Lec 5: Factorial Experiment

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Example

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- A: Types of material, 3 levels.
- B: Temperatures, 3 levels.

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In general, factorial designs are most efficient for the study of the effects of two or more factors.

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In each complete trial or replication of the experiment all possible combinations of the levels of the factors are investigated.

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Question

What's the advantage of the factorial design comparing with two-single factor experient?

Advantages of factorial design

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- The two-factor factorial gives estimates with higher precision (given the same number of experimental units).
- The two-factor factorial experiment makes it possible to detect interactions.

Example

■ TABLE 5.1

Life (in hours) Data for the Battery Design Example

Material Type	Temperature (°F) 15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

■ **TABLE 5.2**

General Arrangement for a Two-Factor Factorial Design

		Factor B			
		1	2	. . .	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	$\vdots$				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

■ TABLE 5.3

The Analysis of Variance Table for the Two-Factor Factorial, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A treatments	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B treatments	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	$MS_E = \frac{SS_E}{ab(n-1)}$	
Total	SS_T	$abn-1$		

Manual Calculation

$$SS_A = \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{...}^2}{abn}$$

$$SS_B = \frac{1}{an}\sum_{j=1}^{b} y_{.j.}^2 - \frac{y_{...}^2}{abn}$$

$$SS_{\text{Subtotals}} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2 - \frac{y_{...}^2}{abn}$$

$$SS_{AB} = SS_{\text{Subtotals}} - SS_A - SS_B$$

$$SS_E = SS_T - SS_{AB} - SS_A - SS_B$$

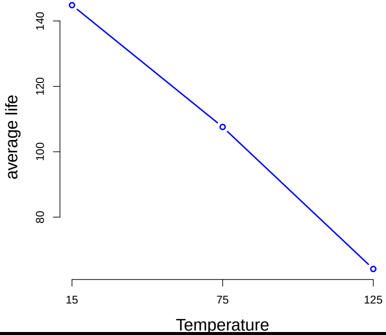
Results for the example

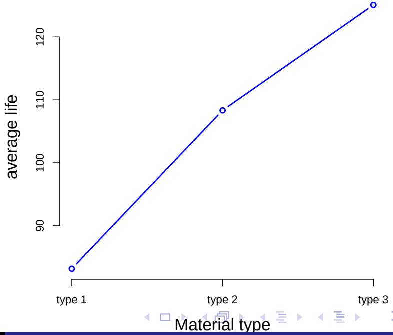
■ **TABLE 5.5**

Analysis of Variance for Battery Life Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Material types	10,683.72	2	5,341.86	7.91	0.0020
Temperature	39,118.72	2	19,559.36	28.97	0.0001
Interaction	9,613.78	4	2,403.44	3.56	0.0186
Error	18,230.75	27	675.21		
Total	77,646.97	35			

	15^o F	70^o F	125^o F	
Type 1	134.75	57.25	57.5	83.16667
Type 2	155.75	119.75	49.5	108.33333
Type 3	144	145.75	85.5	125.08333
	144.83333	107.58333	64.16667	





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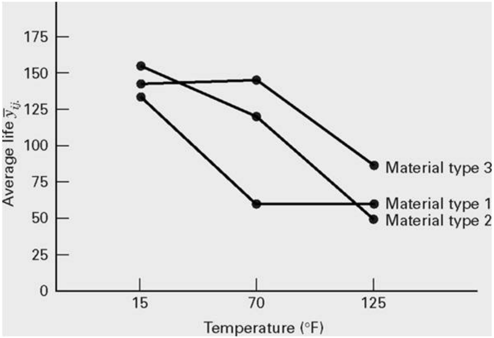

Multiple Comparison

- When interaction is significant, comparisons between the means of one factor (A) may be obscured by the interaction (AB).
- One approach to this situation is to fix factor B at a specific level and apply Tukey's test to the means of factor A at that level.

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Example

$$T_{0.05} = q_{0.05}(a, f)\sqrt{\frac{MS_E}{n}} = q_{0.05}(3, 27)\sqrt{\frac{MS_E}{4}} = 45.71$$

$$\bar{y}_{13.} - \bar{y}_{23.} = 8 \quad \bar{y}_{33.} - \bar{y}_{23.} = 36 \quad \bar{y}_{33.} - \bar{y}_{13.} = 28$$

No interaction in a two-factor model

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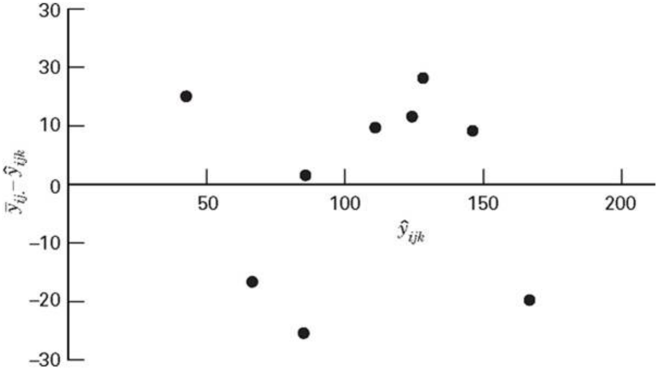
Total	77,646.97	35			

TABLE 5.8

Analysis of Variance for Battery Life Data Assuming No Interaction

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-value
Material types	10,683.72	2	5,341.86	5.95	0.0065
Temperature	39,118.72	2	19,559.36	21.78	0.0001
Error	27,844.52	31	898.21		
Total	77,646.96	35			

Model adequacy checking



One observation per cell case

Model 1

$$y_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ij}$$

$$i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b$$

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The Analysis of Variance Table for the Two-Factor Factorial, Fixed Effects Model

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$$i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b$$

Model 2

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$$

$$i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b$$

TABLE 5.9

Analysis of Variance for a Two-Factor Model, One Observation per Cell

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Square
Rows (A)	$\sum_{i=1}^{a} \frac{y_{i.}^2}{b} - \frac{y_{..}^2}{ab}$	$a - 1$	MS_A	$\sigma^2 + \frac{b\sum \tau_i^2}{a-1}$
Columns (B)	$\sum_{j=1}^{b} \frac{y_{.j}^2}{a} - \frac{y_{..}^2}{ab}$	$b - 1$	MS_B	$\sigma^2 + \frac{a\sum \beta_j^2}{b-1}$
Residual or AB	Subtraction	$(a-1)(b-1)$	MS_{Residual}	$\sigma^2 + \frac{\sum\sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$
Total	$\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{ab}$	$ab - 1$		

Tukey single-degree-of-freedom test

It is helpful in determining whether interaction is present or not. The procedure assumes $(\tau\beta)_{ij} = \gamma\tau_i\beta_j$. Then the test partitions the residual sum of squares into two part. One is

$$SS_N = \frac{[\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}y_{i.}y_{.j} - y_{..}(SS_A + SS_B + \frac{y_{..}^2}{ab})]^2}{abSS_ASS_B},$$

with 1 degree of freedom, and

$$SS_{Error} = SS_{residual} - SS_N,$$

with $(a-1)(b-1)-1$ degrees of freedom. To test the present of interaction, we compute

$$F_0 = \frac{SS_N}{SS_{Error}/[(a-1)(b-1)-1]}$$

If $F_0 > F_{\alpha,1,(a-1)(b-1)-1}$, the hypothesis of no interaction must be rejected.

Question

Is the two-factor factorial model with one observation per cell identical to the randomized complete block model?

Blocking in a Factorial Design

Example

- Study of the battery life vs the factors temperatures and types of material.
- A: Types of material, 3 levels.
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- 9 combinations of A and B.
- 4 observations per combination.
- In total 36 observations.

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- Study of the battery life vs the factors temperatures and types of material.
- A: Types of material, 3 levels.
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- 9 combinations of A and B.
- 4 observations per combination.
- In total 36 observations.
- 4 blocks.

■ **TABLE 5.20**

Analysis of Variance for a Two-Factor Factorial in a Randomized Complete Block

Source of Variation	Sum of Squares	Degrees of Freedom	Expected Mean Square	F_0
Blocks	$\frac{1}{ab}\sum_k y_{..k}^2 - \frac{y_{...}^2}{abn}$	$n-1$	$\sigma^2 + ab\sigma_\delta^2$	
A	$\frac{1}{bn}\sum_i y_{i..}^2 - \frac{y_{...}^2}{abn}$	$a-1$	$\sigma^2 + \frac{bn\sum \tau_i^2}{a-1}$	$\frac{MS_A}{MS_E}$
B	$\frac{1}{an}\sum_j y_{.j.}^2 - \frac{y_{...}^2}{abn}$	$b-1$	$\sigma^2 + \frac{an\sum \beta_j^2}{b-1}$	$\frac{MS_B}{MS_E}$
AB	$\frac{1}{n}\sum_i\sum_j y_{ij.}^2 - \frac{y_{...}^2}{abn} - SS_A - SS_B$	$(a-1)(b-1)$	$\sigma^2 + \frac{n\sum\sum(\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
Error	Subtraction	$(ab-1)(n-1)$	σ^2	
Total	$\sum_i\sum_j\sum_k y_{ijk}^2 - \frac{y_{...}^2}{abn}$	$abn-1$		

Source	DF	Anova SS	MeanSquare	F Value	Pr > F
temperature	2	39118.722	19559.361	26.26	<.0001
material	2	10683.722	5341.861	7.17	0.0036
Interaction	4	9613.778	2403.444	3.23	0.0297
block	3	354.972	118.324		
Error	24	17875.778	744.824		
Total	35	77646.972			

Three factorial experiment

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_r + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

$$i = 1, 2, \cdot, a$$

$$j = 1, 2, \cdot, b$$

$$k = 1, 2, \cdot, c$$

$$l = 1, 2, \cdot, n$$

TABLE 5.12

The Analysis of Variance Table for the Three-Factor Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Square	F_0
A	SS_A	$a-1$	MS_A	$\sigma^2 + \frac{bcn\sum \tau_i^2}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	MS_B	$\sigma^2 + \frac{acn\sum \beta_j^2}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
C	SS_C	$c-1$	MS_C	$\sigma^2 + \frac{abn\sum \gamma_k^2}{c-1}$	$F_0 = \frac{MS_C}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	$\sigma^2 + \frac{cn\sum\sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	$\sigma^2 + \frac{bn\sum\sum (\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$F_0 = \frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	$\sigma^2 + \frac{an\sum\sum (\beta\gamma)_{jk}^2}{(b-1)(c-1)}$	$F_0 = \frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	$\sigma^2 + \frac{n\sum\sum\sum (\tau\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$	$F_0 = \frac{MS_{ABC}}{MS_E}$
Error	SS_E	$abc(n-1)$	MS_E	σ^2	
Total	SS_T	$abcn-1$			

Example

- Soft drink bottling
- Fill correct volume
- Response variable: difference from correct volume

- Carbonation (10%, 12%, 14%)
- Pressure (25psi, 30psi)
- Line speed (200bpm, 250bpm)

■ **TABLE 5.13**

Fill Height Deviation Data for Example 5.3

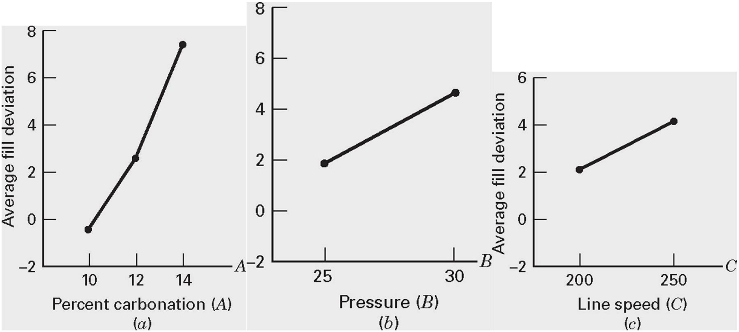
Percent Carbonation (*A*)	**Operating Pressure (*B*): 25 psi, Line Speed (*C*): 200**		**25 psi, Line Speed (*C*): 250**		**30 psi, Line Speed (*C*): 200**		**30 psi, Line Speed (*C*): 250**		$y_{i\ldots}$
	−3		−1		−1		1		
10	−1	(−4)	0	(−1)	0	(−1)	1	(2)	−4
	0		2		2		6		
12	1	(1)	1	(3)	3	(5)	5	(11)	20
	5		7		7		10		
14	4	(9)	6	(13)	9	(16)	11	(21)	59
$B \times C$ Totals $y_{.jk.}$	6		15		20		34		$75 = y_{\ldots}$
$y_{.j..}$	21				54				

$A \times B$ Totals $y_{ij..}$

A	*B* = 25	*B* = 30
10	−5	1
12	4	16
14	22	37

$A \times C$ Totals $y_{i.k.}$

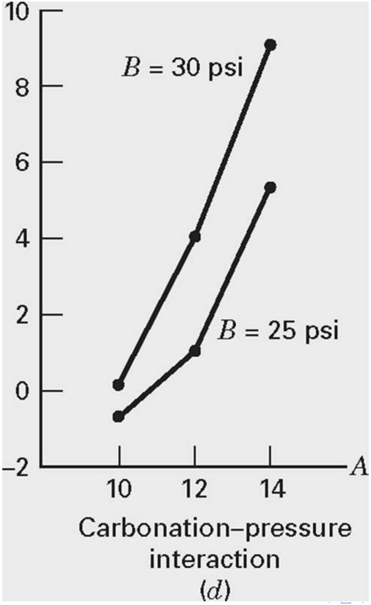
A	*C* = 200	*C* = 250
10	−5	1
12	6	14
14	25	34


8
6
4
2
0
−2
Average fill deviation
10
12
14
A
Percent carbonation (A)
(a)
8
6
4
2
0
−2
25
30
B
Pressure (B)
(b)
6
4
2
0
−2
Average fill deviation
200
250
C
Line speed (C)
(c)

■ TABLE 5.14

Analysis of Variance for Example 5.3

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Percentage of carbonation (*A*)	252.750	2	126.375	178.412	<0.0001
Operating pressure (*B*)	45.375	1	45.375	64.059	<0.0001
Line speed (*C*)	22.042	1	22.042	31.118	0.0001
AB	5.250	2	2.625	3.706	0.0558
AC	0.583	2	0.292	0.412	0.6713
BC	1.042	1	1.042	1.471	0.2485
ABC	1.083	2	0.542	0.765	0.4867
Error	8.500	12	0.708		
Total	336.625	23			



10
8
6
4
2
0
−2
B = 30 psi
B = 25 psi
10
12
14
A
Carbonation–pressure interaction
(d)