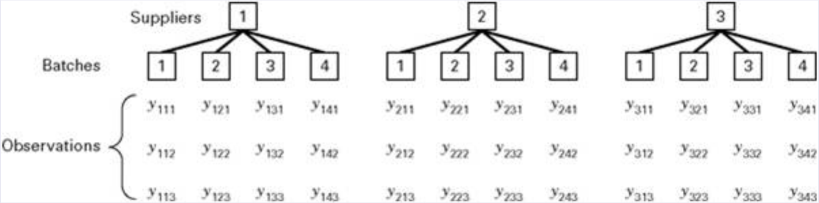
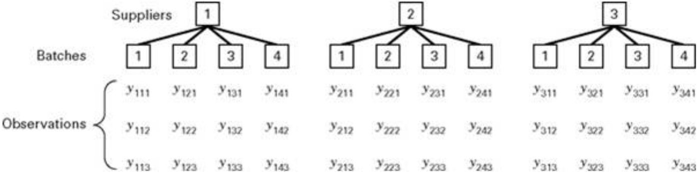
An arrangement of experiment with the levels of factor B under the levels of factor A.

Example



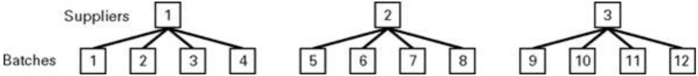
Nested Factors



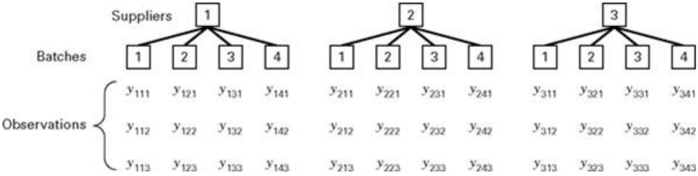
Question?

Why is this a nested design, rather than a factorial design?

Nested Factors



Statistical Model for Nested Design



ANOVA table

■ **TABLE 14.2**
Analysis of Variance Table for the Two-Stage Nested Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square
A	$bn \sum (\bar{y}_{i..} - \bar{y}_{...})^2$	$a - 1$	MS_A
B within A	$n \sum \sum (\bar{y}_{ij.} - \bar{y}_{i..})^2$	$a(b - 1)$	$MS_{B(A)}$
Error	$\sum \sum \sum (y_{ijk} - \bar{y}_{ij.})^2$	$ab(n - 1)$	MS_E
Total	$\sum \sum \sum (y_{ijk} - \bar{y}_{...})^2$	$abn - 1$	

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The appropriate statistics for testing the effects of factors A and B depend on whether A and B are fixed or random.

Fixed or random?

- A: Supplier; B(A): Batch

Fixed or random?

- A: Supplier; B(A): Batch
- A: Year; B(A): Experiment

Fixed or random?

- A: Supplier; B(A): Batch
- A: Year; B(A): Experiment
- A: School; B(A): Pupil

Fixed or random?

- A: Supplier; B(A): Batch
- A: Year; B(A): Experiment
- A: School; B(A): Pupil
- A: Shop; B(A): Product

Expected Values

■ **TABLE 14.1**

Expected Mean Squares in the Two-Stage Nested Design

$E(MS)$	A Fixed B Fixed	A Fixed B Random	A Random B Random
$E(MS_A)$	$\sigma^2 + \frac{bn\sum\tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + \frac{bn\sum\tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + bn\sigma_\tau^2$
$E(MS_{B(A)})$	$\sigma^2 + \frac{n\sum\sum\beta_{j(i)}^2}{a(b-1)}$	$\sigma^2 + n\sigma_\beta^2$	$\sigma^2 + n\sigma_\beta^2$
$E(MS_E)$	σ^2	σ^2	σ^2

Expected Values

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$E(MS_E)$	σ^2	σ^2	σ^2

This table gives information about how to test hypotheses and how to estimate variance components.

Example

■ **TABLE 14.3**
Coded Purity Data for Example 14.1 (Code: y_{ijk} = Purity − 93)

		Supplier 1				Supplier 2				Supplier 3			
	Batches	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**
		1	−2	−2	1	1	0	−1	0	2	−2	1	3
		−1	−3	0	4	−2	4	0	3	4	0	−1	2
		0	−4	1	0	−3	2	−2	2	0	2	2	1
Batch totals	$y_{ij.}$	0	−9	−1	5	−4	6	−3	5	6	0	2	6
Supplier totals	$y_{i..}$	−5				4				14			

ANOVA

■ TABLE 14.4
Analysis of Variance for the Data in Example 14.1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Square	F_0	P-Value
Suppliers	15.06	2	7.53	$\sigma^2 + 3\sigma_\beta^2 + 6\sum\tau_i^2$	0.97	0.42
Batches (within suppliers)	69.92	9	7.77	$\sigma^2 + 3\sigma_\beta^2$	2.94	0.02
Error	63.33	24	2.64	σ^2		
Total	148.31	35				

ANOVA

TABLE 14.5

Incorrect Analysis of the Two-Stage Nested Design in Example 14.1 as a Factorial (Suppliers Fixed, Batches Random)

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Suppliers (*S*)	15.06	2	7.53	1.02	0.42
Batches (*B*)	25.64	3	8.55	3.24	0.04
$S \times B$ interaction	44.28	6	7.38	2.80	0.03
Error	63.33	24	2.64		
Total	148.31	35			

Compare

TABLE 14.4
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$S \times B$ interaction	44.28	6	7.38	2.80	0.03
Error	63.33	24	2.64		
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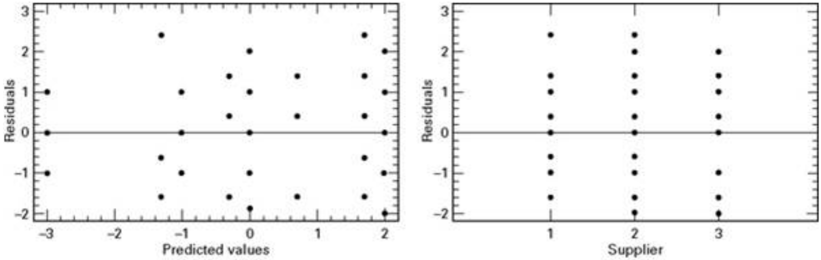
Model validation

Model validation



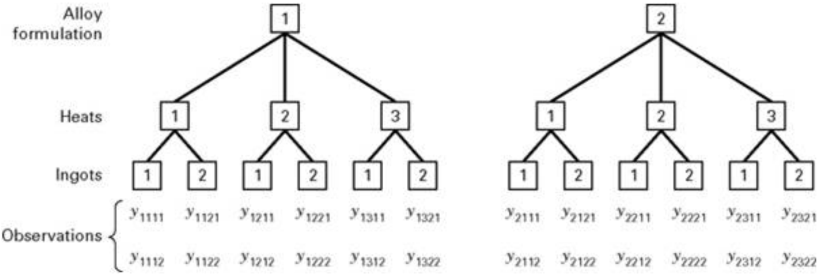
(a) Plot of residuals versus the predicted values

(b) Plot of residuals versus supplier

Three Stage Nested Design

Example

- Suppose a foundry wishes to investigate the hardness of 2 formulations of a metal alloy.
- 3 heat of each alloy formulation are prepared.
- 2 ingots are selected at random.



■ TABLE 14.7

Analysis of Variance for the Three-Stage Nested Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square
A	$bcn \sum_i (\bar{y}_{i...} - \bar{y}_{....})^2$	$a - 1$	MS_A
B (within A)	$cn \sum_i \sum_j (\bar{y}_{ij..} - \bar{y}_{i...})^2$	$a(b - 1)$	$MS_{B(A)}$
C (within B)	$n \sum_i \sum_j \sum_k (\bar{y}_{ijk.} - \bar{y}_{ij..})^2$	$ab(c - 1)$	$MS_{C(B)}$
Error	$\sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \bar{y}_{ijk.})^2$	$abc(n - 1)$	MS_E
Total	$\sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \bar{y}_{....})^2$	$abcn - 1$	

■ TABLE 14.8

Expected Mean Squares for a Three-Stage Nested Design with *A* and *B* Fixed and *C* Random

Model Term	Expected Mean Square
τ_i	$\sigma^2 + n\sigma_\gamma^2 + \frac{bcn \sum \tau_i^2}{a-1}$
$\beta_{j(i)}$	$\sigma^2 + n\sigma_\gamma^2 + \frac{cn \sum \sum \beta_{j(i)}^2}{a(b-1)}$
$\gamma_{k(ij)}$	$\sigma^2 + n\sigma_\gamma^2$
$\epsilon_{l(ijk)}$	σ^2

Nested-factorial Designs

Example

- An industrial engineer is studying the hand insertion of electronic components on printed circuit boards to improve the speed of the assembly operation.
- 3 assembly fixtures.
- 2 workplace layout.
- 4 operator for each fixture-layout combination.

TABLE 14.9
Assembly Time Data for Example 14.2

	Layout 1				Layout 2				
Operator	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**	$y_{i..}$
Fixture 1	22	23	28	25	26	27	28	24	404
	24	24	29	23	28	25	25	23	
Fixture 2	30	29	30	27	29	30	24	28	447
	27	28	32	25	28	27	23	30	
Fixture 3	25	24	27	26	27	26	24	28	401
	21	22	25	23	25	24	27	27	
Operator totals, $y_{.jk.}$	149	150	171	149	163	159	151	160	
Layout totals, $y_{.j..}$		619				633			$1252 = y_{....}$

Mixed Interactions

Interactions between a fixed factor and a random factor

- Restricted model: The sums of the interaction effects over the level of the fixed factor equals 0.
- Unrestricted model.

■ TABLE 14.10

Expected Mean Squares for Example 14.2

Model Term	Expected Mean Square
τ_i	$\sigma^2 + 2\sigma_{\tau\gamma}^2 + 8\sum \tau_i^2$
β_j	$\sigma^2 + 6\sigma_{\gamma}^2 + 24\sum \beta_j^2$
$\gamma_{k(j)}$	$\sigma^2 + 6\sigma_{\gamma}^2$
$(\tau\beta)_{ij}$	$\sigma^2 + 2\sigma_{\tau\gamma}^2 + 4\sum\sum (\tau\beta)_{ij}^2$
$(\tau\gamma)_{ik(j)}$	$\sigma^2 + 2\sigma_{\tau\gamma}^2$
$\epsilon_{(ijk)l}$	σ^2

■ TABLE 14.11
Analysis of Variance for Example 14.2

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Fixtures (*F*)	82.80	2	41.40	7.54	0.01
Layouts (*L*)	4.08	1	4.09	0.34	0.58
Operators (within layouts), *O*(*L*)	71.91	6	11.99	5.15	<0.01
FL	19.04	2	9.52	1.73	0.22
FO(*L*)	65.84	12	5.49	2.36	0.04
Error	56.00	24	2.33		
Total	299.67	47			

■ **TABLE 14.12**

Minitab Balanced ANOVA Analysis of Example 14.2 Using the Restricted Model

Analysis of Variance (Balanced Designs)

Factor	Type	Levels	Values			
Layout	fixed	2	1	2		
Operator(Layout)	random	4	1	2	3	4
Fixture	fixed	3	1	2	3	

Analysis of Variance for Time

Source	DF	SS	MS	F	P
Layout	1	4.083	4.083	0.34	0.581
Operator(Layout)	6	71.917	11.986	5.14	0.002
Fixture	2	82.792	41.396	7.55	0.008
Layout*Fixture	2	19.042	9.521	1.74	0.218
Fixture*Operator(Layout)	12	65.833	5.486	2.35	0.036
Error	24	56.000	2.333		
Total	47	299.667			

Source	Variance component	Error term	Expected Mean Square for Each Term (using restricted model)
1 Layout		2	(6) + 6(2) + 24Q[1]
2 Operator(Layout)	1.609	6	(6) + 6(2)
3 Fixture		5	(6) + 2(5) + 16Q[3]
4 Layout*Fixture		5	(6) + 2(5) + 8Q[4]
5 Fixture*Operator(Layout)	1.576	6	(6) + 2(5)
6 Error	2.333		(6)

Split-Plot design

In some multifactor factorial experiments, we may be unable to completely randomize the order of the runs.

Why is this a split-plot design?

Example

- A paper manufacturer wish to study the effect of two factors on the tensile strength of the paper.
- 3 pulp preparation methods; 4 temperatures; 3 replicates

TABLE 14.14
The Experiment on the Tensile Strength of Paper

	Replicate (or Block) 1			Replicate (or Block) 2			Replicate (or Block) 3		
Pulp Preparation Method	**1**	**2**	**3**	**1**	**2**	**3**	**1**	**2**	**3**
Temperature (°F)									
200	30	34	29	28	31	31	31	35	32
225	35	41	26	32	36	30	37	40	34
250	37	38	33	40	42	32	41	39	39
275	36	42	36	41	40	40	40	44	45

■ **TABLE 14.15**
Expected Mean Squares for Split-Plot Design

	Model Term	Expected Mean Square
Whole plot	τ_i	$\sigma^2 + ab\sigma_\tau^2$
	β_j	$\sigma^2 + b\sigma_{\tau\beta}^2 + \frac{rb\sum \beta_j^2}{a-1}$
	$(\tau\beta)_{ij}$	$\sigma^2 + b\sigma_{\tau\beta}^2$
Subplot	γ_k	$\sigma^2 + a\sigma_{\tau\gamma}^2 + \frac{ra\sum \gamma_k^2}{(b-1)}$
	$(\tau\gamma)_{ik}$	$\sigma^2 + a\sigma_{\tau\gamma}^2$
	$(\beta\gamma)_{jk}$	$\sigma^2 + \sigma_{\tau\beta\gamma}^2 + \frac{r\sum\sum (\beta\gamma)_{jk}^2}{(a-1)(b-1)}$
	$(\tau\beta\gamma)_{ijk}$	$\sigma^2 + \sigma_{\tau\beta\gamma}^2$
	$\epsilon_{(ijk)h}$	σ^2 (not estimable)

■ TABLE 14.16

Analysis of Variance for the Split-Plot Design Using the Tensile Strength Data from Table 14.14

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Replicates (or blocks)	77.55	2	38.78		
Preparation method (*A*)	128.39	2	64.20	7.08	0.05
Whole plot error (replicates (or blocks) × *A*)	36.28	4	9.07		
Temperature (*B*)	434.08	3	144.69	41.94	<0.01
Replicates (or blocks) × *B*	20.67	6	3.45		
AB	75.17	6	12.53	2.96	0.05
Subplot error (replicates (or blocks) × *AB*)	50.83	12	4.24		
Total	822.97	35			

Split-split-plot design

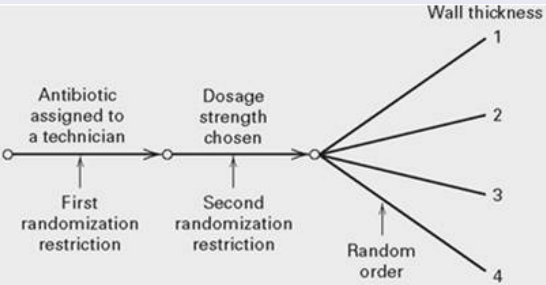
Split-split-plot design

Example

- A researcher is studying the absorbtion times of a particular type of antibiotic capsule.
- 3 technicians.
- 3 dosage strengths.
- 4 capsule wall thicknesses.
- 4 replicates (4 days).

Split-split-plot design

Example



		Technician								
		1			2			3		
Blocks	Dosage strength	1	2	3	1	2	3	1	2	3

The strip-split-plot design