Lec 6: Random Effects Models

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November 25, 2011

Random Factor

The treatment levels are a random sample from a lager population of treatment effect.

Example

- A textile company weaves on a fabric on a large number of looms,
- Response: the strength of the fabric,
- Factor: looms,
- 4 looms

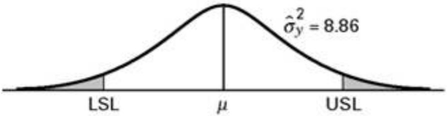
■ TABLE 13.1

Strength Data for Example 13.1

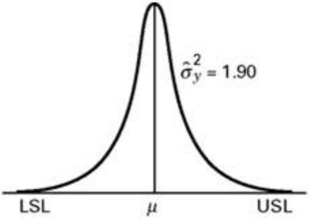
	Observations				
Looms	1	2	3	4	$y_{i.}$
1	98	97	99	96	390
2	91	90	93	92	366
3	96	95	97	95	383
4	95	96	99	98	388
					$1527 = y_{..}$

■ TABLE 13.2
Analysis of Variance for the Strength Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Looms	89.19	3	29.73	15.68	<0.001
Error	22.75	12	1.90		
Total	111.94	15			



(a) Variability of process output.



(b) Variability of process output if $\sigma_\tau^2 = 0$.

Two random factors

Gauge capability study

Statistically designed experiments are frequently used to investigate the sources of variability that affect a system. A common industrial application is to use a designed experiment to study the components of variability in a measurement system.

Example for two random factors model

- An instrument or gauge is used to measure a critical dimension on a part.
- 20 parts have been selected form the production process.
- 3 operators.
- Measure each part 2 times.

■ **TABLE 13.3**
The Measurement Systems Capability Experiment in Example 13.2

Part Number	Operator 1		Operator 2		Operator 3	
1	21	20	20	20	19	21
2	24	23	24	24	23	24
3	20	21	19	21	20	22
4	27	27	28	26	27	28
5	19	18	19	18	18	21
6	23	21	24	21	23	22
7	22	21	22	24	22	20
8	19	17	18	20	19	18
9	24	23	25	23	24	24
10	25	23	26	25	24	25
11	21	20	20	20	21	20
12	18	19	17	19	18	19
13	23	25	25	25	25	25
14	24	24	23	25	24	25
15	29	30	30	28	31	30
16	26	26	25	26	25	27
17	20	20	19	20	20	20
18	19	21	19	19	21	23
19	25	26	25	24	25	25
20	19	19	18	17	19	17

■ TABLE 13.4

Analysis of Variance (Minitab Balanced ANOVA) for Example 13.2

Analysis of Variance (Balanced Designs)

Factor	Type	Levels	Values						
part	random	20	1	2	3	4	5	6	7
			8	9	10	11	12	13	14
			15	16	17	18	19	20	
operator	random	3	1	2	3				

Analysis of Variance for y

Source	DF	SS	MS	F	P
part	19	1185.425	62.391	87.65	0.000
operator	2	2.617	1.308	1.84	0.173
part*operator	38	27.050	0.712	0.72	0.861
Error	60	59.500	0.992		
Total	119	1274.592			

Source	Variance component	Error term	Expected Mean Square for Each Term (using unrestricted model)
1 part	10.2798	3	(4) + 2(3) + 6(1)
2 operator	0.0149	3	(4) + 2(3) + 40(2)
3 part*operator	-0.1399	4	(4) + 2(3)
4 Error	0.9917		(4)

■ TABLE 13.5

Analysis of Variance for the Reduced Model, Example 13.2

Analysis of Variance (Balanced Designs)

Factor	Type	Levels	Values						
part	random	20	1	2	3	4	5	6	7
			8	9	10	11	12	13	14
			15	16	17	18	19	20	
operator	random	3	1	2	3				

Analysis of Variance for y

Source	DF	SS	MS	F	P
part	19	1185.425	62.391	70.64	0.000
operator	2	2.617	1.308	1.48	0.232
Error	98	86.550	0.883		
Total	119	1274.592			

Source	Variance component	Error term	Expected Mean Square for Each Term (using unrestricted model)
1 part	10.2513	3	(3) + 6(1)
2 operator	0.0106	3	(3) + 40(2)
3 error	0.8832		(3)

Moment estimators of variance component

Pros

- Easy to calculate in sample case

Moment estimators of variance component

Pros

- Easy to calculate in sample case

Cons

- May give negative estimates.
- No nice distributional properties.

Mixed model

Mixed model

Example

- An instrument or gauge is used to measure a critical dimension on a part.
- 20 parts have been selected form the production process.
- 3 operators, suppose only three operator use this gauge.
- Measure each part 2 times.

Mixed model

Mixed model

Result

■ **TABLE 13.6**

Analysis of Variance (Minitab) for the Mixed Model in Example 13.3. The Restricted Model is Assumed

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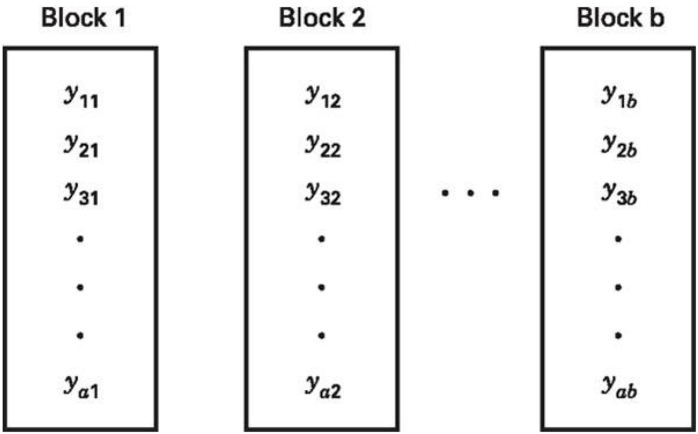
Analysis of Variance (Balanced Designs)

Factor     Type   Levels Values
part       random     20      1      2      3      4      5      6      7
                              8      9     10     11     12     13     14
                             15     16     17     18     19     20
operator   fixed       3      1      2      3

Analysis of Variance for y

Source             DF        SS        MS        F        P
part               19  1185.425    62.391    62.92    0.000
operator            2     2.617     1.308     1.84    0.173
part*operator      38    27.050     0.712     0.72    0.861
Error              60    59.500     0.992
Total             119  1274.592

Source            Variance  Error  Expected Mean Square for Each Term
                  component  term  (using restricted model)
1 part             10.2332     4   (4) + 6(1)
2 operator                     3   (4) + 2(3) + 40Q[2]
3 part*operator    -0.1399     4   (4) + 2(3)
4 Error             0.9917         (4)
```


Block 1
Block 2
Block b
y_{11}
y_{21}
y_{31}
y_{a1}
y_{12}
y_{22}
y_{32}
y_{a2}
y_{1b}
y_{2b}
y_{3b}
y_{ab}

BIBD

In balanced incomplete block experiments, all pairs of treatment appear together in a block equally times (λ)

Block	I	II	III	IV	V
	A	A	A	A	B
	B	B	B	C	C
	C	C	D	D	D
	D	E	E	E	E

BIBD

Block	I	II	III	IV	V
	A	A	A	A	B
	B	B	B	C	C
	C	C	D	D	D
	D	E	E	E	E

- λ: number of concurrences, $\lambda = 3$
- t: number of treatment, $t = 5$
- r: number of replicates, $r = 4$
- k: block size, $k = 4$

General cases

Block	Treatment
I	A
I	B
II	A
II	C
III	A
III	B
III	C
IV	A
IV	B
IV	C

$$\mathbf{Y} = \mathbf{1}\mu + \mathbf{X}\tau + \mathbf{Z}\beta + \varepsilon$$

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The incidence matrix

Block	Treatment
I	A
I	B
II	A
II	C
III	A
III	B
III	C
IV	A
IV	B
IV	C

The incidence matrix $\mathbf{N_0}$

$$\mathbf{N_0} = \mathbf{X}'\mathbf{Z} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$\mathbf{N_0}$ has one row for each treatment and one column for each block.

Replication and block size vector

Block	Treatment
I	A
I	B
II	A
II	C
III	A
III	B
III	C
IV	A
IV	B
IV	C

Also define the replication vector $\mathbf{r}$

$$\mathbf{r} = \mathbf{X}'\mathbf{1} = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$$

and the block size vector $\mathbf{k}$

$$\mathbf{k} = \mathbf{Z}'\mathbf{1} = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

The concurrence matrix

Block	Treatment
I	A
I	B
II	A
II	C
III	A
III	B
III	C
IV	A
IV	B
IV	C

The concurrence matrix $\mathbf{N_0 N_0'}$ is

$$\mathbf{N_0 N_0'} = \begin{pmatrix} 4 & 3 & 3 \\ 3 & 3 & 2 \\ 3 & 2 & 3 \end{pmatrix}$$

The diagonal equals to $\mathbf{r}$.
The other elements are the number of times pairs of treatments occur together in a block

Example

- A chemical engineer thinks that the time of reaction for chemical process is a function of the type of catalyst employed.
- Treatment factor: Catalyst; 4 levels
- Blocking factor: Batch of material

TABLE 4.21
Balanced Incomplete Block Design for Catalyst Experiment

Treatment (Catalyst)	Block (Batch of Raw Material) 1	2	3	4	$y_{i.}$
1	73	74	—	71	218
2	—	75	67	72	214
3	73	75	68	—	216
4	75	—	72	75	222
$y_{.j}$	221	224	207	218	$870 = y_{..}$

Parameter	Intrablock Estimate	Interblock Estimate	Combined Estimate
τ_1	−1.12	10.50	−1.09
τ_2	−0.88	−3.50	−0.88
τ_3	−0.50	−0.50	−0.50
τ_4	2.50	−6.50	2.47