# Lec 10: Fractions of $2^k$ Factorial Design

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# Fraction of $2^k$ experiments

- **Screening**: Some of the factors may influence the results. We want to figure out which.
- The number of combinations, $2^k$, is too large for a complete investigation.

# Fraction of $2^k$ experiments

- **Screening**: Some of the factors may influence the results. We want to figure out which.
- The number of combinations, $2^k$, is too large for a complete investigation.
- We can only investigate a **fraction** (subset) of the $2^k$ combinations.
- One-Half fraction of the $2^k$ design: $2^{k-1}$.
- One-Quarter fraction of the $2^k$ design: $2^{k-2}$.

# Example

■ **TABLE 8.1**

**Plus and Minus Signs for the $2^3$ Factorial Design**

| Treatment Combination | Factorial Effect | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $I$ | $A$ | $B$ | $C$ | $AB$ | $AC$ | $BC$ | $ABC$ |
| $a$ | + | + | − | − | − | − | + | + |
| $b$ | + | − | + | − | − | + | − | + |
| $c$ | + | − | − | + | + | − | − | + |
| $abc$ | + | + | + | + | + | + | + | + |
| $ab$ | + | + | + | − | + | − | − | − |
| $ac$ | + | + | − | + | − | + | − | − |
| $bc$ | + | − | + | + | − | − | + | − |
| (1) | + | − | − | − | + | + | + | − |

# Example

| Treatment Combination | Factorial Effect | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $I$ | $A$ | $B$ | $C$ | $AB$ | $AC$ | $BC$ | $ABC$ |
| $a$ | + | + | − | − | − | − | + | + |
| $b$ | + | − | + | − | − | + | − | + |
| $c$ | + | − | − | + | + | − | − | + |
| $abc$ | + | + | + | + | + | + | + | + |

- Notice that the $2^{3-1}$ design is formed by selecting only those treatment combination that has a "+" in ABC column.
- ABC is called a **generator** or a **word**.

# Example

| Treatment Combination | Factorial Effect | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $I$ | $A$ | $B$ | $C$ | $AB$ | $AC$ | $BC$ | $ABC$ |
| $a$ | + | + | − | − | − | − | + | + |
| $b$ | + | − | + | − | − | + | − | + |
| $c$ | + | − | − | + | + | − | − | + |
| $abc$ | + | + | + | + | + | + | + | + |

- $[A] = \frac{1}{2}(a - b - c + abc) = [BC]$
- $[B] = \frac{1}{2}(-a + b - c + abc) = [AC]$
- $[C] = \frac{1}{2}(-a - b + c + abc) = [AB]$

A and BC, B and AC, C and AB are **aliases**.

# How to find the alias relationships

- Use the **defining relation**: $I = ABC$.
- The one-half fraction with $I = +ABC$ is called the **principal fraction**.
- Multiplying any column by defining relation yields the aliases for the column.

$$A \cdot I = A \cdot ABC = A^2BC = BC$$

$$B \cdot I = B \cdot ABC = AB^2C = AC$$

$$C \cdot I = C \cdot ABC = ABC^2 = AB$$

# The principal fraction

- Defining relation: $I = ABC$
- $A = BC$, $[A] \longrightarrow A + BC$
- $B = AC$, $[B] \longrightarrow B + AC$
- $C = AB$, $[C] \longrightarrow C + AB$

# Alternative Fraction

**Plus and Minus Signs for the $2^3$ Factorial Design**

| Treatment Combination | Factorial Effect | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | *I* | *A* | *B* | *C* | *AB* | *AC* | *BC* | *ABC* |
| *ab* | + | + | + | − | + | − | − | − |
| *ac* | + | + | − | + | − | + | − | − |
| *bc* | + | − | + | + | − | − | + | − |
| (1) | + | − | − | − | + | + | + | − |

- The defining relation $I = -ABC$
- The aliases relationship: $A = -BC$, $B = -AC$, $C = -AB$
-

$$[A]' \longrightarrow A - BC$$
$$[B]' \longrightarrow B - AC$$
$$[C]' \longrightarrow C - AB$$

# Design of Resolution

## Resolution

A design is of resolution $R$ is no p-factors effect is aliased with another effect containing less than $R - p$ factors.

## Example

- $2^{3-1}$ experiment.
- Main effects (1-factor effect) are aliased with 2-factor (3-1 factors) interactions.
- The one-half fraction of the $2^3$ design with the defining relation $I = ABC$ is a $2^{3-1}_{III}$.

# Resolution $III$ designs

*No main effects are aliased with any other main effect, but main effects are aliased with two-factor interactions and some two-factor interactions may be aliased with each other.*

# Resolution $IV$ designs

*No main effect is aliased with any other main effect or with any two-factor interaction, but two-factor interactions are aliased with each other.*

For example: A $2^{4-1}$ design with $I = ABCD$

# Resolution $V$ designs

*No main effect or two-factor interaction is aliased with any other main effect or two-factor interaction, but two-factor interactions are aliased with three-factor interaction.*

For example: A $2^{5-1}$ design with $I = ABCDE$

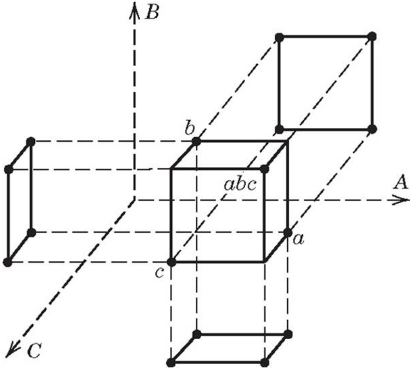
# A easy way to determine the resolutions of the designs

In general, the resolution of a two-level fractional factorial design is equal to the smallest number of letters in the shortest word in the defining relation.

- A $2^{3-1}$ design with $I = ABC$ is a resolution $III$ designs.
- A $2^{4-1}$ design with $I = ABCD$ is a resolution $IV$ designs.
- A $2^{5-1}$ design with $I = ABCDE$ is a resolution $V$ designs.

# Projection of Fractions into Factorials

Any fractional factorial design of resolution R contains complete factorial design in any subset of $R - 1$ factors.



# General construction of one-quarter fraction

1. Write down a basic design consisting of the runs associated with a full factorial in $k-2$ factors
2. Add two additional columns with interactions involving the first $k-2$ factors. Thus a one-quarter fraction of the $2^k$ design has two generators (P, Q).
3. The complete defining relation is $I = P = Q = PQ$

# Example

**TABLE 8.9**
Construction of the $2^{6-2}$ Design with the Generators $I = ABCE$ and $I = BCDF$

| | Basic Design | | | | | |
|---|---|---|---|---|---|---|
| Run | $A$ | $B$ | $C$ | $D$ | $E = ABC$ | $F = BCD$ |
| 1 | − | − | − | − | − | − |
| 2 | + | − | − | − | + | − |
| 3 | − | + | − | − | + | + |
| 4 | + | + | − | − | − | + |
| 5 | − | − | + | − | + | + |
| 6 | + | − | + | − | − | + |
| 7 | − | + | + | − | − | − |
| 8 | + | + | + | − | + | − |
| 9 | − | − | − | + | − | + |
| 10 | + | − | − | + | + | + |
| 11 | − | + | − | + | + | − |
| 12 | + | + | − | + | − | − |
| 13 | − | − | + | + | + | − |
| 14 | + | − | + | + | − | − |
| 15 | − | + | + | + | − | + |
| 16 | + | + | + | + | + | + |

# Choosing a Design

- In former example, $E$ and $F$ were constructed through $E = ABC$ $F = BCD$
- Would it be better to construct $E$ and $F$ through $E = ACB$ $F = ABCD$?

# Choosing a Design

- In former example, $E$ and $F$ were constructed through $E = ABC$ $F = BCD$
- Would it be better to construct $E$ and $F$ through $E = ACB$ $F = ABCD$?

A reasonable criterion is to select the generator has the highest possible resolution.

# Minimal abberation

■ **TABLE 8.13**

**Three Choices of Generators for the $2_{IV}^{7-2}$ Design**

| **Design A Generators:** $F = ABC, G = BCD$ $I = ABCF = BCDG = ADFG$ | **Design B Generators:** $F = ABC, G = ADE$ $I = ABCF = ADEG = BCDEFG$ | **Design C Generators:** $F = ABCD, G = ABDE$ $I = ABCDF = ABDEG = CEFG$ |
|---|---|---|
| Aliases (two-factor interactions) | Aliases (two-factor interactions) | Aliases (two-factor interactions) |
| $AB = CF$ | $AB = CF$ | $CE = FG$ |
| $AC = BF$ | $AC = BF$ | $CF = EG$ |
| $AD = FG$ | $AD = EG$ | $CG = EF$ |
| $AG = DF$ | $AE = DG$ | |
| $BD = CG$ | $AF = BC$ | |
| $BG = CD$ | $AG = DE$ | |
| $AF = BC = DG$ | | |

# Minimal abberation

■ **TABLE 8.13**

**Three Choices of Generators for the $2^{7-2}_{IV}$ Design**

| Design *A* Generators: $F = ABC, G = BCD$ $I = ABCF = BCDG = ADFG$ | Design *B* Generators: $F = ABC, G = ADE$ $I = ABCF = ADEG = BCDEFG$ | Design *C* Generators: $F = ABCD, G = ABDE$ $I = ABCDF = ABDEG = CEFG$ |
|---|---|---|
| Aliases (two-factor interactions) | Aliases (two-factor interactions) | Aliases (two-factor interactions) |
| $AB = CF$ | $AB = CF$ | $CE = FG$ |
| $AC = BF$ | $AC = BF$ | $CF = EG$ |
| $AD = FG$ | $AD = EG$ | $CG = EF$ |
| $AG = DF$ | $AE = DG$ | |
| $BD = CG$ | $AF = BC$ | |
| $BG = CD$ | $AG = DE$ | |
| $AF = BC = DG$ | | |

Design $C$ minimizes the number of words in the defining relation that are of minimum length. We call such a design a **minimum aberration design**.

■ **TABLE 8.14**

**Selected $2^{k-p}$ Fractional Factorial Designs**

| Number of Factors, $k$ | Fraction | Number of Runs | Design Generators |
|---|---|---|---|
| 3 | $2^{3-1}_{III}$ | 4 | $C = \pm AB$ |
| 4 | $2^{4-1}_{IV}$ | 8 | $D = \pm ABC$ |
| 5 | $2^{5-1}_{V}$ | 16 | $E = \pm ABCD$ |
| | $2^{5-2}_{III}$ | 8 | $D = \pm AB$ |
| | | | $E = \pm AC$ |
| 6 | $2^{6-1}_{VI}$ | 32 | $F = \pm ABCDE$ |
| | $2^{6-2}_{IV}$ | 16 | $E = \pm ABC$ |
| | | | $F = \pm BCD$ |
| | $2^{6-3}_{III}$ | 8 | $D = \pm AB$ |
| | | | $E = \pm AC$ |
| | | | $F = \pm BC$ |
| 7 | $2^{7-1}_{VII}$ | 64 | $G = \pm ABCDEF$ |
| | $2^{7-2}_{IV}$ | 32 | $F = \pm ABCD$ |
| | | | $G = \pm ABDE$ |
| | $2^{7-3}_{IV}$ | 16 | $E = \pm ABC$ |
| | | | $F = \pm BCD$ |
| | | | $G = \pm ACD$ |
| | $2^{7-4}_{III}$ | 8 | $D = \pm AB$ |
| | | | $E = \pm AC$ |
| | | | $F = \pm BC$ |
| | | | $G = \pm ABC$ |
| 8 | $2^{8-2}_{V}$ | 64 | $G = \pm ABCD$ |
| | | | $H = \pm ABEF$ |
| | $2^{8-3}_{IV}$ | 32 | $F = \pm ABC$ |
| | | | $G = \pm ABD$ |
| | | | $H = \pm BCDE$ |
| | $2^{8-4}_{IV}$ | 16 | $E = \pm BCD$ |
| | | | $F = \pm ACD$ |
| | | | $G = \pm ABC$ |
| | | | $H = \pm ABD$ |
| 9 | $2^{9-2}_{VI}$ | 128 | $H = \pm ACDFG$ |
| | | | $J = \pm BCEFG$ |
| | $2^{9-3}_{IV}$ | 64 | $G = \pm ABCD$ |
| | | | $H = \pm ACEF$ |
| | | | $J = \pm CDEF$ |
| | $2^{9-4}_{IV}$ | 32 | $F = \pm BCDE$ |
| | | | $G = \pm ACDE$ |
| | | | $H = \pm ABDE$ |
| | | | $J = \pm ABCE$ |

| Number of Factors, $k$ | Fraction | Number of Runs | Design Generators |
|---|---|---|---|
| | $2^{9-5}_{III}$ | 16 | $E = \pm ABC$ |
| | | | $F = \pm BCD$ |
| | | | $G = \pm ACD$ |
| | | | $H = \pm ABD$ |
| | | | $J = \pm ABCD$ |
| 10 | $2^{10-3}_{V}$ | 128 | $H = \pm ABCG$ |
| | | | $J = \pm BCDE$ |
| | | | $K = \pm ACDF$ |
| | $2^{10-4}_{IV}$ | 64 | $G = \pm BCDF$ |
| | | | $H = \pm ACDF$ |
| | | | $J = \pm ABDE$ |
| | | | $K = \pm ABCE$ |
| | $2^{10-5}_{IV}$ | 32 | $F = \pm ABCD$ |
| | | | $G = \pm ABCE$ |
| | | | $H = \pm ABDE$ |
| | | | $J = \pm ACDE$ |
| | | | $K = \pm BCDE$ |
| | $2^{10-6}_{III}$ | 16 | $E = \pm ABC$ |
| | | | $F = \pm BCD$ |
| | | | $G = \pm ACD$ |
| | | | $H = \pm ABD$ |
| | | | $J = \pm ABCD$ |
| | | | $K = \pm AB$ |
| 11 | $2^{11-5}_{IV}$ | 64 | $G = \pm CDE$ |
| | | | $H = \pm ABCD$ |
| | | | $J = \pm ABF$ |
| | | | $K = \pm BDEF$ |
| | | | $L = \pm ADEF$ |
| | $2^{11-6}_{IV}$ | 32 | $F = \pm ABC$ |
| | | | $G = \pm BCD$ |
| | | | $H = \pm CDE$ |
| | | | $J = \pm ACD$ |
| | | | $K = \pm ADE$ |
| | | | $L = \pm BDE$ |
| | $2^{11-7}_{III}$ | 16 | $E = \pm ABC$ |
| | | | $F = \pm BCD$ |
| | | | $G = \pm ACD$ |
| | | | $H = \pm ABD$ |
| | | | $J = \pm ABCD$ |
| | | | $K = \pm AB$ |

| Number of Factors, $k$ | Fraction | Number of Runs | Design Generators |
|---|---|---|---|
| | | | $L = \pm AC$ |
| 12 | $2^{12-8}_{III}$ | 16 | $E = \pm ABC$ |
| | | | $F = \pm ABD$ |
| | | | $G = \pm ACD$ |
| | | | $H = \pm BCD$ |
| | | | $J = \pm ABCD$ |
| | | | $K = \pm AB$ |
| | | | $L = \pm AC$ |
| | | | $M = \pm AD$ |
| 13 | $2^{13-9}_{III}$ | 16 | $E = \pm ABC$ |
| | | | $F = \pm ABD$ |
| | | | $G = \pm ACD$ |
| | | | $H = \pm BCD$ |
| | | | $J = \pm ABCD$ |
| | | | $K = \pm AB$ |
| | | | $L = \pm AC$ |
| | | | $M = \pm AD$ |
| | | | $N = \pm BC$ |
| 14 | $2^{14-10}_{III}$ | 16 | $E = \pm ABC$ |
| | | | $F = \pm ABD$ |
| | | | $G = \pm ACD$ |
| | | | $H = \pm BCD$ |
| | | | $J = \pm ABCD$ |
| | | | $K = \pm AB$ |
| | | | $L = \pm AC$ |
| | | | $M = \pm AD$ |
| | | | $N = \pm BC$ |
| | | | $O = \pm BD$ |
| 15 | $2^{15-11}_{III}$ | 16 | $E = \pm ABC$ |
| | | | $F = \pm ABD$ |
| | | | $G = \pm ACD$ |
| | | | $H = \pm BCD$ |
| | | | $J = \pm ABCD$ |
| | | | $K = \pm AB$ |
| | | | $L = \pm AC$ |
| | | | $M = \pm AD$ |
| | | | $N = \pm BC$ |
| | | | $O = \pm BD$ |
| | | | $P = \pm CD$ |

This design is a one-sixteenth fraction of the $2^7$. 8 runs have 7 degree of freedom to estimate the 7 main effects.

| | Basic Design | | | | | | |
|---|---|---|---|---|---|---|---|
| Run | $A$ | $B$ | $C$ | $D = AB$ | $E = AC$ | $F = BC$ | $G = ABC$ |
| 1 | − | − | − | + | + | + | − |
| 2 | + | − | − | − | − | + | + |
| 3 | − | + | − | − | + | − | + |
| 4 | + | + | − | + | − | − | − |
| 5 | − | − | + | + | − | − | + |
| 6 | + | − | + | − | + | − | − |
| 7 | − | + | + | − | − | + | − |
| 8 | + | + | + | + | + | + | + |

$$[A] \rightarrow A + BD + CE + FG$$

$$[B] \rightarrow B + AD + CF + EG$$

$$\cdots$$

Suppose along with the principal fraction a second frational design with the signs reversed in the column for factor D is also run. The column for D in the second fraction is $- + + - - + + + -$ Then

$$[A]' \rightarrow A - BD + CE + FG$$

$$[B]' \rightarrow B - AD + CF + EG$$

$$\cdots$$

# Single-factor fold over

From the two linear combination of effects $\frac{1}{2}([i]+[i]')$ and $\frac{1}{2}([i]-[i]')$ we have:

| $i$ | From $\frac{1}{2}([i]+[i]')$ | From $\frac{1}{2}([i]-[i]')$ |
|---|---|---|
| $A$ | $A + CE + FG$ | $BD$ |
| $B$ | $B + CF + EG$ | $AD$ |
| $C$ | $C + AE + BF$ | $DG$ |
| $D$ | $D$ | $AB + CG + EF$ |
| $E$ | $E + AC + BG$ | $DF$ |
| $F$ | $F + BC + AG$ | $DE$ |
| $G$ | $G + BE + AF$ | $CD$ |

# Full fold over

| Run | Basic Design A | B | C | $D = AB$ | $E = AC$ | $F = BC$ | $G = ABC$ |
|---|---|---|---|---|---|---|---|
| 1 | − | − | − | + | + | + | − |
| 2 | + | − | − | − | − | + | + |
| 3 | − | + | − | − | + | − | + |
| 4 | + | + | − | + | − | − | − |
| 5 | − | − | + | + | − | − | + |
| 6 | + | − | + | − | + | − | − |
| 7 | − | + | + | − | − | + | − |
| 8 | + | + | + | + | + | + | + |

| Run | Basic Design A | B | C | $D = -AB$ | $E = -AC$ | $F = -BC$ | $G = ABC$ |
|---|---|---|---|---|---|---|---|
| 1 | + | + | + | − | − | − | + |
| 2 | − | + | + | + | + | − | − |
| 3 | + | − | + | + | − | + | − |
| 4 | − | − | + | − | + | + | + |
| 5 | + | + | − | − | + | + | − |
| 6 | − | + | − | + | − | + | + |
| 7 | + | − | − | + | + | − | + |
| 8 | − | − | − | − | − | − | − |

# Full fold over

$$[A] \rightarrow A + BD + CE + FG$$

$$[B] \rightarrow B + AD + CF + EG$$

$$\cdots$$

And

$$[A]' \rightarrow A - BD - CE - FG$$

$$[B]' \rightarrow B - AD - CF - EG$$

$$\cdots$$

# Full fold over

By combining this second fraction with the original one, we obtain the following

| i | From $\frac{1}{2}([i]+[i]')$ | From $\frac{1}{2}([i]-[i]')$ |
|---|---|---|
| A | A | BD+CE+FG |
| B | B | AD+CF+EG |
| C | C | AE+BF+DG |
| D | D | AB+CG+EF |
| E | E | AC+BG+DF |
| F | F | BC+AG+DE |
| G | G | CD+BE+AF |

# Fold over

Resolution $III$:

- Full fold over: Provides estimates of all main effects.
- Single-factor fold over: Provides estimates of the selected factors and its two-factor interaction.

# Example for Alias structure

For a $2^{3-1}$ design with defining relation $I = ABC$.

$$\text{Reduced Model:} y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

$$\boldsymbol{\beta_1} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$\mathbf{X_1} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Full Model: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \varepsilon$

$$\boldsymbol{\beta_1} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \boldsymbol{\beta_2} = \begin{pmatrix} \beta_{12} \\ \beta_{13} \\ \beta_{23} \end{pmatrix}$$

$$\mathbf{X_1} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \mathbf{X_2} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$E(\hat{\boldsymbol{\beta}}_{\mathbf{1}}) = \boldsymbol{\beta}_1 + (\mathbf{X}_1'\mathbf{X}_1)^{-\mathbf{1}}\mathbf{X}_1'\mathbf{X}_2\boldsymbol{\beta}_2 = \boldsymbol{\beta}_1 + \mathbf{A}\boldsymbol{\beta}_2$$

$$E\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 + \beta_{23} \\ \beta_2 + \beta_{13} \\ \beta_3 + \beta_{12} \end{pmatrix}$$