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1 AN INTRODUCTION TO CONSUMER PRICE INDEX METHODOLOGY

1.1 A price index is a measure of the proportionate, or percentage, changes in a set of prices over time. A consumer price index (CPI) measures changes in the prices of goods and services that households consume. Such changes affect the real purchasing power of consumers' incomes and their welfare. As the prices of different goods and services do not all change at the same rate, a price index can only reflect their average movement. A price index is typically assigned a value of unity, or 100, in some reference period and the values of the index for other periods of time are intended to indicate the average proportionate, or percentage, change in prices from this price reference period. Price indices can also be used to measure differences in price levels between different cities, regions or countries at the same point in time.

1.2 Much of this manual and the associated economic literature on price indices is concerned with two basic questions:

- Exactly what set of prices should be covered by the index?
- What is the most appropriate way in which to average their movements?

These two questions are addressed in the early sections of this introduction.

1.3 Consumer price indices (CPIs) are index numbers that measure changes in the prices of goods and services purchased or otherwise acquired by households, which households use directly, or indirectly, to satisfy their own needs and wants. Consumer price indices can be intended to measure either the rate of price inflation as perceived by households, or changes in their cost of living (that is, changes in the amounts that the households need to spend in order to maintain their standard of living). There need be no conflict between these two objectives. In practice, most CPIs are calculated as weighted averages of the percentage price changes for a specified set, or "basket", of consumer products, the weights reflecting their relative importance in household consumption in some period. Much depends on how appropriate and timely the weights are.

1.4 This chapter provides a general introduction to, and overview of, the methodology for compiling CPIs. It provides a summary of the relevant theory and practice of index number compilation that is intended to facilitate the reading and understanding of the detailed chapters that follow, some of which are inevitably quite technical. It describes all the various steps involved in CPI compilation starting with the basic concept, definition and purpose of a CPI, followed by the sampling procedures and survey methods used to collect and process the price data, and finishing with a summary of the actual calculation of the index and its dissemination.

1.5 An introductory presentation of CPI methodology has to start with the basic concept of a CPI and the underlying index number theory, including the properties and behaviour of the various kinds of index number that are, or might be, used for CPI purposes. In principle, it is necessary to settle what type of index to calculate before going on to consider the best way in which to estimate it in practice, taking account of the resources available.

1.6 The main topics covered in this chapter are as follows:
- the origins and uses of CPIs;
- basic index number theory, including the axiomatic and economic approaches to CPIs;
- elementary price indices and aggregate CPIs;
- the transactions, activities and households covered by CPIs;
- the collection and processing of the prices, including adjusting for quality change;
- the actual calculation of the CPI;
- potential errors and bias;
- organization, management and dissemination policy.

In contrast, in this manual, the chapters dealing with index theory come later on; thus the presentation in this chapter does not follow the same order as the corresponding chapters of the manual.

1.7 It is not the purpose of this introduction to provide a complete summary of the contents of the manual. The objective is rather to provide a short presentation of the core methodological issues with which readers need to be acquainted before tackling the detailed chapters that follow. Some special topics, such as the treatment of certain individual products whose prices cannot be directly observed, are not considered here as they are not central to CPI methodology.

The origins and uses of consumer price indices

1.8 CPIs must serve a purpose. The precise way in which they are defined and constructed depends very much on what they are meant to be used for, and by whom. As explained in Chapter 15, CPIs have a long history dating back to the eighteenth century. Laspeyres and Paasche indices, which are still widely used today, were first proposed in the 1870s. They are explained below. The concept of the cost of living index was introduced early in the twentieth century.

1.9 Traditionally, one of the main reasons for compiling a CPI was to compensate wage-earners for inflation by adjusting their wage rates in proportion to the percentage change in the CPI, a procedure known as indexation. For this reason, official CPIs tended to become the responsibility of ministries of labour, but most are now compiled by national statistical offices. A CPI that is specifically intended to be used to index wages is known as a compensation index.

1.10 CPIs have three important characteristics. They are published *frequently*, usually every month but sometimes every quarter. They are available *quickly*, usually about two weeks after the end of the month or quarter. They are also usually *not revised*. CPIs tend to be closely monitored and attract a lot of publicity.

1.11 As CPIs provide timely information about the rate of inflation, they have also come to be used for a wide variety of purposes in addition to indexing wages. For example:

- CPIs are widely used to index pensions and social security benefits.
- CPIs are also used to index other payments, such as interest payments or rents, or the prices of bonds.

- CPIs are also commonly used as a proxy for the general rate of inflation, even though they measure only consumer inflation. They are used by some governments or central banks to set inflation targets for purposes of monetary policy.
- The price data collected for CPI purposes can also be used to compile other indices, such as the price indices used to deflate household consumption expenditures in national accounts, or the purchasing power parities used to compare real levels of consumption in different countries.

1.12 These varied uses can create conflicts of interest. For example, using a CPI as an indicator of general inflation may create pressure to extend its coverage to include elements that are not goods and services consumed by households, thereby changing the nature and concept of the CPI. It should also be noted that because of the widespread use of CPIs to index a wide variety of payments – not just wages, but social security benefits, interest payments, private contracts, etc. – extremely large sums of money turn on their movements, enough to have a significant impact on the state of government finances. Thus, small differences in the movements of CPIs resulting from the use of slightly different formulae or methods can have considerable financial implications. CPI methodology is important in practice and not just in theory.

Choice of index number

1.13 The first question is to decide on the kind of index number to use. The extensive references dealing with index theory in the bibliography reflect the fact that there is a very large literature on this subject. Many different kinds of mathematical formulae have been proposed over the past two centuries. While there may be no single formula that would be preferred in all circumstances, most economists and compilers of CPIs seem to be agreed that, in principle, the index formula should belong to a small class of indices called *superlative* indices. A superlative index may be expected to provide an approximation to a cost of living index. A characteristic feature of a superlative index is that it treats the prices and quantities in both periods being compared symmetrically. Different superlative indices tend to have similar properties, yield similar results and behave in very similar ways. Because of their properties of symmetry, some kind of superlative index is also likely to be seen as desirable, even when the CPI is not meant to be a cost of living index.

1.14 When a monthly or quarterly CPI is first published, however, it is invariably the case that there is not sufficient information on the quantities and expenditures in the current period to make it possible to calculate a symmetric, or superlative, index. While it is necessary to resort to second-best alternatives in practice, being able to make a rational choice between the various possibilities means having a clear idea of the target index that would be preferred in principle. The target index can have a considerable influence on practical matters such as the frequency with which the weights used in the index should be updated.

1.15 A comprehensive, detailed, rigorous and up-to-date discussion of the relevant index number theory is provided in Chapters 15 to 23 of the manual. The following sections provide a summary of this material. Proofs of the various propositions or theorems stated in this chapter are to be found in the later chapters, to which the reader may refer for further explanation.

Price indices based on baskets of goods and services

1.16 The purpose of an index number may be explained as comparing the *values* of households' expenditures on consumer goods and services in two time periods. Knowing that expenditures have increased by 5 per cent is not very informative if we do not know how much of this change is attributable to changes in the *prices* of the goods and services, and how much to changes in the *quantities* purchased. The purpose of an index number is to decompose proportionate or percentage changes in value aggregates into their overall components of price and quantity change. A CPI is intended to measure the price component of the change in households' consumption expenditures. One way to do this is to measure the change in the value of an aggregate, holding the quantities constant.

Lowe indices

1.17 One very wide, and popular, class of price indices is obtained by defining the index as the percentage change, between the periods compared, in the total cost of purchasing a given set of quantities, generally described as a "basket". The meaning of such an index is easy to grasp and to explain to users. This class of index is called a Lowe index in this manual, after the index number pioneer who first proposed it in 1823 (see Chapter 15). Most statistical offices make use of some kind of Lowe index in practice.

1.18 Let there be *n* products in the basket with prices p_i and quantities q_i, and let the two periods compared be 0 and *t*. The Lowe index, P_{Lo}, is defined as follows:

$$P_{Lo} \equiv \frac{\sum_{i=1}^{n} p_i^t q_i}{\sum_{i=1}^{n} p_i^0 q_i}$$

1.19 In principle, any set of quantities could serve as the basket. The basket does not have to be restricted to the quantities purchased in one or other of the two periods compared, or indeed any actual period of time. The quantities could, for example, be arithmetic or geometric averages of the quantities in the two periods. For practical reasons, the basket of quantities used for CPI purposes usually has to be based on a survey of household consumption expenditures conducted in an earlier period than either of the two periods whose prices are compared. For example, a monthly CPI may run from January 2000 onwards, with January 2000 = 100, but the quantities may be derived from an annual expenditure survey made in 1997 or 1998, or even spanning both those years. As it takes a long time to collect and process expenditure data, there is usually a considerable time lag before such data can be introduced into the calculation of CPIs. The basket may also refer to a year, whereas the index may be compiled monthly or quarterly.

1.20 The period whose quantities are actually used in a CPI is described as the *weight reference period* and it will be denoted as period *b*. Period 0 is the *price reference period*. As just noted, *b* is likely to precede 0, at least when the index is first published, and this is assumed here, but *b* could be any period, including one between 0 and *t*, if the index is calculated some time after *t*. The Lowe index using the quantities of period *b* can be written as follows:

$$P_{Lo} \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} \equiv \sum_{i=1}^{n} \left(p_i^t / p_i^0\right) s_i^{0b} \qquad \text{where} \qquad s_i^{0b} = \frac{p_i^0 q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} \qquad (1.1)$$

The index can be written, and calculated, in two ways: either as the ratio of two value aggregates, or as an arithmetic weighted average of the price ratios, or *price relatives*, p_i^t / p_i^0, for the individual products using the hybrid expenditure shares s_i^{0b} as weights. The expenditures are described as *hybrid* because the prices and quantities belong to two different time periods, 0 and b respectively. The hybrid weights may be obtained by updating the actual expenditure shares in period b, namely $p_i^b q_i^b / \sum p_i^b q_i^b$, for the price changes occurring between periods b and 0 by multiplying them by the price relatives b and 0, namely p_i^0 / p_i^b. Lowe indices are widely used for CPI purposes.

Laspeyres and Paasche indices

1.21 Any set of quantities could be used in a Lowe index, but there are two special cases which figure very prominently in the literature and are of considerable importance from a theoretical point of view. When the quantities are those of the price reference period, that is when $b = 0$, the *Laspeyres* index is obtained. When quantities are those of the other period, that is when $b = t$, the *Paasche* index is obtained. It is necessary to consider the properties of Laspeyres and Paasche indices, and also the relationships between them, in more detail.

1.22 The Laspeyres price index, P_L, is defined as:

$$P_L = \frac{\sum_{i=1}^{n} p_i^t q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0} \equiv \sum_{i=1}^{n} \left(p_i^t / p_i^0 \right) s_i^0 \qquad (1.2)$$

where s_i^0 denotes the share of the *actual* expenditure on commodity i in period 0: that is, $p_i^0 q_i^0 / \sum p_i^0 q_i^0$.

1.23 The Paasche index, P_P, is defined as:

$$P_P = \frac{\sum_{i=1}^{n} p_i^t q_i^t}{\sum_{i=1}^{n} p_i^0 q_i^t} \equiv \left\{ \sum_{i=1}^{n} \left(p_i^t / p_i^0 \right)^{-1} s_i^t \right\}^{-1} \qquad (1.3)$$

where s_i^t denotes the actual share of the expenditure on commodity i in period t; that is, $p_i^t q_i^t / \sum p_i^t q_i^t$. Notice that the Paasche index is a weighted *harmonic* average of the price relatives that uses the actual expenditure shares in the later period t as weights. It follows from equation (1.1) that the Paasche index can also be expressed as a weighted arithmetic average of the price relatives using hybrid expenditure weights, in which the quantities of t are valued at the prices of 0.

Decomposing current value changes using Laspeyres and Paasche indices

1.24 Laspeyres and Paasche quantity indices are defined in a similar way to the price indices, simply by interchanging the p and q values in formulae (1.2) and (1.3). They summarize changes over time in the flow of quantities of goods and services consumed. A Laspeyres quantity index values the quantities at the fixed prices of the earlier period, while the Paasche quantity index uses the prices of the later period. The ratio of the values of the expenditures in two periods (V) reflects the combined effects of both price and quantity changes. When Laspeyres and Paasche indices are used, the value change can be exactly decomposed into a price index times a quantity index only if the Laspeyres price (quantity) index is matched with the Paasche quantity (price) index. Let P_{La} and Q_{La} denote the Laspeyres price and quantity indices and let P_{Pa} and Q_{Pa} denote the Paasche price and quantity indices: then, $P_{La} Q_{Pa} \equiv V$ and $P_{Pa}\, Q_{La} \equiv V$.

1.25 Suppose, for example, a time series of household consumption expenditures at current prices in the national accounts is to be deflated by a price index to show changes in real consumption. To generate a series of consumption expenditures at constant base period prices (whose movements are identical with those of the Laspeyres volume index), the consumption expenditures at current prices must be deflated by a series of Paasche price indices.

Ratios of Lowe and Laspeyres indices

1.26 The Lowe index is transitive. The ratio of two Lowe indices using the same set of q^b values is also a Lowe index. For example, the ratio of the Lowe index for period t+1 with price reference period 0 divided by that for period t also with price reference period 0 is:

$$\frac{\sum_{i=1}^{n} p_i^{t+1} q_i^b \Big/ \sum_{i=1}^{n} p_i^0 q_i^b}{\sum_{i=1}^{n} p_i^t q_i^b \Big/ \sum_{i=1}^{n} p_i^0 q_i^b} = \frac{\sum_{i=1}^{n} p_i^{t+1} q_i^b}{\sum_{i=1}^{n} p_i^t q_i^b} = P_{Lo}^{t,t+1} \qquad (1.4)$$

This is a Lowe index for period t+1 with period t as the price reference period. This kind of index is, in fact, widely used to measure short-term price movements, such as between t and t+1, even though the quantities may date back to some much earlier period b.

1.27 A Lowe index can also be expressed as the ratio of two Laspeyres indices. For example, the Lowe index for period t with price reference period 0 is equal to the Laspeyres index for period t with price reference period b divided by the Laspeyres index for period 0 also with price reference period b. Thus,

$$P_{Lo} = \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} = \frac{\sum_{i=1}^{n} p_i^t q_i^b \Big/ \sum_{i=1}^{n} p_i^b q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b \Big/ \sum_{i=1}^{n} p_i^b q_i^b} = \frac{P_{La}^t}{P_{La}^0} \qquad (1.5)$$

Updated Lowe indices

1.28 It is useful to have a formula that enables a Lowe index to be calculated directly as a chain index, in which the index for period t+1 is obtained by updating the index for period t. Because Lowe indices are transitive, the Lowe index for period t+1 with price reference period 0 can be written as the product of the Lowe index for period t with price reference period 0 multiplied by the Lowe index for period t+1 with price reference period t. Thus,

$$\frac{\sum_{i=1}^{n} p_i^{t+1} q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} = \left[\frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b}\right]\left[\frac{\sum_{i=1}^{n} p_i^{t+1} q_i^b}{\sum_{i=1}^{n} p_i^t q_i^b}\right]$$

$$= \left[\frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b}\right]\left[\sum_{i=1}^{n} \left(\frac{p_i^{t+1}}{p_i^t}\right) s_i^{tb}\right] \qquad (1.6)$$

where the expenditure weights s_i^{tb} are hybrid weights defined as:

$$s_i^{tb} \equiv p_i^t q_i^b \Big/ \sum_{i=1}^{n} p_i^t q_i^b \qquad (1.7)$$

1.29 Hybrid weights of the kind defined in equation (1.7) are often described as *price-updated* weights. They can be obtained by adjusting the original expenditure weights $p_i^b q_i^b / \sum p_i^b q_i^b$ by the price relatives p_i^t / p_i^b. By price-updating the expenditure weights from *b* to *t* in this way, the index between *t* and *t*+1 can be calculated directly as a weighted average of the price relatives p_i^{t+1} / p_i^t without referring back to the price reference period 0. The index can then be linked on to the value of the index in the preceding period *t*.

Interrelationships between fixed basket indices

1.30 Consider first the interrelationship between the Laspeyres and the Paasche indices. A well-known result in index number theory is that if the price and quantity changes (weighted by values) are *negatively* correlated, then the Laspeyres index exceeds the Paasche index. Conversely, if the weighted price and quantity changes are *positively* correlated, then the Paasche index exceeds the Laspeyres index. The proof is given in Appendix 15.1 of Chapter 15.

1.31 As consumers are usually price-takers, they typically react to price changes by substituting goods or services that have become *relatively* cheaper for those that have become *relatively* dearer. This is known as the *substitution effect*, a phenomenon that figures prominently in this manual and the wider literature on index numbers. Substitution tends to generate a negative correlation between the price and quantity relatives, in which case the Laspeyres index is greater than the Paasche index, the gap between them tending to widen over time.

1.32 In practice, however, statistical offices do not calculate Laspeyres or Paasche indices but instead usually calculate Lowe indices as defined in equation (1.1). The question then arises of how the Lowe index relates to the Laspeyres and Paasche indices. It is shown in the text of Chapter 15, and also in Appendix 15.2, that if there are persistent long-term trends in relative prices and if the substitution effect is operative, the Lowe index will tend to exceed the Laspeyres, and therefore also the Fisher and the Paasche indices. Assuming that period *b* precedes period 0, the ranking under these conditions will be:

Lowe ≥ Laspeyres ≥ Fisher ≥ Paasche

Moreover, the amount by which the Lowe exceeds the other three indices will tend to increase, the further back in time period *b* is in relation to period 0.

1.33 The positioning of period *b* is crucial. Given the assumptions about long-term price trends and substitution, a Lowe index will tend to increase as period *b* is moved backwards in time, or to decrease as period *b* is moved forwards in time. While *b* may have to precede 0 when the index is first published, there is no such restriction on the positioning of *b* as price and quantity data become available for later periods with passage of time. Period *b* can then be moved forwards. If *b* is positioned midway between 0 and *t*, the quantities are likely to be equi-representative of both periods, assuming that there is a fairly smooth transition from the relative quantities of 0 to those of *t*. In these circumstances, the Lowe index is likely to be close to the Fisher and other superlative indices, and cannot be presumed to have either an upward or a downward bias. These points are elaborated further below, and also in Chapter 15.

1.34 It is important that statistical offices take these relationships into consideration in deciding upon their policies. There are obviously practical advantages and financial savings from continuing to make repeated use over many years of the same fixed set of quantities to calculate a CPI. However, the amount by which such a CPI exceeds some conceptually preferred target index, such as a cost of living index (COLI), is likely to get steadily larger the further back in time the period *b* to which the quantities refer. Most users are likely to interpret the difference as upward bias. A large bias may undermine the credibility and acceptability of the index.

Young index

1.35 Instead of holding constant the quantities of period *b*, a statistical office may calculate a CPI as a weighted arithmetic average of the individual price relatives, holding constant the revenue shares of period *b*. The resulting index is called a *Young* index in this manual, again after another index number pioneer. The Young index is defined as follows:

$$P_{Yo} \equiv \sum_{i=1}^{n} s_i^b \left(\frac{p_i^t}{p_i^0} \right) \quad where \quad s_i^b \equiv \frac{p_i^b q_i^b}{\sum_{i=1}^{n} p_i^b q_i^b} \qquad (1.8)$$

In the corresponding Lowe index, equation (1.1), the weights are hybrid revenue shares that value the quantities of *b* at the prices of 0. As already explained, the price reference period 0 is usually later than the weight reference period *b* because of the time needed to collect and process and revenue data. In that case, a statistical office has the choice of assuming that either the quantities of period *b* remain constant or the expenditure shares in period *b* remain constant. Both cannot remain constant if prices change between *b* and 0. If the expenditure shares actually remained constant between periods *b* and 0, the quantities must have changed inversely in response to the price changes, which implies an elasticity of substitution of unity.

1.36 Whereas there is a presumption that the Lowe index will tend to exceed the Laspeyres index, it is more difficult to generalize about the relationship between the Young index and the Laspeyres index. The Young could be greater or less than the Laspeyres depending on how sensitive the quantities are to changes in relative prices. It is shown in Chapter 15 that with high elasticities of substitution (greater than unity) the Young will tend to exceed the Laspeyres, whereas with low elasticities the Young will tend to be less than the Laspeyres.

1.37 As explained later in this chapter, the Lowe index may be preferred to the Young index because the Young index has some undesirable properties that cause it to fail some critical index number tests (see also Chapter 16).

Geometric Young, Laspeyres and Paasche indices

1.38 In the geometric version of the Young index, a weighted geometric average is taken of the price relatives using the expenditure shares of period *b* as weights. It is defined as follows:

$$P_{GYo} \equiv \prod_{i=1}^{n} \left(\frac{p_i^t}{p_i^0} \right)^{s_i^b} \qquad (1.9)$$

where s_i^b is defined as above. The geometric Laspeyres is the special case in which $b = 0$; that is, the expenditure shares are those of the price reference period 0. Similarly, the geometric Paasche uses the expenditure shares of period *t*. It should be noted that these geometric indices cannot be expressed as the ratios of value aggregates in which the quantities are fixed. They are not basket indices and there are no counterpart Lowe indices.

1.39 It is worth recalling that for any set of positive numbers the arithmetic average is greater than, or equal to, the geometric average, which in turn is greater than, or equal to, the harmonic average, the equalities holding only when the numbers are all equal. In the case of unitary cross-elasticities of demand and constant expenditure shares, the geometric Laspeyres and Paasche indices coincide. In this case, the ranking of the indices must be the ordinary Laspeyres ≥ the geometric Laspeyres and Paasche ≥ the ordinary Paasche, because the indices are, respectively, arithmetic, geometric and harmonic averages of the same price relatives which all use the same set of weights.

1.40 The geometric Young and Laspeyres indices have the same information requirements as their ordinary arithmetic counterparts. They can be produced on a timely basis. Thus, these geometric indices must be treated as serious practical possibilities for purposes of CPI calculations. As explained later, the geometric indices are likely to be less subject than their arithmetic counterparts to the kinds of index number biases discussed in later sections. Their main disadvantage may be that, because they are not fixed basket indices, they are not so easy to explain or justify to users.

Symmetric indices

1.41 A symmetric index is one that makes equal use of the prices and quantities in both the periods compared and treats them in a symmetric manner. There are three particular symmetric indices that are widely used in economic statistics. It is convenient to introduce them at this point. As already noted, these three indices are also superlative indices.

1.42 The first is the *Fisher price index*, P_F, defined as the *geometric* average of the Laspeyres and Paasche indices; that is,

$$P_F \equiv \sqrt{P_L P_P} \qquad (1.10)$$

1.43 The second is the *Walsh price index*, P_W. This is a basket index whose quantities consist of *geometric* averages of the quantities in the two periods; that is,

$$P_W \equiv \frac{\sum_{i=1}^{n} p_i^t \sqrt{q_i^t q_i^0}}{\sum_{i=1}^{n} p_i^0 \sqrt{q_i^t q_i^0}} \qquad (1.11)$$

By taking a *geometric* rather than an arithmetic average of the quantities, equal weight is given to the *relative* quantities in both periods. The quantities in the Walsh index can be regarded as being equi-representative of both periods.

1.44 The third index is the *Törnqvist price index*, P_T, defined as a *geometric* average of the price relatives weighted by the average expenditure shares in the two periods.

$$P_T = \prod_{i=1}^{n} \left(p_i^t / p_i^0 \right)^{\sigma_i} \qquad (1.12)$$

where σ_i is the arithmetic average of the share of expenditure on product *i* in the two periods.

$$\sigma_i = \frac{s_i^t + s_i^0}{2} \qquad (1.13)$$

where the s_i values are defined as in equations (1.2) and (1.3) above.

1.45 The theoretical attractions of these indices become more apparent in the following sections on the axiomatic and economic approaches to index numbers.

Fixed base versus chain indices

1.46 This topic is examined in Chapter 15. When a time series of Lowe or Laspeyres indices is calculated using a fixed set of quantities, the quantities become progressively out of date and increasingly irrelevant to the later periods for which prices are being compared. The base period, in which quantities are set, has to be updated sooner or later and the new index series linked to the old. Linking is inevitable in the long run.

1.47 In a chain index, each link consists of an index in which each period is compared with the preceding one, the weight and price reference periods being moved forward each period. Any index number formula can be used for the individual links in a chain index. For example, it is possible to have a chain index in which the index for *t+1* on *t* is a Lowe index defined as $\sum p^{t+1}q^{t-j} / \sum p^{t}q^{t-j}$. The quantities refer to some period that is *j* periods earlier than the price reference period *t.* The quantities move forward one period as the price reference period moves forward one period. If $j = 0$, the chain Lowe becomes a chain Laspeyres, while if $j = -1$, it becomes a chain Paasche.

1.48 The CPIs in some countries are, in fact, annual chain Lowe indices of this general type, the quantities referring to some year or years that precede the price reference period 0 by a fixed period. For example, the 12 monthly indices from January 2000 to January 2001, with January 2000 as the price reference period, could be Lowe indices based on price-updated expenditures for 1998. The 12 indices from January 2001 to January 2002 are then based on price updated expenditures for 1999, and so on.

1.49 The expenditures lag behind the January price reference period by a fixed interval, moving forward a year each January as the price reference period moves forward one year. Although, for practical reasons, there has to be a time lag between the quantities and the prices when the index is first published, it is possible to recalculate the monthly indices for the current year later, using current expenditure data when they eventually become available. In this way, it is possible for the long-run index to be an annually chained monthly index, with contemporaneous annual weights. This method is explained in more detail in Chapter 9. It is used by one statistical office.

1.50 A chain index has to be "path dependent". It must depend on the prices and quantities in all the intervening periods between the first and last periods in the index series. Path dependency can be advantageous or disadvantageous. When there is a gradual economic transition from the first to the last period, with smooth trends in relative prices and quantities, chaining will tend to reduce the index number spread between the Lowe, Laspeyres and Paasche indices, thereby making the movements in the index less dependent on the choice of index number formula.

1.51 If there are fluctuations in the prices and quantities in the intervening periods, however, chaining may not only increase the index number spread but also distort the measure of the overall change between the first and last periods. For example, suppose all the prices in the last period return to their initial levels in period 0, which implies that they must have fluctuated in between; a chain Laspeyres index will not return to 100. It will tend to be greater than 100. If the cycle is repeated with all the prices periodically returning to their original levels, a chain Laspeyres index will tend to drift further and further above 100 even though there may be no long-term upward trend in the prices. Chaining is therefore not advised when prices fluctuate. When monthly prices are subject to regular and substantial seasonal fluctuations, for example, monthly chaining cannot be recommended. Seasonal fluctuations cause serious problems, which are analysed in Chapter 22. While a number of countries update their expenditure weights annually, the 12-monthly indices within each year are not chain indices but Lowe indices using fixed annual quantities.

1.52 *The Divisia index*. If the prices and quantities are continuous functions of time, it is possible to partition the change in their total value over time into price and quantity components following the method of Divisia. As shown in Chapter 15, the Divisia index may be derived mathematically by differentiating value (i.e. price multiplied by quantity) with respect to time to obtain two components: a relative-value-weighted price change and a relative-value-weighted quantity change. These two components are defined to be price and quantity indices, respectively. The Divisia is essentially a theoretical index. In practice, prices can be recorded only at discrete intervals, even if they vary continuously with time. A chain index may, however, be regarded as a discrete approximation to a Divisia. The Divisia index itself offers limited practical guidance about the kind of index number formula to choose for the individual links in a chain index.

Axiomatic and stochastic approaches to index numbers

1.53 Various *axiomatic approaches* to index numbers are explained in Chapter 16. These approaches seek to determine the most appropriate functional form for an index by specifying a number of axioms, or tests, that the index ought to satisfy. They throw light on the properties possessed by different kinds of indices, some of which are not intuitively obvious. Indices that fail to satisfy certain basic or fundamental axioms, or tests, may be rejected completely because they are liable to behave in unacceptable ways. An axiomatic approach may also be used to rank indices on the basis of their desirable, and undesirable, properties.

First axiomatic approach

1.54 The first approach is the traditional test approach pioneered by Irving Fisher. The price and quantity indices are defined as functions of the two vectors of prices and two vectors of quantities relating to the two periods compared. The prices and quantities are treated as independent variables, whereas in the economic approach to index numbers considered later in this chapter the quantities are assumed to be functions of the prices.

1.55 Chapter 16 starts by considering a set of 20 axioms, but only a selection of them is given here by way of illustration.

T1: *positivity* – the price index and its constituent vectors of prices and quantities should be positive.

T3: *identity test* – if the price of every product is identical in both periods, then the price index should equal unity, no matter what the quantity vectors are.

T5: *proportionality in current prices* – if all prices in period t are multiplied by the positive number λ, then the new price index should be λ times the old price index; i.e., the price index function is (positively) homogeneous of degree one in the components of the period t price vector.

T10: *invariance to changes in the units of measurement* (commensurability test) – the price index does not change if the units in which the products are measured are changed.

T11: *time reversal test* – if all the data for the two periods are interchanged, then the resulting price index should equal the reciprocal of the original price index.

T14: *mean value test for prices* – the price index lies between the highest and the lowest price relatives.

T16: *Paasche and Laspeyres bounding test* – the price index lies between the Laspeyres and Paasche indices.
T17: *monotonicity in current prices* – if any period *t* price is increased, then the price index must increase.

1.56 Some of the axioms or tests can be regarded as more important than others. Indeed, some of the axioms seem so inherently reasonable that it might be assumed that any index number actually in use would satisfy them. For example, test T10, the commensurability test, says that if the unit of quantity in which a product is measured is changed, say, from a gallon to a litre, the index must be unchanged. One index that does not satisfy this test is the *Dutot* index, which is defined as the ratio of the arithmetic means of the prices in the two periods. As explained later, this is a type of elementary index that is in fact widely used in the early stages of CPI calculation.

1.57 Consider, for example, the average price of salt and pepper. Suppose it is decided to change the unit of measurement for pepper from grams to ounces while leaving the units in which salt is measured (for example, kilos) unchanged. As an ounce is equal to 28.35 grams, the absolute value of the price of pepper increases by over 28 times, which effectively increases the weight of pepper in the Dutot index by over 28 times.

1.58 When the products covered by an index are heterogeneous and measured in different physical units, the value of any index that does not satisfy the commensurability test depends on the purely arbitrary choice of units. Such an index must be unacceptable conceptually. If the prices refer to a strictly homogeneous set of products that all use the same unit of measurement, the test becomes irrelevant.

1.59 Another important test is T11, the time reversal test. In principle, it seems reasonable to require that the same result should be obtained whichever of the two periods is chosen as the price reference period: in other words, whether the change is measured forwards in time, i.e., from 0 to *t*, or backwards in time from *t* to 0. The Young index fails this test because an arithmetic average of a set of price relatives is not equal to the reciprocal of the arithmetic average of the reciprocals of the price relatives. The fact that the *conceptually* arbitrary decision to measure the change in prices forwards from 0 and *t* gives a different result from measuring backwards from *t* to 0 is seen by many users as a serious disadvantage. The failure of the Young index to satisfy the time reversal test needs to be taken into account by statistical offices.

1.60 Both the Laspeyres and Paasche fail the time reversal test for the same reasons as the Young index. For example, the formula for a Laspeyres calculated backwards from *t* to 0, P_{BL}, is:

$$P_{BL} = \frac{\sum_{i=1}^{n} p_i^0 q_i^t}{\sum_{i=1}^{n} p_i^t q_i^t} \equiv \frac{1}{P_P} \qquad (1.14)$$

This index is identical to the reciprocal of the (forwards) Paasche, not to the reciprocal of the (forwards) Laspeyres. As already noted, the (forwards) Paasche tends to register a smaller increase than the (forwards) Laspeyres so that the Laspeyres index cannot satisfy the time reversal test. The Paasche index also fails the time reversal test.

1.61 In contrast, the Lowe index satisfies the time reversal test *provided* that the quantities q_i^b remain fixed when the price reference period is changed from 0 to t. The quantities of a Laspeyres index are, however, those of the price reference period *by definition*, and must change whenever the price reference period is changed. The basket for a forwards Laspeyres is different from that for a backwards Laspeyres, and the Laspeyres fails the time reversal test in consequence.

1.62 Similarly, the Lowe index is transitive whereas the Laspeyres and Paasche indices are not. Assuming that a Lowe index uses a fixed set of quantities, q_i^b, whatever the price reference period, it follows that

$Lo^{0,t} = Lo^{0,t-k} Lo^{t-k,t}$

where $Lo^{0,t}$ is the Lowe index for period t with period 0 as the price reference period. The Lowe index that compares t directly with 0 is the same as that calculated indirectly as a chain index through period t-k.

1.63 If, on the other hand, the Lowe index is defined in such a way that quantities vary with the price reference period, as in the index $\sum p^{t+1}q^{t-j} / \sum p^t q^{t-j}$ considered earlier, the resulting chain index is not transitive. The chain Laspeyres and chain Paasche indices are special cases of this index.

1.64 In the real world, the quantities do change and the whole point of chaining is to enable the *quantities* to be continually updated to take account of the changing universe of products. Achieving transitivity by arbitrarily holding the quantities constant, especially over a very long period of time, does not compensate for the potential biases introduced by using out-of-date quantities.

Ranking of indices using the first axiomatic approach

1.65 In Chapter 16 it is shown not only that the Fisher price index satisfies all the 20 axioms listed but also, more remarkably, that it is the only possible index that can satisfy all 20 axioms. Thus, on the basis of this particular set of axioms, the Fisher clearly dominates other indices.

1.66 In contrast to Fisher, the other two symmetric (and superlative) indices defined in equations (1.11) and (1.12) above do not emerge so well from the 20 tests. In Chapter 16, it is shown that the Walsh price index fails four tests while the Törnqvist index fails nine tests. Nevertheless, the Törnqvist and the Fisher may be expected to approximate each other quite closely numerically when the data follow relatively smooth trends, as shown in Chapter 19.

1.67 One limitation of the axiomatic approach is that the list of axioms is inevitably somewhat arbitrary. Some axioms, such as the Paasche and Laspeyres bounding test failed by both Törnqvist and Walsh, could be regarded as dispensable. Additional axioms or tests can be envisaged, and two further axioms are considered below. Another problem with a simple application of the axiomatic approach is that it is not sufficient to know which tests are failed. It is also necessary to know how badly an index fails. Failing badly one major test, such as the commensurability test, might be considered sufficient to rule out an index, whereas failing several minor tests marginally may not be very disadvantageous.

Some further tests

1.68 Consider a further symmetry test. Reversing the roles of prices and quantities in a price index yields a quantity index of the same functional form as the price index. The *factor* *reversal test* requires that the product of this quantity index and the original price index should be identical with the change in the value of the aggregate in question. The test is important if, as stated earlier, price and quantity indices are intended to enable changes in the values of aggregates over time to be factored into their price and quantity components in an economically meaningful way. Another interesting result given in Chapter 16 is that the Fisher index is the only price index to satisfy four minimal tests: T1 (positivity), T11 (time reversal test), T12 (quantity reversal test) and T21 (factor reversal test). As the factor reversal test implicitly assumes that the prices and quantities must refer either to period 0 or to period *t*, it is not relevant to a Lowe index in which three periods are involved, *b*, 0 and *t*.

1.69 As shown earlier, the product of the Laspeyres price (quantity) index and the Paasche quantity (price) index is identical with the change in the total value of the aggregate in question. Thus, Laspeyres and Paasche indices may be said to satisfy a weak version of the factor reversal test in that dividing the value change by a Laspeyres (Paasche) price index does lead to a meaningful quantity index, i.e., the Paasche (Laspeyres), even though the functional forms of the price and quantity indices are not identical.

1.70 Another test discussed in Chapter 16 is the *additivity test*. This is more important from the perspective of quantity than price indices. Price indices may be used to deflate value changes to obtain implicit quantity changes. The results may be presented for sub-aggregates such as broad categories of household consumption. Just as expenditure aggregates at current prices are, by definition, obtained simply by summing individual expenditures, it is reasonable to expect that the changes in the sub-aggregates of a quantity index should add up to the changes in the totals – the additivity test. Quantity indices such as Laspeyres and Paasche that use a common set of prices to value quantities in both periods must satisfy the additivity test. Similarly, the Lowe quantity index defined as $\sum p^j q^t / \sum p^j q^0$ is also additive. The Geary–Khamis quantity index (see Annex 4) used to make international comparisons of real consumption and gross domestic product (GDP) between countries is an example of such a Lowe quantity index. It uses an arithmetically weighted average of the prices in the different countries as the common price vector p^j to compare the quantities in different countries.

1.71 Similarly, an average of the prices in two periods can be used to value the quantities in intertemporal indices. If the quantity index is also to satisfy the time reversal test, the average must be symmetrical. The *invariance to proportional changes in current prices test* (which corresponds to test T7 listed in Chapter 16, except that the roles of prices and quantities are reversed) requires that a quantity index depend only on the *relative*, not the absolute, level of the prices in each period. The Walsh quantity index satisfies this test, is additive and satisfies the time reversal test as well. It emerges as a quantity index with some very desirable properties.

1.72 Although the Fisher index itself is not additive, it is possible to decompose the overall *percentage change* in a Fisher price, or quantity, index into additive components that reflect the percentage change in each price or quantity. A similar multiplicative decomposition is possible for a Törnqvist price or quantity index.

The stochastic approach and a second axiomatic approach

1.73 Before considering a second axiomatic approach, it is convenient to take the stochastic approach to price indices. The stochastic approach treats the observed price *changes* or *relatives* as if they were a random sample drawn from a defined universe whose mean can be interpreted as the general rate of inflation. There can, however, be no single unique rate of inflation. Many possible universes can be defined, depending on which particular sets of expenditures or transactions the user is interested in. Clearly, the sample mean depends on the choice of universe from which the sample is drawn. Specifying the universe is similar to specifying the scope of a CPI. The stochastic approach addresses issues such as the appropriate form of average to take and the most efficient way to estimate it from a sample of price relatives, once the universe has been defined.

1.74 The stochastic approach is particularly useful when the universe is reduced to a single type of product. Because of market imperfections, there may be considerable variation in the prices at which the same product is sold in different outlets and also in the price changes observed. In practice, statistical offices have to estimate the average price change for a single product from a sample of price observations. Important methodological issues are raised, which are discussed in some detail in Chapter 7 and Chapter 20.

The unweighted stochastic approach

1.75 In Chapter 16, the unweighted stochastic approach to index number theory is explained. If simple random sampling has been used, equal weight may be given to each sampled price relative. Suppose each price relative can be treated as the sum of two components: a common inflation rate and a random disturbance with a zero mean. Using least squares or maximum likelihood, the best estimate of the common inflation rate is the unweighted *arithmetic* mean of price relatives, an index formula known as the *Carli* index. This index is the unweighted version of the Young index and is discussed further below, in the context of elementary price indices.

1.76 If the random component is multiplicative, not additive, the best estimate of the common inflation rate is given by the unweighted *geometric* mean of price relatives, known as the *Jevons* index. The Jevons index may be preferred to the Carli on the grounds that it satisfies the time reversal test, whereas the Carli does not. As explained below, this fact may be decisive when determining the functional form to be used to estimate the elementary indices compiled in the early stages of CPI calculations.

The weighted stochastic approach

1.77 As explained in Chapter 16, a *weighted* stochastic approach can be applied at an aggregative level covering sets of different products. As the products may be of differing economic importance, equal weight should not be given to each type of product. The products may be weighted on the basis of their share in the total value of the expenditures, or other transactions, in some period or periods. In this case, the index (or its logarithm) is the expected value of a random sample of price relatives (or their logarithms) whose probability of selection is proportional to the expenditure on that type of product in some period, or periods. Different indices are obtained depending on which expenditure weights are used and on whether the price relatives or their logarithms are used.

1.78 Suppose a sample of price relatives is randomly selected with the probability of selection proportional to the expenditure on that type of product in the price reference period 0. The expected price change is then the Laspeyres price index for the universe. Other indices may, however, also be obtained using the weighted stochastic approach. Suppose both periods are treated symmetrically and the probabilities of selection are made proportional to the arithmetic mean expenditure shares in both periods 0 and *t*. When these weights are applied to the logarithms of the price relatives, the expected value of the logarithms is the

Törnqvist index, also known as the Törnqvist–Theil index. From an axiomatic viewpoint, the choice of a symmetric average of the expenditure shares ensures that the time reversal test is satisfied, while the choice of the arithmetic mean, as distinct from some other symmetric average, may be justified on the grounds that the fundamental proportionality in current prices test, T5, is thereby satisfied.

1.79 By focusing on price changes, the Törnqvist index emerges as an index with some very desirable properties. This suggests a second axiomatic approach to indices, in which the focus is shifted from the individual prices and quantities used in the traditional axiomatic approach, to price changes and values shares.

A second axiomatic approach

1.80 A second axiomatic approach is examined in Chapter 16 in which a price index is defined as a function of the two sets of prices, or their ratios, and two sets of values. Provided the index is invariant to changes in units of measurement, i.e., satisfies the commensurability test, it makes no difference whether individual prices or their ratios are specified. A set of 17 axioms is postulated which are similar to the 20 axioms considered in the first axiomatic approach.

1.81 It is shown in Appendix 16.1 that the Törnqvist, or Törnqvist–Theil, is the only price index to satisfy all 17 axioms, just as the Fisher price index is the only index to satisfy all 20 tests in the first approach. However, the Törnqvist index does not satisfy the factor reversal test, so that the implicit quantity index obtained by deflating the change in value by the Törnqvist price index is not the Törnqvist quantity index. The implicit quantity index is therefore not "best" in the sense of satisfying the 17 axioms when these are applied to the quantity, rather than price, indices.

1.82 Zero prices may cause problems for indices based on price ratios, and especially for geometric averages of price ratios. In particular, if any price tends to zero, one test that may be applied is that the price index ought not to tend to zero or plus infinity. The Törnqvist does not meet this test. It is therefore proposed in Chapter 16 that when using the Törnqvist index, care should be taken to bound the prices away from zero in order to avoid a meaningless index number.

1.83 Finally, Chapter 16 examines the axiomatic properties of the Lowe and Young indices. The Lowe index emerges quite well from the axiomatic approach, satisfying both the time reversal and circularity tests. On the other hand, the Young index, like the Laspeyres and Paasche indices, fails both tests. As already explained, however, the attractiveness of the Lowe index depends more on how relevant the fixed quantity weights are to the two periods being compared, that is on the positioning of period *b,* than its axiomatic properties.

1.84 Although the "best" indices emerging from the two axiomatic approaches, namely Fisher and Törnqvist, are not the same, they have much in common. As already noted, they are both symmetric indices and they are both superlative indices. Although their formulae are different, they may be expected to behave in similar ways and register similar price movements. The same *type* of indices keep emerging as having desirable properties whichever approach to index theory is adopted, a conclusion that is reinforced by the economic approach to index numbers, which is explained in Chapter 17.

Cost of living index

1.85 Approaching the consumer price index from the standpoint of economic theory has led to the development of the concept of a cost of living index (COLI). The theory of the COLI was first developed by Konus (1924). It rests on the assumption of optimizing behaviour on the part of a rational consumer. The COLI for such a consumer has been defined succinctly as the ratio of the minimum expenditures needed to attain the given level of utility, or welfare, under two different price regimes. A more precise definition and explanation are given in Chapter 17.

1.86 Whereas a Lowe index measures the change in the cost of purchasing a fixed basket of goods and services resulting from changes in their prices, a COLI measures the change in the *minimum* cost of maintaining a given level of utility, or welfare, that results from changes in the prices of the goods and services consumed.

1.87 A COLI is liable to possible misinterpretation because households' welfare depends on a variety of physical and social factors that have no connection with prices. Events may occur that impinge directly on welfare, such as natural or man-made disasters. When such events occur, households may need to increase their consumption of goods and services in order to compensate for the loss of welfare caused by those events. Changes in the costs of consumption triggered by events *other than changes in prices* are irrelevant for a CPI that is not merely intended to measure changes in the prices of consumer goods and services but is generally interpreted by users as measuring price changes, and only price changes. In order to qualify as a CPI, a COLI must therefore hold constant not only the consumer's preferences but all the non-price factors that affect the consumer's welfare and standard of living. If a CPI is intended to be a COLI it must be *conditional* on:

- a particular level of utility or welfare;
- a particular set of consumer preferences;
- a particular state of the physical and social environment.

Of course, Lowe indices are also conditional as they depend on the particular basket of goods and services selected.

1.88 Lowe indices and COLIs have in common the fact that they may both be defined as the ratios of expenditures in two periods. However, whereas, by definition, the quantities are fixed in Lowe indices, they vary in response to changes in relative prices in COLIs. In contrast to the fixed basket approach to index theory, the economic approach explicitly recognizes that the quantities consumed are actually dependent on the prices. In practice, rational consumers may be expected to adjust the *relative* quantities they consume in response to changes in *relative* prices. A COLI assumes that a consumer seeking to minimize the cost of maintaining a given level of utility will make the necessary adjustments. The baskets of goods and services in the numerator and denominator of a COLI are not therefore exactly the same.

1.89 The observed expenditure of a rational consumer in the selected base period may be assumed to be the minimum expenditure needed to achieve the level of utility enjoyed in that period. In order to calculate a COLI based on that period, it is necessary to know what would be the minimum expenditure needed to attain precisely the same level of utility if the prices prevailing were those of the second period, other things remaining equal. The quantities purchased under these assumed conditions are likely to be *hypothetical*. They will not be the quantities actually consumed in the second period if other factors, including the resources available to the consumer, have changed.

1.90 The quantities required for the calculation of the COLI in at least one of the periods are not likely to be observable in practice. The COLI is not an operational index that can be calculated directly. The challenge is therefore to see whether it is possible to find methods of estimating a COLI indirectly or at least to find upper and lower bounds for the index. There is also considerable interest in establishing the relationship between a COLI and Lowe indices, including Laspeyres and Paasche, that can be calculated.

Upper and lower bounds on a cost of living index

1.91 It follows from the definition of a Laspeyres index that, if the consumer's income were to change by the same proportion as the change in the Laspeyres index, the consumer must have the possibility of purchasing the same basket of products as in the base period. The consumer cannot be worse off. However, if *relative* prices have changed, a utility-maximizing consumer would not continue to purchase the same quantities as before. The consumer would be able to achieve a *higher level* of utility by substituting, at least marginally, products that have become relatively cheaper for those that have become dearer. As a COLI measures the change in the minimum expenditures needed to maintain a constant level of utility, the COLI based on the first period will increase by less than the Laspeyres index.

1.92 By a similar line of reasoning, it follows that when relative prices change, the COLI based on the second period must increase by more than the Paasche index. As explained in more detail in Chapter 17, the Laspeyres index provides an upper bound to the COLI based on the first period and the Paasche a lower bound to the COLI based on the second period. It should be noted that there are two different COLIs involved here: one based on the first period and the other based on the second period. In general, however, the two COLIs are unlikely to differ much.

1.93 Suppose that the theoretical target index is a COLI, but that, for practical reasons, the CPI is actually calculated as a Lowe index in which the quantities refer to some period *b* that precedes the price reference period 0. One important conclusion to be drawn from this preliminary analysis is that as the Lowe may be expected to exceed the Laspeyres, assuming long-term price trends and substitution, while the Laspeyres may in turn be expected to exceed the COLI, the widely used Lowe index may be expected to have an upward bias. This point has had a profound influence on attitudes towards CPIs in some countries. The bias results from the fact that, by definition, fixed basket indices, including Laspeyres, do not permit any substitution between products in response to changes in relative prices. It is therefore usually described as "substitution bias". A Paasche index would be expected to have a downward substitution bias.

Some special cases

1.94 The next step is to establish whether there are special conditions under which it may be possible to measure a COLI exactly. In Chapter 17 it is shown that if the consumer's preferences are homothetic – that is, each indifference curve has the same shape, each being a uniform enlargement, or contraction, of each other – then the COLI is independent of the utility level on which it is based. The Laspeyres and Paasche indices provide upper and lower bounds to the *same* COLI.

1.95 One interesting special case occurs when the preferences can be represented by the so-called "Cobb–Douglas" function in which the cross-elasticities of demand between the various products are all unity. Consumers adjust the relative quantities they consume inversely in proportion to the changes in relative prices so that expenditure shares remain constant. With Cobb–Douglas preferences, the geometric Laspeyres provides an exact measure of the COLI. As the expenditure shares remain constant over time, all three *geometric* indices – Young, Laspeyres and Paasche – coincide with each other and with the COLI. Of course, the arithmetic versions of these indices do not coincide in these circumstances, because the baskets in periods *b*, 0 and *t* are all different as substitutions take place in response to changes in relative prices.

1.96 One of the more famous results in index number theory is that if the preferences can be represented by a homogeneous quadratic utility function, the Fisher index provides an exact measure of the COLI (see Chapter 17). Even though consumers' preferences are unlikely to conform exactly with this particular functional form, this result does suggest that, in general, the Fisher index is likely to provide a close approximation to the underlying unknown COLI and certainly a much closer approximation than either the arithmetic Laspeyres or Paasche indices.

Estimating COLIs by superlative indices

1.97 The intuition – that the Fisher index approximates the COLI – is corroborated by the following line of reasoning. Diewert (1976) noted that a homogeneous quadratic is a flexible functional form that can provide a second-order approximation to other twice-differentiable functions around the same point. He then described an index number formula as *superlative* when it is exactly equal to the COLI based on a certain functional form *and* when that functional form is flexible, e.g., a homogeneous quadratic. The derivation of these results, and further explanation, is given in detail in Chapter 17. In contrast to the COLI based on the true but unknown utility function, a superlative index is an actual index number that can be calculated. The practical significance of these results is that they provide a theoretical justification for expecting a superlative index to provide a fairly close approximation to the underlying COLI in a wide range of circumstances.

1.98 *Superlative indices as symmetric indices.* The Fisher is by no means the only example of a superlative index. In fact, there is a whole family of superlative indices. It is shown in Chapter 17 that any quadratic mean of order *r* is a superlative index for each value of $r \neq 0$. A quadratic mean of order *r* price index P^r is defined as follows:

$$P^r \equiv \frac{\sqrt[r]{\sum_{i=1}^{n} s_i^0 \left(\frac{p_i^t}{p_i^0}\right)^{r/2}}}{\sqrt[r]{\sum_{i=1}^{n} s_i^t \left(\frac{p_i^0}{p_i^t}\right)^{r/2}}} \qquad (1.15)$$

where s_i^0 and s_i^t are defined as in equations (1.2) and (1.3) above.

1.99 The symmetry of the numerator and denominator of equation (1.15) should be noted. A distinctive feature of equation (1.15) is that it treats the price changes and expenditure shares in both periods symmetrically, whatever value is assigned to the parameter *r*. Three special cases are of interest:

– when $r = 2$, equation (1.1) reduces to the Fisher price index;

- when $r = 1$ it is equivalent to the Walsh price index;
- in the limit as $r \to 0$, it equals the Törnqvist index.

These indices were introduced earlier as examples of indices that treat the information available in both periods *symmetrically*. Each was originally proposed long before the concept of a superlative index was developed.

1.100 *Choice of superlative index.* Chapter 17 addresses the question of which superlative formula to choose in practice. As each may be expected to approximate to the same underlying COLI, it may be inferred that they ought also to approximate to each other. The fact that they are all symmetric indices reinforces this conclusion. These conjectures tend to be borne out in practice by numerical calculations. So long as the parameter r does not lie far outside the range 0 to 2, superlative indices tend to be very close to each other. In principle, however, there is no limit on r and it has recently been shown that as r becomes larger, the formula tends to assign increasing weight to the extreme price relatives and the resulting superlative indices may diverge significantly from each other. Only when the absolute value of r is small, as in the case of the three commonly used superlative indices (Fisher, Walsh and Törnqvist), is the choice of superlative index unimportant.

1.101 Both the Fisher and the Walsh indices date back nearly a century. The Fisher index owes its popularity to the axiomatic, or test, approach, which Fisher himself was instrumental in developing. As already noted, it dominates other indices using the first axiomatic approach, while the Törnqvist dominates using the second axiomatic approach outlined above. The fact that the Fisher and the Törnqvist are both superlative indices whose use can be justified on economic grounds suggests that, from a theoretical point of view, it may not be possible to improve on them for CPI purposes.

Representativity bias

1.102 The fact that the Walsh index is a Lowe index that is also superlative suggests that the bias in other Lowe indices depends on the extent to which their quantities deviate from those in the Walsh basket. This can be viewed from another angle.

1.103 As the quantities in the Walsh basket are *geometric* averages of the quantities in the two periods, equal importance is assigned to the *relative*, as distinct from the absolute, quantities in both periods. The Walsh basket may therefore be regarded as being the basket that is most representative of *both* periods. If equal importance is attached to the consumption patterns in the two periods, the optimal basket for a Lowe index ought to be the most representative basket. The Walsh index then becomes the conceptually preferred target index for a Lowe index.

1.104 Suppose that period b, for which the quantities are actually used in the Lowe index, lies midway between 0 and t. In this case, assuming fairly smooth trends in the relative quantities, the actual basket in period b is likely to approximate to the most representative basket. Conversely, the further away that period b is from the midpoint between 0 and t, the more the relative quantities of b are likely to diverge from those in the most representative basket. In this case, the Lowe index between periods 0 and t that uses period b quantities is likely to exceed the Lowe index that uses the most representative quantities by an amount that becomes progressively larger the further back in time period b is positioned. The excess constitutes "bias" if the latter index is the target index. The bias can be attributed to the fact that the period b quantities tend to become increasingly unrepresentative of a comparison between 0 and *t* the further back period *b* is positioned. The underlying economic factors responsible are, of course, exactly the same as those that give rise to bias when the target index is the COLI. Thus, certain kinds of indices can be regarded as biased without invoking the concept of a COLI. Conversely, the same kinds of indices tend to emerge as being preferred, whether or not the objective is to estimate a cost of living bias.

1.105 If interest is focused on short-term price movements, the target index is an index between consecutive time periods *t* and *t*+1. In this case, the most representative basket has to move forward one period as the index moves forward. Choosing the most representative basket implies chaining. Similarly, chaining is also implied if the target index is a COLI between t and *t*+1. In practice, the universe of products is continually changing as well. As the most representative basket moves forward, it is possible to update the set of products covered, as well as take account of changes in the relative quantities of products that were covered previously.

Data requirements and calculation issues

1.106 As superlative indices require price and expenditure data for both periods, and as expenditure data are usually not available for the current period, it is not feasible to calculate a superlative CPI, at least at the time that a CPI is first published. In practice, CPIs tend to be Lowe indices with fixed quantities or annually updated chain Lowe indices. In the course of time, however, the requisite expenditure data may become available, enabling a superlative CPI to be calculated subsequently. Users will find it helpful for superlative CPIs to be published retrospectively as they make it possible to evaluate the properties and behaviour of the official index. Superlative CPIs can be treated as supplementary indices that complement, rather than replace, the original indices, if the policy is not to revise the official index.

1.107 Chapter 17 notes that, in practice, CPIs are usually calculated in stages (see also Chapters 9 and 20) and addresses the question of whether indices calculated this way are consistent in aggregation: that is, have the same values whether calculated in a single operation or in two stages. The Laspeyres index is shown to be exactly consistent, but the superlative indices are not. The widely used Fisher and Törnqvist indices are nevertheless shown to be approximately consistent.

Allowing for substitution

1.108 Chapter 17 examines one further index proposed recently, the Lloyd–Moulton index, P_{LM}, defined as follows:

$$P_{LM} \equiv \left\{ \sum_{i=1}^{n} s_i^0 \left(\frac{p_i^t}{p_i^0} \right)^{1-\sigma} \right\}^{\frac{1}{1-\sigma}} \qquad \sigma \neq 1 \qquad (1.16)$$

The parameter σ, which must be non-negative, is the elasticity of substitution between the products covered. It reflects the extent to which, on average, the various products are believed to be substitutes for each other. The advantage of this index is that it may be expected to be free of substitution bias to a reasonable degree of approximation, while requiring no more data than a Lowe or Laspeyres index. It is therefore a practical possibility for CPI calculation, even for the most recent periods, although it is likely to be difficult to obtain a satisfactory, acceptable estimate of the numerical value of the elasticity of substitution, the parameter used in the formula.

Aggregation issues

1.109 It has been assumed up to this point that the COLI is based on the preferences of a single representative consumer. Chapter 18 examines the extent to which the various conclusions reached above remain valid for CPIs that are actually compiled for groups of households. The general conclusion is that essentially the same relationships hold at an aggregate level, although some additional issues arise which may require additional assumptions.

1.110 One issue is how to weight individual households. Aggregate indices that weight households by their expenditures are called "plutocratic", while those that assign the same weight to each household are called "democratic". Another question is whether, at any one point of time, there is a single set of prices or whether different households face different prices. In general, when defining the aggregate indices it is not necessary to assume that all households are confronted by the same set of prices, although the analysis is naturally simplified if there is only a single set.

1.111 A plutocratic aggregate COLI assumes that each individual household minimizes the cost of attaining a given level of utility when confronted by two different sets of prices, the aggregate COLI being defined as the ratio of the aggregate minimum costs over all households. As in the case of a single household, it is recognized that the aggregate COLI that is appropriate for CPI purposes must be *conditional* on the state of a particular set of environmental variables, typically those in one or other of the periods compared. The environment must be understood in a broad sense to refer not only to the physical environment but also to the social and political environment.

1.112 Like the index for a single representative consumer, an aggregate COLI cannot be calculated directly, but it may be possible to calculate aggregate Laspeyres and Paasche indices that bound their respective COLIs from above or below. If there is only one single set of national prices, the aggregate plutocratic Laspeyres index reduces to an ordinary aggregate Laspeyres index. As the aggregate plutocratic Laspeyres and Paasche can, in principle, be calculated, so can the aggregate plutocratic Fisher index. It is argued in Chapter 18 that this should normally provide a good approximation to the aggregate plutocratic COLI.

1.113 Chapter 18 finally concludes that, in principle, both democratic and plutocratic Laspeyres, Paasche and Fisher indices could be constructed by a statistical agency, provided that information on household-specific price relatives and expenditures is available for both periods. If expenditure information is available only for the first period, then only the Laspeyres democratic and plutocratic indices can be constructed. The data requirements are rather formidable, however. The required data are unlikely to be available for *individual* households in practice and, even if they were to be, they could be subject to large errors.

Illustrative numerical data

1.114 Chapter 19 presents some numerical examples using an artificial data set. The purpose is not to illustrate the methods of calculation as such, but rather to demonstrate how different index number formulae can yield very different numerical results. Hypothetical but economically plausible prices, quantities and expenditures are given for six commodities over five periods of time. In general, differences between the different formulae tend to increase with the variance of the price relatives. They also depend on the extent to which the prices follow smooth trends or fluctuate.

1.115 The numerical results are striking. For example, the Laspeyres index over the five periods registers an increase of 44 per cent while the Paasche falls by 20 per cent. The two commonly used superlative indices, Törnqvist and Fisher, register increases of 25 per cent and 19 per cent respectively, an index number spread of only 6 points compared with the 64-point gap between the Laspeyres and Paasche. When the indices are chained, the chain Laspeyres and Paasche indices register increases of 33 per cent and 12 per cent respectively, reducing the gap between the two indices from 64 to 21 points. The chained Törnqvist and Fisher indices register increases of 22.26 per cent and 22.24 per cent respectively, being virtually identical numerically. These results show that the choice of index formula and method does matter.

Seasonal products

1.116 As explained in Chapter 22, the existence of seasonal products poses some intractable problems and serious challenges for CPI compilers and users. Seasonal products are products that are either:

- not available during certain seasons of the year; or
- are available throughout, but their prices or quantities are subject to regular fluctuations that are synchronized with the season or time of the year.

There are two main sources of seasonal fluctuations: the climate and custom. Month-to-month movements in a CPI may sometimes be so dominated by seasonal influences that it is difficult to discern the underlying trends in prices. Conventional seasonal adjustment programmes may be applied, but these may not always be satisfactory. The problem is not confined to interpreting movements in the CPI, as seasonality creates serious problems for the compilation of a CPI when some of the products in the basket regularly disappear and reappear, thereby breaking the continuity of the price series from which the CPI is built up. There is no panacea for seasonality. A consensus on what is best practice in this area has not yet been formed. Chapter 22 examines a number of different ways in which the problems may be tackled using an artificial data set to illustrate the consequences of using different methods.

1.117 One possibility is to exclude seasonal products from the index, but this may be an unacceptable reduction in the scope of the index, as seasonal products can account for a significant proportion of total household consumption. Assuming seasonal products are retained, one solution is to switch the focus from month-to-month movements in the index to changes between the same month in successive years. In some countries, it is common for the media and other users, such as central banks, to focus on the annual rate of inflation between the most recent month and the same month in the previous year. This year-on-year figure is much easier to interpret than month-to-month changes, which can be somewhat volatile, even in the absence of seasonal fluctuations.

1.118 This approach is extended in Chapter 22 to the concept of a rolling year-on-year index that compares the prices for the most recent 12 months with the corresponding months in the price reference year. The resulting *rolling year indices* can be regarded as seasonally adjusted price indices. They are shown to work well using the artificial data set. Such an index can be regarded as a measure of inflation for a year that is centred around a month that is six months earlier than the last month in the rolling index. For some purposes, this time lag may be disadvantageous, but in Chapter 22 it is shown that under certain conditions the current month year-on-year monthly index, together with the previous month's year-on-year monthly index, can successfully predict the rolling year index that is centred on the current month. Of course, rolling year indices and similar analytic constructs are not intended to replace the monthly or quarterly CPI but to provide supplementary information that can be extremely useful to users. They can be published alongside the official CPI.

1.119 Various methods of dealing with the breaks in price series caused by the disappearance and reappearance of seasonal products are examined in Chapter 22. However, this remains an area in which more research needs to be done.

Elementary price indices

1.120 As explained in Chapters 9 and 20, the calculation of a CPI proceeds in stages. In the first stage, *elementary price indices* are estimated for the *elementary expenditure aggregates* of a CPI. In the second stage, these elementary indices are aggregated, or averaged, to obtain higher-level indices using the elementary expenditure aggregates as weights. An elementary aggregate consists of the expenditures on a small and relatively homogeneous set of products defined within the consumption classification used in the CPI. As explained in Chapter 6, statistical offices usually select a set of representative products within each aggregate and then collect samples of their prices from a number of different outlets. The elementary aggregates serve as strata for sampling purposes.

1.121 The prices collected at the first stage are typically not prices observed in actual transactions between different economic units, but the prices at which the products are offered for sale in retail outlets of one kind or another. In principle, however, a CPI measures changes in the prices paid by households. These prices may actually vary during the course of a month, which is typically the time period to which the CPI relates. In principle, therefore, the first step should be to average the prices at which some product is sold during the period, bearing in mind that the price may vary even for the same product sold in the same outlet. In general, this is not a practical possibility. However, when the outlet is an electronic point of sale at which all the individual products are "scanned" as they are sold, the values of the transactions are actually recorded, thereby making it possible to calculate an average price instead of simply recording the offer price at a single point of time. Some use of scanner data is already made for CPI purposes and it may be expected to increase over the course of time.

1.122 Once the prices are collected for the representative products in a sample of outlets, the question arises of what is the most appropriate formula to use to estimate an elementary price index. This topic is considered in Chapter 20. It was comparatively neglected until a number of papers in the 1990s provided much clearer insights into the properties of elementary indices and their relative strengths and weaknesses. The quality of a CPI depends heavily on the quality of the elementary indices which are the building blocks from which CPIs are constructed.

1.123 Prices are collected for the same product in the same outlet over a succession of time periods. An elementary price index is therefore typically calculated from two sets of matched price observations. Here it is assumed that there are no missing observations and no changes in the quality of the products sampled so that the two sets of prices are perfectly matched. The treatment of new and disappearing products, and of quality change, is a separate and complex issue in its own right. It is outlined below, and discussed in detail in Chapters 7, 8 and 21.

Weights within elementary aggregates

1.124 In most cases, the price indices for elementary aggregates are calculated without the use of explicit expenditure weights. Whenever possible, however, weights should be used that reflect the relative importance of the sampled items, even if the weights are only approximate. In many cases, the elementary aggregate is simply the lowest level at which any reliable weighting information is available. In this case, the elementary index has to be calculated without the use of weights. Even in this case, however, it should be noted that when the items are selected with probabilities proportional to the size of some relevant variable such as sales, for example, weights are implicitly introduced by the sampling selection procedure.

1.125 For certain elementary aggregates, information about sales of particular items, market shares and regional weights may be used as explicit weights within an elementary aggregate. Weights within elementary aggregates may be updated independently, and possibly more often than the elementary aggregates themselves (which serve as weights for the higher-level indices).

1.126 For example, assume that the number of suppliers of a certain product, such as petrol, is limited. The market shares of the suppliers may be known from business survey statistics and can be used as weights in the calculation of an elementary aggregate price index for petrol. As another example, prices for water may be collected from a number of local water supply services where the population in each local region is known. The relative size of the population in each region may then be used as a proxy for the relative consumption expenditures to weight the price in each region to obtain the elementary aggregate price index for water.

Interrelationships between different elementary index formulae

1.127 Useful insights into the properties of various formulae that have been used, or considered, for elementary price indices may be gained by examining the mathematical interrelationships between them. Chapter 20 provides a detailed analysis of such relationships. As it is assumed that there are no explicit weights available, the various formulae considered all make use of unweighted averages: that is, *simple* averages in which the various items are *equally* weighted. There are two basic options for an elementary index:

– some kind of simple average of the price ratios or relatives;
– the ratio of some kind of simple average of the prices in the two periods.

In the case of a geometric average, the two methods coincide, as the geometric average of the price ratios or relatives is identical to the ratio of the geometric average prices.

1.128 Using the first of the above options, three possible elementary price indices are:
– a simple arithmetic average of the price relatives, known as the *Carli* index, or P_C; the Carli is the unweighted version of the Young index;
– a simple geometric average of the price relatives, known as the *Jevons* index, or P_J; the Jevons is the unweighted version of the geometric Young index;
– a simple harmonic average of the price relatives, or P_H.

As noted earlier, for any set of positive numbers the arithmetic average is greater than, or equal to, the geometric average, which in turn is greater than, or equal to, the harmonic average, the equalities holding only when the numbers are all equal. It follows that $P_C \geq P_J \geq P_H$.

1.129 It is shown in Chapter 20 that the gaps between the three indices widen as the variance of the price relatives increases. The choice of formula becomes more important the greater the diversity of the price movements. P_J can be expected to lie approximately halfway between P_C and P_H.

1.130 Using the second of the options, three possible indices are:

– the ratio of the simple arithmetic average prices, known as the *Dutot* index, or P_D;
– the ratio of the simple geometric averages, again the Jevons index, or P_J;
– the ratio of the simple harmonic averages, or P_H.

The ranking of *ratios* of different kinds of average are not predictable. For example, the Dutot, P_D, could be greater or less than the Jevons, P_J.

1.131 The Dutot can also be expressed as a weighted average of the price relatives in which the prices of period 0 serve as the weights:

$$P_D \equiv \frac{\sum_{i=1}^{n} p_i^t \Big/ n}{\sum_{i=1}^{n} p_i^0 \Big/ n} = \frac{\sum_{i=1}^{n} p_i^0 \left(\dfrac{p_i^t}{p_i^0} \right)}{\sum_{i=1}^{n} p_i^0} \qquad (1.17)$$

As compared with the Carli, which is a simple average of the price relatives, the Dutot gives more weight to the price relatives for the products with high prices in period 0. It is nevertheless difficult to provide an economic rationale for this kind of weighting. Prices are not expenditures. If the products are homogeneous, very few quantities are likely to be purchased at high prices if the same products can be purchased at low prices. If the products are heterogeneous, the Dutot should not be used anyway, as the quantities are not commensurate and not additive.

1.132 While it is useful to establish the interrelationships between the various indices, they do not actually help decide which index to choose. However, as the differences between the various formulae tend to increase with the dispersion of the price relatives, it is clearly desirable to define the elementary aggregates in such a way as to try to minimize the variation in the price movements within each aggregate. The less variation there is, the less difference the choice of index formula makes. As the elementary aggregates also serve as strata for sampling purposes, minimizing the variance in the price relatives within the strata will also reduce the sampling error.

Axiomatic approach to elementary indices

1.133 One way to decide between the various elementary indices is to exploit the axiomatic approach outlined earlier. A number of tests are applied to the elementary indices in Chapter 20.

1.134 The Jevons index, P_J, satisfies all the selected tests. It dominates the other indices in the way that the Fisher tends to dominate other indices at an aggregative level. The Dutot index, P_D, fails only one, the commensurability test. This failure is critical, however. It reflects the fundamental point made earlier that when the quantities are not additive from an economic viewpoint, the prices are also not additive and hence cannot be meaningfully averaged. However, P_D performs well when the sampled products are homogeneous. The key issue for the Dutot is therefore how heterogeneous are the products within the elementary aggregate. If the products are not sufficiently homogeneous for their quantities to be additive, the Dutot should not be used.

1.135 Although the Carli index, P_C, has been widely used in practice, the axiomatic approach shows it to have some undesirable properties. In particular, as the unweighted version of the Young index, it fails the time reversal and transitivity tests. This is a serious disadvantage, especially as elementary indices are often monthly chain indices. A consensus has emerged that the Carli may be unsuitable because it is liable to have a significant upward bias. This is illustrated by numerical example in Chapter 9. Its use is not sanctioned for the Harmonized Indices of Consumer Prices used within the European Union. Conversely, the harmonic average of the price relatives, P_H, is liable to have an equally significant downward bias; anyway, it does not seem to be used in practice.

1.136 Based on the axiomatic approach, the Jevons emerges as the preferred index, but its use may not be appropriate in all circumstances. If one observation is zero, the geometric mean is zero. The Jevons is sensitive to extreme falls in prices; it may be necessary to impose upper and lower bounds on the individual price relatives when using the Jevons.

Economic approach to elementary indices

1.137 The economic approach to elementary indices is explained in Chapter 20. The sampled products for which prices are collected are treated as if they constituted a basket of goods and services purchased by rational utility-maximizing consumers. The objective is then to estimate a conditional cost of living index covering the set of products in question.

1.138 It should be noted, however, that the differences between the prices of the sampled products do not necessarily mean that the products are qualitatively different. If markets were perfect, relative prices should reflect relative costs of production and relative utilities. In fact, price differences may occur simply because of market imperfections. For example, exactly the same products may be bought and sold at different prices in different outlets simply because consumers lack information about the prices charged in other outlets. Producers may also practise price discrimination, charging different customers different prices for exactly the same products. Price discrimination is widespread in many service industries. When the price differences are a result of market imperfections, consumers cannot be expected to react to changes in the relative prices of products in the same way as they would if they were well informed and had free choice.

1.139 In any case, assuming there is no information about quantities or expenditures within an elementary aggregate, it is not possible to calculate any kind of superlative index. So the conditional cost of living index at the level of an elementary aggregate can be estimated only on the assumption that certain special conditions apply.

1.140 There are two special cases of some interest. The first case is where the underlying preferences are so-called Leontief preferences. With these preferences *relative* quantities remain fixed whatever the relative prices. No substitutions are made in response to changes in relative prices. The cross-elasticities of demand are zero. With Leontief preferences, a Laspeyres index provides an exact measure of the cost of living index. In this case, the Carli calculated for a random sample would provide an estimate of the cost of living index provided that the items were selected with probabilities proportional to the population expenditure shares. It might appear that if the items were selected with probabilities proportional to the population quantity shares, the sample Dutot would provide an estimate of the population Laspeyres. However, assuming that the basket for the Laspeyres index contains a number of heterogeneous products whose quantities are not additive, the quantity shares, and hence the probabilities, are undefined.

1.141 The second case is one already considered above, namely when the preferences can be represented by a Cobb-Douglas function. As already explained, with these preferences, the geometric Laspeyres would provide an exact measure of the cost of living index. In this case, the Carli calculated for a random sample would provide an unbiased estimate of the cost of living index, provided that the items were selected with probabilities proportional to the population expenditure shares.

1.142 On the economic approach, the choice between the sample Jevons and the sample Carli rests on which is likely to approximate the more closely to the underlying COLI: in other words, on whether the demand cross-elasticities are likely to be closer to unity or zero, on average. In practice, the cross-elasticities could take on any value ranging up to plus infinity for an elementary aggregate in which the sampled products were strictly homogeneous, i.e., perfect substitutes.. It should be noted that in the limiting case in which the sampled products are homogeneous, there is only a single kind of product and therefore no index number problem: the price index is given by the ratio of the unit values in the two periods. It may be conjectured that, on average, the cross-elasticities are likely to be closer to unity than zero for most elementary aggregates so that, in general, the Jevons index is likely to provide a closer approximation to the cost of living index than the Carli. In this case, the Carli must be viewed as having an upward bias.

1.143 It is worth noting that the use of the Jevons index does not imply, or assume, that expenditure shares remain constant. Obviously, a geometric average of the price relatives can be calculated whatever changes do or do not occur in the expenditure shares, in practice. What the economic approach shows is that *if* the expenditure shares remain constant (or roughly constant), *then* the Jevons can be expected to provide a good estimate of the underlying cost of living index. The insight provided by the economic approach is that the Jevons is likely to provide a closer approximation to the cost of living index than the Carli because a significant amount of substitution is more likely than no substitution, especially as elementary aggregates should be deliberately constructed in such a way as to group together similar items that are close substitutes for each other.

1.144 An alternative to the Jevons, P_J, would be a geometric average of P_C and P_H, an index labelled P_{CSWD} in Chapter 20. This could be justified on grounds of treating the data in both periods symmetrically without invoking any particular assumption about the form of the underlying preferences. It is also shown in Chapter 20 that the geometric average of P_C and P_H is likely to be very close to P_J, so that the latter may be preferred on the grounds that it is a simpler concept and easier to compile.

1.145 It may be concluded that, based on the economic approach, as well as the axiomatic approach, the Jevons emerges as the preferred index in general, although there may be cases in which little or no substitution takes place within the elementary aggregate and the Carli might be preferred. The index compiler must make a judgement on the basis of the nature of the products actually included in the elementary aggregate.

1.146 The above discussion has also thrown light on some of the sampling properties of the elementary indices. If the products in the sample are selected with probabilities proportional to expenditures in the price reference period:

– the sample (unweighted) Carli index provides an unbiased estimate of the population Laspeyres;
– the sample (unweighted) Jevons index provides an unbiased estimate of the population geometric Laspeyres.

These results hold irrespective of what the underlying cost of living index may be.

Concepts, scope and classifications

1.147 The purpose of Chapter 3 of the manual is to define and clarify a number of basic concepts underlying a CPI and to explain the scope of the index: that is, the set of goods and services and the set of households that the index is intended to cover, in principle. Chapter 3 also examines the structure of the classification of consumer goods and services used.

1.148 While the general purpose of a CPI is to measure changes in the prices of *consumption* goods and services, there are a number of concepts that need to be defined precisely before an operational definition of a CPI can be arrived at. The concept of consumption is an imprecise one that can be interpreted in several different ways, each of which may lead to a different CPI. It is also necessary to decide whether the index is meant to cover all consumers, i.e., all households, or just a particular group of households. The scope of a CPI is inevitably influenced by what is intended, or believed, to be the main use of the index. Compilers also need to remember that the index may be used as proxy for a general price index and used for purposes other than those for which it is intended.

1.149 The word “consumer” can be used to refer both to a type of economic unit and to a type of product. To avoid confusion here, the term *consumption* good or service will be used where necessary, rather than *consumer* good or service. A consumption good or service provides utility to its user. It may be defined as *a good or service that members of households use, directly or indirectly, to satisfy their own personal needs and wants*. “Utility” should be interpreted in a broad sense. It is simply the generic, technical term preferred by economists for the benefit or welfare that individuals or households derive from the use of a consumer good or service.

1.150 A CPI is generally understood to be a price index that measures changes in the prices of consumption goods and services acquired and used by households. In principle, more broadly based price indices can be defined whose scope extends beyond consumption goods and services to include the prices of physical assets such as land or dwellings. Such indices may be useful as broad measures of inflation as perceived by households, but most CPIs are confined to consumption goods and services. These may include the prices of the flows of services provided by assets such as dwellings, even though the assets themselves may be excluded. In any case, the prices of financial assets such as bonds, shares or other marketable securities purchased by households are generally regarded as being outside the scope of a CPI.

Acquisitions and uses

1.151 The times at which households acquire and use consumption goods or services are generally not the same. Goods are typically acquired at one point in time and used at some other point in time, or even used repeatedly over an extended period of time. The time of acquisition of a *good* is the moment at which the legal or effective economic ownership of the good passes to the consumer. In a market situation, this is the point at which the purchaser incurs a liability to pay. A *service* is acquired at the time that the producer provides it, no change of ownership being involved. The time at which acquisitions are recorded, and their prices, should also be consistent with the way in which the same transactions are recorded in the expenditure data used for weighting purposes.

1.152 The time at which payment is made may be determined mainly by institutional arrangements and administrative convenience. When payments are not made in cash, there may be a significant lapse of time before the consumer's bank account is debited for a purchase paid for by cheque, by credit card or similar arrangements. The time at which these debits are eventually made is irrelevant for the recording of the acquisitions and the prices. On the other hand, when the acquisition of a good or service is financed by the creation of a new financial asset at the time of acquisition, such as a loan to the purchaser, two economically separate transactions are involved, the purchase/sale of the good or service and the creation of the asset. The price to be recorded is the price payable at the time of acquisition, however the purchase is financed. Of course, the provision of finance may affect the price payable. The subsequent repayments of any debt incurred by the purchaser and the associated interest payments are financial transactions that are quite distinct from the purchase of the good or service whose price has to be recorded. The explicit or implicit interest payments payable on the amount depend on the capital market, the nature of the loan, its duration, the creditworthiness of the purchaser, and so on. These points are explained in more detail in Chapter 3.

1.153 The distinction between the *acquisition* and the *use* of a consumer good or service outlined above has led to two different concepts of a CPI being proposed:

- A CPI may be intended to measure the average change between two time periods in the prices of the consumer goods and services acquired by households.
- Alternatively, a CPI may be intended to measure the average change between two time periods in the prices of the consumer goods and services used by households to satisfy their needs and wants.

The distinction between time of acquisition and time of use is particularly important for durable goods and certain kinds of services.

1.154 *Durable and non-durable goods.* A "non-durable" good might be better described as a *single use* good. For example, food or drink are used once only to satisfy hunger or thirst. Many so-called non-durable consumer goods are in fact extremely durable physically. Households may hold substantial stocks of non-durables, such as many foodstuffs and fuel, for long periods of time before they are used.

1.155 The distinguishing feature of a durable consumption good is that it is durable under use. Consumer durables can be used repeatedly or continuously to satisfy the needs or wants of consumers over a long period of time, possibly many years: for example, furniture or vehicles. For this reason, a durable is often described as providing a flow of services to the consumer over the period it is used (see also Box 14.3 of Chapter 14). There is a close parallel between the definitions of consumer durables and fixed assets. Fixed assets are defined in national accounts as goods that are used repeatedly or continuously over long periods of time in processes of production: for example, buildings or other structures, machinery and equipment.

1.156 A list of the different kinds of consumer durables distinguished in the Classification of Individual Consumption according to Purpose (COICOP) is given in Chapter 3. Of course, some durables last much longer than others, the less durable ones being described as "semi-durables" in COICOP: for example, clothing. It should be noted that dwellings are classified as fixed assets, not durable consumption goods, and are therefore not included in COICOP. Dwellings are used to *produce* housing services. These services are consumed by tenants or owner-occupiers, as the case may be, and are therefore included in COICOP.

1.157 Many services are durable and are also not fully consumed, or used up, at the time they are acquired. Some services bring about long-lasting improvements from which the consumers derive enduring benefits. The condition and quality of life of persons receiving medical treatments such as hip replacements or cataract surgery, for example, are substantially and permanently improved. Similarly, consumers of educational services can derive lifetime benefits from them. Expenditures on education and health also share with durable goods the characteristic that they are also often so costly that they have to be financed by borrowing or by running down other assets.

1.158 Expenditures on durable goods and durable services are liable to fluctuate, whereas using up such goods and services is likely to be a fairly steady process. However, the using up cannot be directly observed and valued. It can only be estimated by making assumptions about the timing and duration of the flows of benefits. Partly because of the conceptual and practical difficulties involved in measuring uses, statistical offices tend to adopt the acquisitions approach to consumer durables in both their national accounts and CPIs.

1.159 *A consumer price index based on the acquisitions approach*. Households may acquire goods and services for purposes of consumption in four main ways. They may:

- purchase them in monetary transactions;
- produce them themselves for their own consumption;
- receive them as payments in kind in barter transactions, particularly as remuneration in kind for work done;
- receive them as free gifts, or transfers, from other economic units.

1.160 The broadest possible scope for goods and services based on the acquisitions approach would be one covering all four categories, irrespective of who bears the costs. It would therefore include all *social transfers in kind* in the form of education, health, housing and other goods and services provided free of charge, or at nominal prices, to individual households by governments or non-profit institutions (NPIs). Total acquisitions are equivalent to the total actual individual consumption of (non-institutional) households, as defined in the SNA (see Chapter 14). *Collective* services provided by governments to the community as whole, such as public administration and defence, are not included and are outside the scope of a CPI.

1.161 From the point of view of the government or NPI that provides and pays for them, social transfers are valued either by the market prices paid for them or by the costs of producing them. From the point of view of the receiving households they have zero or nominal prices. For CPI purposes, the appropriate price is that paid by the household. The price paid by the government belongs in a price index for government expenditures. When households incur zero expenditures, the services provided free carry zero weight in a CPI. However, when governments and NPIs introduce charges for goods or services that were previously provided free, the increase from a zero to a positive price could be captured by a CPI, as explained in Chapter 3.

1.162 *Expenditures versus acquisitions.* Expenditures need to be distinguished from acquisitions. Expenditures are incurred by the economic units that bear the costs. Households do not incur expenditures on social transfers in kind, so the scope of households' expenditures is generally narrower than the scope of their acquisitions. Moreover, not all expenditures are monetary. A *monetary expenditure* occurs when a household pays in cash, by cheque or credit card, or otherwise incurs a financial liability to pay. Only monetary expenditures generate monetary prices that can be observed and recorded for CPI purposes.

1.163 *Non-monetary expenditures* occur when households pay, but in other ways than cash. There are three important categories of non-monetary expenditures:

- In barter transactions, households exchange consumption goods and services among themselves. As the values of the goods and services surrendered as payments constitute negative expenditures, the expenditures should cancel out so that barter transactions between households carry zero weight on aggregate. They can be ignored in practice for CPI purposes.
- When employees are remunerated in kind, they purchase the goods or services, but pay with their labour, not cash. Monetary values can be imputed for the expenditures implicitly incurred by the households.
- Similarly, when households produce goods and services for themselves, they incur the costs, some of which may be monetary in the form of purchased inputs. The monetary values of the implicit expenditures on the outputs produced can be imputed on the basis of the corresponding market prices. If such imputed prices were to be included in the CPI, the prices of the inputs would have to be excluded to avoid double counting.

1.164 *A hierarchy of consumption aggregates.* A hierarchy of possible consumption aggregates may be envisaged, as explained in Chapter 14:

– total acquisitions of goods and services by households;
– *less* social transfers in kind = households' total expenditures;
– *less* non-monetary expenditures = households' monetary expenditures.

The choice of consumption aggregate is a policy matter. For example, if the main reason for compiling a CPI is to measure inflation, the scope of the index might be restricted to household monetary expenditures on consumption, inflation being essentially a monetary phenomenon. Prices cannot be collected for the consumer goods and services involved in non-monetary expenditures, although they can be estimated on the basis of the prices observed in corresponding monetary transactions. The European Union's Harmonized Indices of Consumer Prices, which are specifically intended to measure inflation within the EU, are confined to monetary expenditures.

Unconditional and conditional cost of living indices

1.165 Cost of living indices, or COLIs, are explained in Chapters 15 and 17. As also noted in Chapter 3, the scope of a COLI depends on whether it is conditional or unconditional. The welfare of a household depends not only on the utility derived from the goods and services it consumes, but on the social, political and physical environment in which the household lives. An *unconditional* cost of living index measures the change in the minimum cost of maintaining a given level of welfare in response to changes in any of the factors that affect welfare, whereas a *conditional* cost of living index measures the change in the minimum cost of maintaining a given level of utility or welfare resulting from changes in consumer prices, holding the environmental factors constant.

1.166 An unconditional COLI may be a more comprehensive *cost of living* index than a conditional COLI, but it is not a more comprehensive *price* index. An unconditional index does not include any more price information than a conditional index and it does not give more insight into the impact of price changes on welfare. On the contrary, the impact of the price changes is diluted and obscured the more environmental variables are included within the scope of an unconditional index. In order to qualify as a price index, a COLI must be conditional.

Specific types of transactions

1.167 Given that conceptually, a CPI is an index that measures changes in the prices of consumption goods and services, expenditures on items that are not consumption goods and services fall outside the scope of the CPI; for example, expenditures on assets such as land or bonds, shares and other financial assets. Similarly, payments that do not involve any flows of goods or services in return for the payments are outside the scope; for example, payments of income taxes or social security contributions.

1.168 *Transfers*. A transfer occurs when one economic unit provides a good, service or asset, including money, to another without receiving any counterpart good, service or asset in return. As no good or service is acquired when a household makes a transfer, the transfer must be outside the scope. For this reason, compulsory cash transfers, such as payments of direct taxes on income or wealth, must be outside the scope of a CPI. It is not always clear, however, whether certain payments to government are transfers or purchases of services. For example, payments to obtain certain kinds of licences are sometimes taxes under another name, whereas in other cases the government may provide a service by exercising some kind of supervisory, regulatory or control function. Gifts or donations must be transfers and therefore outside the scope. On the other hand, subscriptions to clubs and societies which provide their members with some kind of service in return are included. Tips and gratuities can be borderline cases. When they are effectively an expected, even obligatory, part of the payment for a service they are not transfers and should be treated as part of the price paid.

1.169 *Undesirable or illegal goods or services*. All goods and services that households *willingly* buy on the market to satisfy their own needs and wants should be included, even if most people might regard them as undesirable or even if they are prohibited by law. Of course, illegal goods and services may have to be excluded in practice because the requisite data cannot be collected.

1.170 *Financial transactions*. Financial transactions occur when one kind of financial asset is exchanged for another, bearing in mind that money is itself a financial asset. For example, the purchase of a bond or share is a financial transaction. Borrowing is a financial transaction in which cash is exchanged, the counterpart being the creation of a financial asset or liability.

1.171 No consumption occurs when a financial transaction takes place, even though financial transactions may be undertaken in order to facilitate future consumption. Financial transactions as such are not covered by CPIs because, by definition, no goods are exchanged, nor services provided, in financial transactions. However, some "financial" transactions may not be entirely financial because they may include an explicit or implicit service charge in addition to the provision of an asset, such as a loan. As a service charge constitutes the purchase of a service by the household, it should be included in a CPI, although it may be difficult to separate out the service charge in some cases. For example, foreign exchange transactions are financial transactions in which one financial asset is exchanged for another. Changes in the price of a foreign currency in terms of the domestic currency resulting from changes in the exchange rate are outside the scope of a CPI. On the other hand, the commission charges associated with the exchange of currencies are included as payments for the services rendered by the foreign exchange dealers.

1.172 Households may borrow in order to make large expenditures on durables or houses, but also to finance large educational or health expenses, or even expensive holidays. Whatever the purpose of the borrowing, the financial transaction in which the loan is contracted is outside the scope of a CPI. The treatment of the interest payable on loans is a separate issue considered below.

1.173 *Composite transactions*. As just noted, some transactions are composite transactions containing two or more components whose treatment may be quite different for CPI purposes. For example, part of a life insurance premium is a financial transaction leading to the creation of a financial claim and is therefore outside the scope, whereas the remainder consists of a service charge which should be covered by a CPI. The two components are not separately itemized, however.

1.174 As explained in Chapter 3, the treatment of payments of nominal interest is difficult because it may have four conceptually quite different components:

– a pure interest payment;
– a risk premium that depends on the creditworthiness of the borrower;
– a service charge payable to the bank, moneylender or other financial institution engaged in the business of making loans;
– a payment to compensate the creditor for the real holding loss incurred on the principal of the loan during inflation.

The fourth component is clearly outside the scope of a CPI as it is a capital flow. Conversely, the third component, the service charge, should clearly be included. The treatment of the first two components is controversial. When there is significant inflation or a very imperfect capital market, payments of nominal interest may be completely dominated by the last two components, both of which are conceptually quite different from the concept of pure interest. For example, the "interest" charged by a village moneylender may be mostly a high service charge. It may be impossible to decompose the various components of nominal interest in practice. The treatment of nominal interest as a whole remains difficult and somewhat controversial.

Household production

1.175 When households engage in production for the market, the associated business transactions are all outside the scope of a CPI. Expenditures incurred for business purposes are excluded, even though they involve purchases of goods and services that might be used to satisfy the personal needs and wants of members of the household instead.

1.176 Households also produce goods and services for their own consumption, mainly service production such as the preparation of meals, the care of children, the sick or the elderly, the cleaning and maintenance of durables and dwellings, the transportation of household members, and so on. Owner-occupiers produce housing services for their own consumption. Households also grow vegetables, fruit, flowers or other crops for their own use.

1.177 Many of the goods or services purchased by households do not provide utility directly but are used as inputs into the production of other goods and services that do provide utility: for example, raw foodstuffs, fertilizers, cleaning materials, paints, electricity, coal, oil, petrol, and so on.

1.178 In principle, a CPI should record changes in the prices of the outputs from these production activities, as it is the outputs rather than the inputs that are actually consumed and provide utility. However, as the outputs are not themselves purchased, no prices can be observed for them. Prices could be imputed for them equal to the prices they would fetch on the market, but this would make a CPI heavily dependent on assumed rather than collected prices. The pragmatic solution recommended in Chapter 3 is to treat all goods and services purchased on the market to be used exclusively as inputs into the production of other goods and services that are directly consumed by households as if they were themselves consumption goods and services. On this principle, goods such as insecticides and electricity are treated as providing utility indirectly and included in CPIs. This is, of course, the solution usually adopted in practice not only for CPIs but also in national accounts, where most expenditures on inputs into household production are classified as final consumption expenditures.

1.179 In some countries, there is an increasing tendency for households to purchase prepared, take-away meals rather than the ingredients. As the prices of such meals cost more than the sum of the ingredients that the households previously purchased, the weight attached to food consumption increases. This partly reflects the fact that the costs of the households' own labour inputs into the preparation of meals were previously ignored. Various kinds of household service activities that were previously outside the scope of a CPI may be brought within the scope if households choose to pay others to perform the services.

1.180 *Subsistence agriculture and owner-occupied housing*. In the case of two important types of production for own consumption within households, namely agricultural production for own consumption and housing services produced by owner-occupiers, the national accounts do actually try to record the values of the outputs produced and consumed rather than the inputs. Similarly, CPIs may also try to price the outputs rather than the inputs in these two cases.

1.181 In principle, the prices of the outputs from own-account agricultural production may be included in CPIs, even though they are imputed. On the other hand, for households relying on subsistence agriculture, the prices of inputs of agricultural materials purchased on the market may be their main exposure to inflation. Two points may be noted. First, the imputed market value of the output should usually be greater than the costs of the purchased inputs, if only because it should cover the costs of the labour inputs provided by the household. Thus, pricing the purchased inputs rather than the outputs may mean that the consumption of own agricultural production in CPIs does not receive sufficient weight. Second, double counting should be avoided. If the imputed prices of the outputs are included, the actual prices of the inputs consumed should not be included as well.

1.182 In the case of owner-occupied housing, the situation is complicated by the fact that the production requires the use of the capital services provided by a large fixed asset in the form of the dwelling itself. Even if the inputs into the production of housing services are priced for CPI purposes, it is still necessary to impute prices for the inputs of capital services (mainly depreciation plus interest) provided by the dwelling. Some countries therefore prefer to impute the prices of the outputs of housing services actually consumed on the basis of the rents payable for the same kind of dwellings rented on the market. The treatment of owner-occupied housing is complex, and somewhat controversial, and is considered in Chapters 3, 9, 10 and 23, among others.

Coverage of households and outlets

1.183 As explained in Chapter 3, households may be either individual persons or groups of persons living together who make common provision for food or other essentials for living. A CPI may be required to cover:

– *either* the consumption expenditures made by households resident in a particular area, usually a country or region, whether the expenditures are made inside or outside the area – this is called the "national" concept of expenditure;
– *or* the consumption expenditures that take place within a particular area, whether made by households resident in that area or residents of other areas – this is called the "domestic" concept.

Adopting the domestic concept may make it more difficult to collect the relevant disaggregated expenditure data in household surveys. A CPI may also be defined to cover a group of countries, such as the European Union.

1.184 Not all kinds of households have to be included. As explained in Chapter 3, some countries choose to exclude particular categories of households such as very wealthy households or households engaged in agriculture. Some countries also compile different indices designed to cover different groups of households, such as households resident in different regions. Another possibility is to compile a general CPI designed to cover all or most households and, in addition, one or more special indices aimed at particular sections of the community, such as households headed by pensioners. The precise coverage of households is a matter of choice. It is inevitably influenced by what are believed to be the main uses of the index. The set of households actually covered by the CPI is described as the "reference population".

Price variation

1.185 Prices for exactly the same good or service may vary between different outlets, while different prices may sometimes be charged to different types of customers. Prices may also vary during the course of the month to which the index relates. Conceptually, it is necessary to distinguish such pure price variation from price differences that are attributable to differences in the quality of the goods or services offered, although it is not always easy to distinguish between the two in practice. The existence of pure price differences reflects some form of market imperfections, such as consumers' lack of information or price discrimination.

1.186 When pure price differences exist, a change in market conditions may make it possible for some households to switch from purchasing at higher prices to purchasing at lower prices, for example if new outlets open that offer lower prices. The resulting fall in the average price paid by households counts as a price fall for CPI purposes, even though the price charged by each individual outlet may not change. If the prices are collected from the outlets and switches in households' purchasing habits remain unobserved, the CPIs are said to be subject to outlet substitution bias, as explained in more detail in Chapter 11. On the other hand, when the price differences reflect differences in the quality of the goods and services sold in the different outlets, switching from outlets selling at higher prices to outlets selling at lower prices simply means that households are choosing to purchase lower-quality goods or services. In itself, this does not imply any change in price.

Classifications

1.187 As explained in Chapter 3, the classification of household expenditures used in a CPI provides the necessary framework for the various stages of CPI compilation. It provides a structure for purposes of weighting and aggregation, and also a basis for stratifying the samples of products whose prices are collected. The goods and services covered by a CPI may be classified in several ways: not simply on the basis of their physical characteristics but also by the purposes they serve and the degree of similarity of their price behaviour. Product-based and purpose-based classifications differ but can usually be successfully mapped onto each other. In practice, most countries use a hybrid classification system in which the breakdown at the highest level is by purpose while the lower-level breakdowns are by product type. This is the case for the recently revised internationally agreed Classification of Individual Consumption according to Purpose (COICOP), which provides a suitable classification for CPI purposes.

1.188 The first level of classification in COICOP consists of 12 divisions covering total consumption expenditures of households. As just noted, the breakdown into divisions is essentially by purpose. At the second level of disaggregation, the 12 *divisions* are divided into 47 *groups* of products, which are in turn divided into 117 *classes* of products at the third level. Chapter 3 provides a listing of ten classes of goods defined as durables in COICOP. It also gives a list of seven classes described as semi-durables, such as clothing, footwear and household textiles.

1.189 The 117 classes at the lowest level of aggregation of COICOP are not sufficiently detailed for CPI purposes. They can be divided into sub-classes using the sub-classes of the internationally agreed Central Product Classification (CPC). Even some of these may require further breakdown in order to arrive at some of the elementary aggregates used for CPI purposes. In order to be useful for CPI purposes, expenditure weights must be available for the various sub-classes or elementary aggregates. From a sampling perspective, it is desirable for the price movements of the individual products within the elementary aggregates to be as homogeneous as possible. The elementary aggregates may also be divided into strata for sampling purposes, on the basis of location or the type of outlet in which the products are sold.

Consumer price indices and national accounts price deflators

1.190 Appendix 3.1 of Chapter 3 explains the differences between the overall CPI and the deflator for total household consumption expenditures in national accounts. In practice, CPIs may be designed to cover only a subset of the households and a subset of the expenditures covered by the national accounts. Moreover, the index number formulae needed for CPIs and national accounts deflators may be different. These differences mean that the overall CPI is generally not the same as the deflator for total household consumption expenditures in the national accounts. On the other hand, the basic price and expenditure data collected and used for CPI purposes are also widely used to build up the price indices needed to deflate the individual components of household consumption in the national accounts.

Expenditure weights

1.191 As already noted, there are two main stages in the calculation of a CPI. The first is the collection of the price data and the calculation of the elementary price indices. The second is the averaging of the elementary price indices to arrive at price indices at higher levels of aggregation up to the overall CPI itself. Expenditure data are needed for the elementary aggregates that can be used as weights in the second stage. These weights are needed whatever index number formula is used for aggregation purposes. Chapter 4 is concerned with the derivation, and sources, of the expenditure weights.

Household expenditure surveys and national accounts

1.192 The principal data source for household consumption expenditures in most countries is a household expenditure survey (HES). An HES is a sample survey of thousands of households that are asked to keep records of their expenditures on different kinds of consumer goods and services over a specified period of time, such as a week or longer. The size of the sample obviously depends on the resources available, but also on the extent to which it is desired to break down the survey results by region or type of household. HESs are costly operations. This manual is not concerned with the conduct of HESs or with general sampling survey techniques or procedures. There are several standard texts on survey methods to which reference may be made. Household expenditure surveys may be taken at specified intervals of time, such as every five years, or they may be taken each year on a continuing basis.

1.193 HESs can impose heavy burdens on the respondents, who have to keep detailed expenditure records of a kind that they would not normally keep, although this may become easier when supermarkets or other retail outlets provide detailed printouts of purchases. HESs tend to have some systematic biases. For example, many households either deliberately, or unconsciously, understate the amounts of their expenditures on certain “undesirable” products, such as gambling, alcoholic drink, tobacco or drugs. Corrections can be made for such biases. Moreover, the data collected in HESs may also need to be adjusted to bring them into line with the concept of expenditure required by the CPI. For example, the imputed expenditures on the housing services produced and consumed by owner-occupiers are not collected in HESs.

1.194 As explained in Chapter 14, the use of the commodity flow method within the supply and use tables of the SNA enables data drawn from different primary sources to be reconciled and balanced against each other. The commodity flow method may be used to improve estimates of household consumption expenditures derived from expenditure surveys by adjusting them to take account of the additional information provided by statistics on the sales, production, imports and exports of consumer goods and services. By drawing on various sources, the household expenditure data in the national accounts may provide the best estimates of aggregate household expenditures, although the classifications used may not be fine enough for CPI purposes. Moreover, because HESs may be conducted only at intervals of several years, the expenditure data in the national accounts may be more up to date, as national accounts are able to draw upon other kinds of more recent data, such as retail sales and the production and import of consumer goods and services. It is important to note, however, that national accounts should not be viewed as if they were an alternative, independent data source to HESs. On the contrary, HESs provide one of the main sources for the expenditure data on household consumption used to compile national accounts.

1.195 Household expenditure surveys in many countries may not be conducted as frequently as might be desired for CPI, or national accounts, purposes. National HESs can be very costly and onerous for the households, as already noted. They may be conducted only once every five or ten years, or even at longer intervals. In any case, conducting and processing HESs is time-consuming, so the results may not be available for CPI purposes until one or two years after the surveys have been conducted. It is for these practical reasons that CPIs in many countries are Lowe indices that use the quantities of some base period b that may precede the time reference period 0 by a few years and period t by many years.

1.196 Some countries conduct continuous HESs not only in order to update their CPI weights but also to improve their national accounts. Of course, the same panel of households does not have to be retained indefinitely; the panel can be gradually rotated by dropping some households and replacing them by others. Countries that conduct continuous expenditure surveys are able to revise and update their expenditure weights each year so that the CPI becomes a chain index with annual linking. Even with continuous expenditure surveys, however, there is a lag between the time at which the data are collected and the time at which the results are processed and ready for use, so that it is never possible to have survey results that are contemporaneous with the price changes. Thus, even when the weights are updated annually, they still refer to some period that precedes the time reference period. For example, when the price reference period is January 2000, the expenditure weights may refer to 1997 or 1998, or both years. When the price reference period moves forward to January 2001, the weights move forward to 1998 or 1999, and so on. Such an index is a chain Lowe index.

1.197 Some countries prefer to use expenditure weights that are the average rates of expenditure over periods of two or three years in order to reduce “noise” caused by errors of estimation (the expenditure surveys are only samples) or erratic consumer behaviour over short periods of time resulting from events such as booms or recessions, stock market fluctuations, oil shocks, or natural or other disasters.

Other sources for estimating expenditure weights

1.198 If expenditures need to be disaggregated by region for sampling or analytical purposes, it is possible to supplement whatever information may be available by region in HESs by using data from population censuses. Another data source may be food surveys. These are special surveys, conducted in some countries, that focus on households’ expenditures on food products. They can provide more detailed information on food expenditures than that available from HESs.

1.199 Another possible source of information consists of points of purchase (POP) surveys, which are conducted in some countries. A POP survey is designed to provide information about the retail outlets at which households purchase specified groups of goods and services. Households are asked, for each item, about the amounts spent in each outlet and the names and addresses of the outlets. The main use for a POP survey is for selecting the sample of outlets to be used for price collection purposes.

Collection of price data

1.200 As explained in Chapter 9, there are two levels of calculation involved in a CPI. At the lower level, samples of prices are collected and processed to obtain lower-level price indices. These lower-level indices are the elementary indices, whose properties and behaviour are studied in Chapter 20. At the higher level, the elementary indices are averaged to obtain higher-level indices using expenditures as weights. At the higher level, all the index number theory elaborated in Chapters 15 to 18 comes into play.

1.201 Lower-level indices are calculated for elementary aggregates. Depending on the resources available and procedures adopted by individual countries, these elementary aggregates could be sub-classes or micro-classes of the expenditure classification described above. If it is desired to calculate CPIs for different regions, the sub-classes or micro-classes have to be divided into strata referring to the different regions. In addition, in order to improve the efficiency of the sampling procedures used to collect prices, it will usually be desirable, if feasible, to introduce other criteria into the definitions of the strata, such as the type of outlet. When the sub-classes or micro-classes are divided into strata for data collection purposes, the strata themselves become the elementary aggregates. As a weight needs to be attached to each elementary aggregate in order to calculate the higher-level indices, an estimate of the expenditure within each elementary aggregate must be available. Expenditure or quantity data are typically not available within an elementary aggregate, so the elementary indices have to be estimated from price data alone. This may change if scanner data from electronic points of sale become more available.

1.202 Chapter 5 is concerned with sampling strategies for price collection. Chapter 6 is concerned with the methods and operational procedures actually used to collect prices. In principle, the relevant prices for a CPI should be the purchasers' prices actually paid by households, but it is generally neither practical nor cost-effective to try to collect prices each month or quarter directly from households, even though expenditure data are collected directly from households in household expenditure surveys. In practice, the prices that are collected are not actual transaction prices, but rather the prices at which goods and services are offered for sale in outlets such as retail shops, supermarkets or service providers. However, it may become increasingly feasible to collect actual transactions prices as more goods and services are sold through electronic points of sale that record both prices and expenditures.

Random sampling and purposive sampling

1.203 Given that the prices are collected from the sellers, there are two different sampling problems that arise. The first is how to select the individual products within an elementary aggregate whose prices are to be collected. The second is how to select a sample of outlets selling those products. For some products, it may not be necessary to visit retail outlets to collect prices because there may be only a single price applying throughout the country. Such prices may be collected from the central organization responsible for fixing the prices. The following paragraphs refer to the more common situation in which prices are collected from a large number of outlets.

1.204 As explained in Chapter 5, the universe of products from which the sample is taken has several dimensions. The products may be classified not only on the basis of the characteristics and functions that determine their place in COICOP, but also according to the locations and outlets at which they are sold and the times at which they are sold. The fact that the universe is continually changing over time is a major problem, not only for CPIs but also for most other economic statistics. Products disappear to be replaced by other kinds of products, while outlets close and new ones open. The fact that the universe is changing over time creates both conceptual and practical problems, given that the measurement of price changes over time requires some continuity in the products priced. In principle, the price changes recorded should refer to matched products that are identical in both time periods. The problems created when products are not identical are considered in some detail later.

1.205 In designing the sample for price collection purposes, due attention should be paid to standard statistical criteria to ensure that the resulting sample estimates are not only unbiased and efficient in a statistical sense, but also cost-effective. There are two types of bias encountered in the literature on index numbers, namely *sampling bias* as understood here and the *non-sampling biases* in the form of substitution bias or representativity bias, as discussed in Chapter 10. It is usually clear from the context which type of bias is meant.

1.206 There is a large literature on sampling survey techniques to which reference may be made and which need not be summarized here. In principle, it would be desirable to select both outlets and products using random sampling with known probabilities of selection. This ensures that the sample of products selected is not distorted by subjective factors and enables sampling errors to be calculated. Many countries nevertheless continue to rely heavily on the purposive selection of outlets and products, because random sampling may be too difficult and too costly. Purposive selection is believed to be more cost-effective, especially when the sampling frames available are not comprehensive and not well suited to CPI purposes. It may also be cost-effective to collect a "cluster" of prices on different products from the same outlet, instead of distributing the price collection more thinly over a larger number of outlets.

1.207 Efficient sampling, whether random or purposive, requires comprehensive and up-to-date sampling frames. Two types of frames are needed for CPI purposes: one listing the universe of outlets, and the other listing the universe of products. Examples of possible sampling frames for outlets are business registers, central or local government administrative records or telephone directories. When the sampling frames contain the requisite information, it may be possible to increase efficiency by selecting samples of outlets using probabilities that are proportional to the size of some relevant economic characteristic, such as the total value of sales. Sampling frames for products are not always readily available in practice. Possible frames are catalogues or other product lists drawn up by major manufacturers, wholesalers or trade associations, or lists of products that are specific to individual outlets such as large supermarkets.

1.208 Depending on the information available in the sampling frame, it may be possible to group the outlets into strata on the basis of their location and size, as indicated by sales or employees. When there is information about size, it may be possible to increase efficiency by taking a random sample of outlets with probabilities proportional to size. In practice, however, there is also widespread use of purposive sampling.

1.209 In most countries, the selection of most of the individual items to be priced within the selected outlets tends to be purposive, being specified by the central office responsible for the CPI. The central office draws up lists of products that are deemed to be representative of the products within an elementary aggregate. The lists can be drawn up in collaboration with managers of wholesale or large retail establishments, or other experts with practical experience and knowledge. The actual procedures are described in more detail in Chapter 6.

1.210 It has been argued that the purposive selection of products is liable to introduce only a negligible amount of sampling bias, although there is not much conclusive evidence on this matter. In principle, random sampling is preferable and it is also quite feasible. For example, the United States Bureau of Labor Statistics makes extensive use of random selection procedures to select both outlets and products within outlets. When the selection of products is delegated to the individual price collectors, it is essential to ensure that they are well trained and briefed, and closely supervised and monitored.

Methods of price collection

1.211 The previous section focused on the sampling issues that arise when prices have to be collected for a large number of products from a large number of outlets. This section is concerned with some of the more operational aspects of price collection.

1.212 *Central price collection.* Many important prices can be collected directly by the central office responsible for the CPI from the head office of the organization responsible for fixing the prices. When prices are the same throughout the country, collection from individual outlets is superfluous:

- Some tariffs or service charges are fixed nationally and apply throughout the country. This may be the case for public utilities such as water, gas and electricity, postal services and telephone charges, or public transport. The prices or charges can be obtained from the relevant head offices.
- Some national chains of stores or supermarkets may charge the same prices everywhere, in which case the prices can be obtained from their head offices. Even when national chains do not charge uniform prices, there may be only a few minor regional differences in the prices and all the relevant information may be obtainable centrally.
- Many of these prices determined centrally may change very infrequently, perhaps only once or twice or year, so they do not have to be collected monthly. Moreover, many of these prices can be collected by telephone, fax or email and may not require visits to the head offices concerned.

1.213 *Scanner data*. One important new development is the increasing availability in many countries of large amounts of very detailed "scanner" data obtained from electronic points of sale. Such data are collated by commercial databases. Scanner data are up to date and comprehensive. An increasingly large proportion of all goods sold are being scanned as they pass through electronic points of scale.

1.214 The potential benefits of using scanner data are obviously considerable and could ultimately have a significant impact on the way in which price data are collected for CPI purposes. Not enough experience is yet available to provide general guidelines about the use of scanner data. Clearly, statistical offices should monitor developments in this field closely and explore the possibility of exploiting this major new source of data. Scanner data also increase the scope for using improved methods of quality adjustment, including hedonic methods, as explained in Chapter 7.

1.215 *Local price collection*. When prices are collected from local outlets, the individual products selected for pricing can be determined in two ways. One way is for a specific list of individual products to be determined in advance by the central office responsible for the CPI. Alternatively, the price collector can be given the discretion to choose from a specified range of products. The collector may use some kind of random selection procedure, or select the products that sell the most or are recommended by the shop owner or manager. An individual product selected for pricing in an individual outlet may be described as a sampled product. It may be a good or a service.

1.216 When the list of products is determined in advance by the central office, the objective is usually to select products that are considered to be representative of the larger group of products within an elementary aggregate. The central office also has to decide how loosely or tightly to describe, or specify, the representative products selected for pricing. In theory, the number of different products that might be identified is to some extent arbitrary, depending on the number of economic characteristics that are deemed to be relevant or important. For example, “beef” is a generic term for a group of similar but nevertheless distinct products. There are many different cuts of beef, such as minced beef, stewing steak or rump steak, each of which can be considered a different product and which can sell at very different prices. Furthermore, beef can also be classified according to whether it is fresh, chilled or frozen, and cross-classified again according to whether it comes from domestic or imported animals, or from animals of different ages or breeds.

1.217 Tightening the specifications ensures that the central office has more control over the items actually priced in the outlets, but it also increases the chance that some products may not actually be available in some outlets. Loosening the specifications means that more items may be priced but leaves the individual price collectors with more discretion with regard to the items actually priced. This could make the sample as a whole less representative.

Continuity of price collection

1.218 A CPI is intended to measure pure price changes. The products whose prices are collected and compared in successive time periods should ideally be perfectly *matched*; that is, they should be identical in respect of their physical and economic characteristics. When the products are perfectly matched, the observed price changes are *pure* price changes. When selecting representative products, it is therefore necessary to ensure that enough of them can be expected to remain on the market over a reasonably long period of time in exactly the same form or condition as when first selected. Without continuity, there would not be enough price changes to measure.

1.219 Having identified the items whose prices are to be collected, the normal strategy is to continue pricing exactly those same items for as long as possible. Price collectors can do this if they are provided with very precise, or tight, specifications of the items to be priced. Alternatively, they must keep detailed records themselves of the items that they have selected to price.

1.220 The ideal situation for a price index would be one in which all the products whose prices are being recorded remain on the market indefinitely without any change in their physical and economic characteristics, except of course for the timing of their sale. It is worth noting that many theorems in index number theory are derived on the assumption that exactly the same set of goods and services is available in both the time periods being compared. Most products, however, have only a limited economic life. Eventually, they disappear from the market to be replaced by other products. As the universe of products is continually evolving, the representative products selected initially may gradually account for a progressively smaller share of total purchases and sales. As a whole, they may become less and less representative. As a CPI is intended to cover all products, some way has to be found to accommodate the changing universe of products. In the case of consumer durables whose features and designs are continually being modified, some models may have very short lives indeed, being on the market for only a year or less before being replaced by newer models.

1.221 At some point the continuity of the series of price observations may have to be broken. It may become necessary to compare the prices of some products with the prices of other new ones that are very similar but not identical. Statistical offices must then try to eliminate from the observed price changes the estimated effects of the changes in the characteristics of the products whose prices are compared. In other words, they must try to adjust the prices collected for any changes in the quality of the products priced, as explained in more detail below. At the limit, a completely new product may appear that is so different from those existing previously that quality adjustment is not feasible and its price cannot be directly compared with that of any previous product. Similarly, a product may become so unrepresentative or obsolete that it has to be dropped from the index because it is no longer worth trying to compare its price with those of any of the products that have displaced it.

Resampling

1.222 One strategy to deal with the changing universe of products would be to resample, or reselect, at regular intervals the complete set of items to be priced. For example, with a monthly index, a new set of items could be selected each January. Each set of items would be priced until the following January. Two sets have to be priced each January in order to establish a link between each set of 12 monthly changes. Resampling each year would be consistent with a strategy of updating the expenditure weights each year.

1.223 Although resampling may be preferable to maintaining an unchanged sample or selection, it is not used much in practice. Systematically resampling the entire set of products each year would be difficult to manage and costly to implement. Moreover, it does not provide a complete solution to the problem of the changing universe of products, as it does not capture price changes that occur at the moment of time when new products or new qualities are first introduced. Many producers deliberately use the time when products are first marketed to make significant price changes.

1.224 A more practical way in which to keep the sample up to date is to rotate it gradually by dropping certain items and introducing new ones. Items may be dropped for two reasons:

- The product is believed by the price collector or central office to be no longer representative. It appears to account for a steadily diminishing share of the total expenditures within the basic categories in question.

- The product may simply disappear from the market altogether. For example, it may have become obsolete as a result of changing technology or unfashionable because of changing tastes, although it could disappear for other reasons.

1.225 At the same time, new products or new qualities of existing products appear on the market. At some point, it becomes necessary to include them in the list of items priced. This raises the general question of the treatment of quality change and the treatment of new products.

Adjusting prices for quality changes

1.226 The treatment of quality change is perhaps the greatest challenge facing CPI compilers. It is a recurring theme throughout this manual. It presents both conceptual and practical problems for compilers of CPIs. The whole of Chapter 7 is devoted to the treatment of quality change, while Chapter 8 addresses the closely related topic of new goods and item substitution.

1.227 When a sampled product is dropped from the list of products priced in some outlet, the normal practice is to find a new product to replace it in order to ensure that the sample, or selection, of sampled products remains sufficiently comprehensive and representative. If the new product is introduced specifically to replace the old one, it is necessary to establish a link between the series of past price observations on the old item and the subsequent series for the new item. The two series of observations may, or may not, overlap in one or more periods. In many cases, there can be no overlap because the new quality, or model, is only introduced after the one which it is meant to replace is discontinued. Whether or not there is an overlap, the linking of the two price series requires some estimate of the change in quality between the old product and the product selected to replace it.

1.228 However difficult it is to estimate the contribution of the change in quality to the change in the observed price, it must be clearly understood that some estimate has to be made either explicitly or, by default, implicitly. The issue cannot be avoided or bypassed. All statistical offices have limited resources and many may not have the capacity to undertake the more elaborate explicit adjustments for quality change described in Chapter 7. Even though it may not be feasible to undertake an explicit adjustment through lack of data or resources, it is not possible to avoid making some kind of implicit adjustment. Even apparently "doing nothing" necessarily implies some kind of implicit adjustment, as explained below. Whatever the resources available to them, statistical offices must be conscious of the implications of the procedures they adopt.

1.229 Three points are stressed in the introductory section of Chapter 7:

- The pace of innovation is high, and possibly increasing, leading to continual changes in the characteristics of products.
- There is not much consistency between countries in the methods they use to deal with quality change.
- A number of empirical studies have demonstrated that the choice of method does matter, as different methods can lead to very different results.

Evaluation of the effect of quality change on price

1.230 It is useful to try to clarify why one would wish to adjust the observed price change between two items that are similar, but not identical, for differences in their quality. A change in the quality of a good or service occurs when there is a change in some, but not most, of its characteristics. For purposes of a CPI, a quality change must be evaluated from the consumer's perspective. As explained in Chapter 7, the evaluation of the quality change is essentially an estimate of the additional amount that a consumer is willing to pay for the new characteristics possessed by the new quality. This additional amount is not a price increase because it represents the monetary value of the additional satisfaction or utility that is derived from the new quality. Of course, if the old quality is preferred to the new one, consumers would only be willing to buy the new quality if its price were lower.

1.231 Consider the following hypothetical experiment in which a new quality appears alongside an old one. Assume that the two products are substitutes and that the consumer is familiar with the characteristics of the old and the new qualities. Use lower case *p* to refer to prices of the old quality and upper case *P* for the prices of the new quality. Suppose that both qualities are offered to the consumer at the same price, namely the price P_t at which the new quality is actually being sold in period *t.* The consumer is then asked to choose between them and prefers the new quality.

1.232 Suppose next that the price of the old quality is progressively reduced until it reaches p_t*, at which point the consumer becomes indifferent between purchasing the old quality at p_t* and the new quality at P_t. Any further decrease below p_t* causes the consumer to switch back to the old quality. The difference between P_t and p_t* is a measure of the additional value that the consumer places on the new quality as compared with the old quality. It measures the maximum amount that the consumer is willing to pay for the new quality over and above the price of the old quality.

1.233 Let $p_{t\text{-}1}$ denote the actual price at which the old quality was sold in period *t*-1. For CPI purposes, the price increase between the two qualities is not the observed difference $P_t - p_{t\text{-}1}$ but $p_t^* - p_{t\text{-}1}$. It is important to note that p_t*, the hypothetical price for the old quality in period *t*, is directly comparable with the actual price of the old quality in period *t*-1 because both refer to the same identical product. The difference between them is a *pure* price change. The difference between *Pt* and p_t* is not a price change but an evaluation of the difference in the quality of the two items in period *t*. The actual price of the new quality in period *t* needs to be multiplied by the ratio p_t^* / P_t in order to make the comparison between the prices in periods *t*-1 and *t* a comparison between products of equal quality in the eyes of the consumer. The ratio p_t^* / P_t is the required quality adjustment.

1.234 Of course, it is difficult to estimate the quality adjustment in practice, but the first step has to be to clarify conceptually the nature of the adjustment that is required in principle. In practice, producers often treat the introduction of a new quality, or new model, as a convenient opportunity at which to make a significant price change. They may deliberately make it difficult for consumers to disentangle how much of the observed difference in price between the old and the new qualities represents a price change.

1.235 Chapter 7 explains the two possibilities open to statistical offices. One possibility is to make an explicit adjustment to the observed price change on the basis of the different characteristics of the old and new qualities. The other alternative is to make an implicit adjustment by making an assumption about the pure price change; for example, on the basis of price movements observed for other products. It is convenient to take the implicit methods first.

Implicit methods for adjusting for quality changes

1.236 *Overlapping qualities*. Suppose that the two qualities overlap, both being available on the market at time *t*. If consumers are well informed, have a free choice and are collectively willing to buy some of both at the same time, economic theory suggests that the ratio of the prices of the new to the old quality should reflect their relative utilities to consumers. This implies that the difference in price between the old and the new qualities does not indicate any change in price. The price changes up to period *t* can be measured by the prices for the old quality, while the price changes from period *t* onwards can be measured by the prices for the new quality. The two series of price changes are linked in period *t*, the difference in price between the two qualities not having any impact on the linked series.

1.237 When there is an overlap, simple linking of this kind may provide an acceptable solution to the problem of dealing with quality change. In practice, however, this method is not used very extensively because the requisite data are seldom available. Moreover, the conditions may not be consistent with those assumed in the theory. Even when there is an overlap, consumers may not have had time to acquire sufficient knowledge of the characteristics to be able to evaluate the relative qualities properly, especially when there is a substantial change in quality. Not all consumers may have access to both qualities. When the new quality first appears, the market is liable to remain in disequilibrium for some time, as it takes time for consumers to adjust their consumption patterns.

1.238 There may be a succession of periods in which the two qualities overlap before the old quality finally disappears from the market. If the market is temporarily out of equilibrium, the relative prices of the two qualities may change significantly over time so that the market offers alternative evaluations of the relative qualities depending on which period is chosen. When new qualities that embody major new improvements appear on the market for the first time, there is often a tendency for their prices to fall relatively to older qualities before the latter eventually disappear. In this situation, if the price series for the old and new qualities are linked in a single period, the choice of period can have a substantial effect on the overall change in the linked series.

1.239 The statistician has then to make a deliberate judgement about the period in which the relative prices appear to give the best representation of the relative qualities. In this situation, it may be preferable to use a more complex linking procedure which uses the prices for both the new and the old qualities in several periods in which they overlap. However, the information needed for this more complex procedure will never be available if price collectors are instructed only to introduce a new quality when an old one is dropped. In this case, the timing of the switch from the old to the new can have a significant effect on the long-term change in the linked series. This factor must be explicitly recognized and taken into consideration.

1.240 If there is no overlap between the new and the old qualities, the problems just discussed do not arise as no choice has to be made about when to make the link. Other and more difficult problems nevertheless take their place.

1.241 *Non-overlapping qualities*. In the following sections, it is assumed that the overlap method cannot be used because there is a discontinuity between the series of price observations for the old and new qualities. Again, using lower case *p* for the old quality and upper case *P* for the new, it is assumed that the price data available to the index compiler take the following form:

..., p_{t-3}, p_{t-2}, p_{t-1}, P_t, P_{t+1}, P_{t+2}, …

The problem is to estimate the pure price change between t-1 and t in order to have a continuous series of price observations for inclusion in the index. Using the same notation as above:

– price changes up to period t–1 are measured by the series for the old quality;
– the change between t-1 and t is measured by the ratio p^*_t / p_{t-1} where p^*_t is equal to P_t *after* adjustment for the change in quality;
– price changes from period t onwards are measured by the series for the new quality.

1.242 The problem is to estimate p^*_t . This may be done explicitly by one of the methods described later. Otherwise, one of the implicit methods has to be used. These may be grouped into three categories:

- The first solution is to assume that $p^*_t / p_{t-1} = P_t / p_{t-1}$ or $p^*_t = P_t$. No change in quality is assumed to have occurred, so the whole of the observed price increase is treated as a pure price increase. In effect, this contradicts the assumption that there has been a change in quality.
- The second is to assume that $p^*_t / p_{t-1} = 1$, or $p^*_t = p_{t-1}$. No price change is assumed to have occurred, the whole of the observed difference between p_{t-1} and P_t being attributed to the difference in their quality.
- The third is to assume that $p^*_t / p_{t-1} = I$, where I is an index of the price change for a group of similar products, or possibly a more general price index.

1.243 The first two possibilities cannot be recommended as default options to be used automatically in the absence of any adequate information. The use of the first option can only be justified if the evidence suggests that the extent of the quality change is negligible, even though it cannot be quantified more precisely. “Doing nothing”, in other words ignoring the quality change completely, is equivalent to adopting the first solution. Conversely, the second can only be justified if the evidence suggests that the extent of any price change between the two periods is negligible. The third option is likely to be much more acceptable than the other two. It is the kind of solution that is often used in economic statistics when data are missing.

1.244 Elementary indices are typically based on a number of series relating to different sampled products. The particular linked price series relating to the two qualities is therefore usually just one out of a number of parallel price series. What may happen in practice is that the price observations for the old quality are used up to period t-1 and the prices for the new quality from t onwards, the price change between t-1 and t being omitted from the calculations. In effect, this amounts to using the third option: that is, estimating the missing price change on the assumption that it is equal to the average change for the other sampled products within the elementary aggregate.

1.245 It may be possible to improve on this estimate by making a careful selection of the other sampled products whose average price change is believed to be more similar to the item in question than the average for the group of sampled products as a whole. This procedure is described in some detail in Chapter 7, where it is illustrated with a numerical example and described as “targeting” the imputation or estimation.

1.246 The general method of estimating the price on the basis of the average change for the remaining group of products is widely used. It is sometimes described as the “overall” class mean method. The more refined targeted version is the “targeted” mean method. In general, one or other method seems likely to be preferable to either of the first two options listed above, although each case must be considered on its individual merits.

1.247 While the class mean method seems a sensible practical solution, it may nevertheless give biased results, as explained in Chapter 7. The introduction of a new quality is precisely the occasion on which a producer may choose to make a significant price change. Many of the most important price changes may be missed if, in effect, they are assumed to be equal to the average price changes for products not subject to quality change.

1.248 It is necessary, therefore, to try to make an explicit adjustment for the change in quality, at least when a significant quality change is believed to have occurred. Again there are several methods that may be used.

Explicit quality adjustments

1.249 *Quantity adjustments*. The quality change may take the form of a change in the physical characteristics of the product that can easily be quantified, such as change in weight, dimensions, purity, or chemical composition of a product. It is generally a considerable oversimplification to assume that the quality of a product changes in proportion to the size of some single physical characteristic. For example, most consumers are very unlikely to rate a refrigerator that has three times the capacity of a smaller one as being worth three times the price of the latter. Nevertheless it is clearly possible to make some adjustment to the price of a new quality of different size to make it more comparable with the price of an old quality. There is considerable scope for the judicious, or common sense, application of relatively straightforward quality adjustments of this kind. A thorough discussion of quality adjustments based on “size” is given in Chapter 7.

1.250 *Differences in production or option costs*. An alternative procedure may be to try to measure the change in quality by the estimated change in the costs of producing the two qualities. The estimates can be made in consultation with the producers of the goods or services, if appropriate. This method, like the first, is only likely to be satisfactory when the changes take the form of relatively simple changes in the physical characteristics of the good, such as the addition of some new feature, or option, to an automobile. It is not satisfactory when a more fundamental change in the nature of the product occurs as a result of a new discovery or technological innovation. It is clearly inapplicable, for example, when a drug is replaced by another more effective variant of the same drug that also happens to cost less to produce.

1.251 Another possibility for dealing with a quality change that is more complex or subtle is to seek the advice of technical experts. This method is especially relevant when the general consumer may not have the knowledge or expertise to be able to assess or evaluate the significance of all of the changes that may have occurred, at least when they are first made.

1.252 *The hedonic approach*. Finally, it may be possible to systematize the approach based on production or option costs by using econometric methods to estimate the impact of observed changes in the characteristics of a product on its price. In this approach, the market prices of a set of different qualities or models are regressed on what are considered to be the most important physical or economic characteristics of the different models. This approach to the evaluation of quality change is known as *hedonic analysis*. When the characteristics are attributes that cannot be quantified, they are represented by dummy variables. The regression coefficients measure the estimated marginal effects of the various characteristics on the prices of the models and can therefore be used to evaluate the effects of changes in those characteristics, i.e., changes in quality, over time.

1.253 The hedonic approach to quality adjustment can provide a powerful, objective and scientific method of evaluating changes in quality for certain kinds of products. It has been particularly successful in dealing with computers. The economic theory underlying the hedonic approach is examined in more detail in Chapter 21. The application of the method is explained in some detail in Chapter 7. Products can be viewed as bundles of characteristics that are not individually priced, as the consumer buys the bundle as a single package. The objective is to try to "unbundle" the characteristics to estimate how much they contribute to the total price. In the case of computers, for example, three basic characteristics are the processor speed, the size of the RAM and the hard drive capacity. An example of a hedonic regression using these characteristics is given in Chapter 7.

1.254 The results obtained by applying hedonics to computer prices have had a considerable impact on attitudes towards the treatment of quality change in CPIs. They have demonstrated that for goods where there are rapid technological changes and improvements in quality, the size of the adjustments made to the market prices of the products to offset the changes in the quality can largely determine the movements of the elementary price index. For this reason, the manual contains a thorough treatment of the use of hedonics. Chapter 7 provides further analysis, including a comparison showing that the results obtained by using hedonics and matched models can differ significantly when there is a high turnover of models.

1.255 It may be concluded that statistical offices must pay close attention to the treatment of quality change and try to make explicit adjustments whenever possible. The importance of this topic can scarcely be over-emphasized. The need to recognize and adjust for changes in quality has to be impressed on price collectors. Failure to pay proper attention to quality changes can introduce serious biases into a CPI.

Item substitution and new goods

1.256 As noted above, ideally price indices would seek to measure pure price changes between matched products that are identical in the two periods compared. However, as explained in Chapter 8, the universe of products that a CPI has to cover is a dynamic universe that is gradually changing over time. Pricing matched products constrains the selection of products to a static universe of products given by the intersection of the two sets of products existing in the two periods compared. This static universe, by definition, excludes both new products and disappearing products, whose price behaviour is likely to diverge from that of the matched products. Price indices have to try to take account of the price behaviour of new and disappearing products as far as possible.

1.257 A formal consideration and analysis of these problems are given in Appendix 8.1 to Chapter 8. A replacement universe is defined as one that starts with the base period universe but allows new products to enter as replacements as some products disappear. Of course, quality adjustments of the kind discussed above are needed when comparing the prices of the replacement products with those of the products that they replace.

1.258 One way in which to address the underlying problem of the changing universe is by sample rotation. This requires a completely new sample of products to be drawn to replace the existing one. The two samples must overlap in one period that acts as the link period. This procedure can be viewed as a systematic exploitation of the overlap method of adjusting for quality change. It may not therefore deal satisfactorily with all changes in quality that occur, because the relative prices of different goods and services at a single point of time may not provide satisfactory measures of the relative qualities of all the goods and services concerned. Nevertheless, frequent sample rotation helps by keeping the sample up to date and may reduce the extent to which explicit quality adjustments are required. Sample rotation is expensive, however.

New goods and services

1.259 The difference in quality between the original product and the one that it replaces may become so great that the new quality is better treated as a new good, although the distinction between a new quality and a new good is inevitably somewhat arbitrary. As noted in Chapter 8, a distinction is also drawn in the economics literature between evolutionary and revolutionary new goods. An evolutionary new good or service is one that meets existing needs in much more efficient or new ways, whereas a revolutionary new good or service provides completely new kinds of services or benefits. In practice, an evolutionary new good can be fitted into some sub-class of the product or expenditure classification, whereas a revolutionary new good will require some modification to the classification in order to accommodate it.

1.260 There are two main concerns with new goods or services. The first relates to the timing of the introduction of the new product into the index. The second relates to the fact that the mere availability of the new product on the market may bring a welfare gain to consumers, whatever the price at which it is sold initially. Consider, for example, the introduction of the first antibiotic drug, penicillin. The drug provided cures for conditions that previously might have been fatal. The benefit might be virtually priceless to some individuals. One way of gauging how much benefit is gained by the introduction of a new good is to ask how high its price would have to be to reduce the demand for the product to zero. Such a price is called the “demand reservation price”. It could be very high indeed in the case of a new life-saving drug. If the demand reservation price could be estimated, it could be treated as the price in the period just before the new product appeared. The fall between the demand reservation price and the price at which the product actually makes its first appearance could be included in the CPI.

1.261 In practice, of course, statistical offices cannot be expected to estimate demand reservation prices with sufficient reliability for them to be included in a CPI. The concept is nevertheless useful because it highlights the fact that the mere introduction of a new good may bring a significant welfare gain that could be reflected in the CPI, especially if it is intended to be a COLI. In general, any enlargement of the set of consumption possibilities open to consumers has the potential to make them better off, other things being equal.

1.262 It is often the case that new goods enter the market at a higher price than can be sustained in the longer term, so their prices typically tend to fall relatively over the course of time. Conversely, the quantities purchased may be very small initially but increase significantly. These complications make the treatment of new products particularly difficult, especially if they are revolutionary new goods. Because of both the welfare gain from the introduction of a new product and the tendency for the price of a new good to fall after it has been introduced, it is possible that important price reductions may fail to be captured by CPIs because of the technical difficulties created by new products. Chapter 8 concludes by expressing concern about the capacity of CPIs to deal satisfactorily with the dynamics of modern markets. In any case, it is essential that statistical offices are alert to these issues and adopt procedures that take account of them to the maximum extent possible, given the data and resources available to them.

Calculation of consumer price indices in practice

1.263 Chapter 9 provides a general overview of the ways in which CPIs are calculated in practice. The methods used in different countries are by no means all the same, but they have much in common. There is clearly interest from users as well as compilers in knowing how most statistical offices set about calculating their CPIs. The various stages in the calculation process are illustrated by numerical examples. The chapter is descriptive and not prescriptive, although it does try to evaluate the strengths and weaknesses of existing methods. It makes the point that because of the greater insights into the properties and behaviour of indices gained in recent years, it is now recognized that not all existing practices are necessarily optimal.

1.264 As the various stages involved in the calculation process have, in effect, already been summarized in the preceding sections of this chapter, it is not proposed to repeat them all again in this section. It may be useful, however, to give an indication of the nature of the contents of Chapter 9.

Elementary price indices

1.265 Chapter 9 starts by describing how the elementary aggregates are constructed by working down from groups, classes and sub-classes of COICOP, or some equivalent expenditure classification. It reviews the principles underlying the delineation of the elementary aggregates themselves. Elementary aggregates are intended to be as homogeneous as possible, not merely in terms of the physical and economic characteristics of the products covered but also in terms of their price movements.

1.266 Chapter 9 then considers the consequences of using alternative elementary index formulae to calculate the elementary indices. It proceeds by means of a series of numerical examples that use simulated price data for four different products within an elementary aggregate. The elementary indices themselves, and their properties, have already been explained above. An elementary price index may be calculated either as a chain index or as a direct index; that is, either by comparing the price each month, or quarter, with that in the immediately preceding period or with the price in the fixed price reference period. Table 9.1 of Chapter 9 uses both approaches to illustrate the calculation of three basic types of elementary index, Carli, Dutot and Jevons. It is designed to highlight a number of their properties. For example, it shows the effects of “price bouncing” in which the same four prices are recorded for two consecutive months, but the prices are switched between the four products. The Dutot and Jevons indices record no increase but the Carli index registers an increase. Table 9.1 also illustrates the differences between the direct and the chain indices. After six months, each of the four prices is 10 per cent higher than at the start. Each of the three direct indices records a 10 per cent increase, as also do the chained Dutot and Jevons indices because they are transitive. The chained Carli, however, records an increase of 29 per cent, which is interpreted as illustrating the systematic upward bias in the Carli formula resulting from its failure to satisfy the time reversal test.

1.267 It is noted in Chapter 9 that the chaining and direct approaches have different implications when there are missing price observations, quality changes and replacements. The conclusion is that the use of a chain index can make the estimation of missing prices and the introduction of replacement items easier from a computational point of view.

1.268 Chapter 9 also examines the effects of missing price observations, distinguishing between those that are temporarily missing and those that have become permanently unavailable. Table 9.2 contains a numerical example of the treatment of the temporarily missing prices. One possibility is simply to omit the product whose price is missing for one month from the calculation of indices that compare that month with the preceding and following months, and also with the base period. Another possibility is to impute a price change on the basis of the average price for the remaining products, using one or other of the three types of average. The example is a simplified version of the kind of examples that are used in Chapter 7 to deal with the same problem.

1.269 Tables 9.3 and 9.4 illustrate the case in which one product disappears permanently to be replaced by another product. In Table 9.3 there is no overlap between the two products and the options considered are again to omit the products or to impute price changes for them based on averages for the other products. Table 9.4 illustrates the situation in which the products overlap in one month.

1.270 Chapter 9 also considers the possibility that there may be some expenditure weights available within an elementary aggregate, in which case it may be possible to calculate a Laspeyres or a geometric Laspeyres index, these being the weighted versions of the Carli and the Jevons.

Higher-level indices

1.271 Later sections of Chapter 9 illustrate the calculation of the higher-level indices using the elementary price indices and the weights provided by the elementary expenditure aggregates. It is at this stage that the traditional index number theory that was summarized earlier in this chapter and is explained in detail in Chapters 15 to 19 comes into play.

1.272 At the time the monthly CPI is first calculated, the only expenditure weights available must inevitably refer to some earlier period or periods of time. As explained earlier in this chapter, this predisposes the CPI to some form of Lowe or Young index in which the quantities, or expenditures, refer to some weight reference period b which precedes the price reference period 0. Such indices are often loosely described as Laspeyres type indices, but this description is inappropriate. At some later date, however, estimates may become available of the expenditures in both the price reference period 0 and the current period t, so that retrospectively the number of options open is greatly increased. It then becomes possible to calculate both Laspeyres and Paasche type indices, and also superlative indices such as Fisher or Törnqvist. There is some interest in calculating such indices later, if only to see how the original indices compare with the superlative indices. Some countries may wish to calculate retrospective superlative indices for that reason. Although most of the discussion in Chapter 9 focuses on some type of Lowe index because the official index first published will inevitably be of that type, this should not be interpreted as implying that such an index is the only possibility in the longer term.

1.273 *Production and maintenance of higher-level indices*. In practice, the higher-level indices up to and including the overall CPI are usually calculated as Young indices; that is, as weighted averages of the elementary price indices using weights derived from expenditures in some earlier weight reference period. This is a relatively straightforward operation, and a numerical example is given in Table 9.5 of Chapter 9 in which, for simplicity, the weight and price reference periods are assumed to be the same. Table 9.6 illustrates the case in which weight and price reference periods are not the same, and the weights are price updated between weight reference period *b* and the price reference period 0. It illustrates the point that statistical offices have two options when a new price reference period is introduced: they can either preserve the relative quantities of the weight reference period or they can preserve the relative expenditures, but they cannot do both. Price updating preserves the quantities.

1.274 The introduction of new weights is a necessary and integral part of the compilation of a CPI over the long run. Weights have to be updated sooner or later, some countries preferring to update their weights each year. Whenever the weights are changed, the index based on the new weights has to be linked to the index based on the old weights. Thus, the CPI inevitably becomes a chain index over the long term. An example of the linking is given in Table 9.7. Apart from the technicalities of the linking process, the introduction of new weights, especially if carried out at intervals of five years or so, provides an opportunity to undertake a major review of the whole methodology. New products may be introduced into the index, classifications may be revised and updated, while even the index number formula might be changed. Annual chaining facilitates the introduction of new products and other changes on a more regular basis, but in any case some ongoing maintenance of the index is needed whether it is annually chained or not.

1.275 Chapter 9 concludes with a section on data editing, a process that is very closely linked to the actual calculation of the elementary prices indices. Data editing comprises two steps: the detection of possible errors and outliers, and the verifying and correction of the data. Effective monitoring and quality control are needed to ensure the reliability of the basic price data fed into the calculation of the elementary prices indices, on which the quality of the overall index depends.

Organization and management

1.276 The collection of price data is a complex operation involving extensive fieldwork by a large number of individual collectors. The whole process requires careful planning and management to ensure that data collected conform to the requirements laid down by the central office with overall responsibility for the CPI. Appropriate management procedures are described in Chapter 12 of this manual.

1.277 Price collectors should be well trained to ensure that they understand the importance of selecting the right products for pricing. Inevitably, price collectors are bound to use their own discretion to a considerable extent. As already explained, one issue of crucial importance to the quality and reliability of a CPI is how to deal with the slowly evolving set of products with which a price collector is confronted. Products may disappear and have to be replaced by others, but it may also be appropriate to drop some products before they disappear altogether, if they have become unrepresentative. Price collectors need to be provided with appropriate training and very clear instructions and documentation about how to proceed. Clear instructions are also needed to ensure that price collectors collect the right prices when there are sales, special offers or other exceptional circumstances.

1.278 As just noted, the price data collected have also to be subjected to careful checking and editing. Many checks can be carried out by computer, using standard statistical control methods. It may also be useful to send out auditors to accompany price collectors and monitor their work. The various possible checks and controls are explained in detail in Chapter 12.

1.279 Improvements in information technology should obviously be exploited to the fullest extent possible. For example, collectors may use hand-held computers and transmit their results electronically to the central office.

Publication and dissemination

1.280 As noted above and in Chapter 2, the CPI is an extremely important statistic whose movements can influence the central bank's monetary policy, affect stock markets, influence wage rates and social security payments, and so on. There must be public confidence in its reliability, and in the competence and integrity of those responsible for its compilation. The methods used to compile it must therefore be fully documented, transparent and open to public scrutiny. Many countries have an official CPI advisory group consisting of both experts and users. The role of such a group is not just to advise the statistical office on technical matters but also to promote public confidence in the index.

1.281 Users of the index also attach great importance to having the index published as soon as possible after the end of each month or quarter, preferably within two or three weeks. There are also many users who do not wish the index to be revised once it has been published. Thus there is likely to be some trade-off between the timeliness and the quality of the index.

1.282 Publication should be understood to mean the dissemination of the results in any form. In addition to publication in print, or hard copy, the results should be released electronically and be available through the Internet on the web site of the statistical office.

1.283 As explained in Chapter 13, good publication policy goes beyond timeliness, confidence and transparency. The results should be made available to all users, in both the public and the private sectors, at the same time and according to a publication schedule announced in advance. There should be no discrimination among users in the timing of the release of the results. The results should not be subject to governmental scrutiny as a condition for their release, and should be seen to be free from political or other pressures.

1.284 There are many decisions to be taken about the degree of detail in the published data and the different ways in which the results may be presented. Users need to be consulted about these questions. These issues are discussed in Chapter 13. As they do not affect the actual calculation of the index, they need not be pursued further at this point.

USES OF CONSUMER PRICE INDICES 2

2.1 The consumer price index (CPI) is treated as a key indicator of economic performance in most countries. The purpose of this chapter is to explain why CPIs are compiled and what they are used for.

A range of possible consumer price indices

2.2 As noted in Chapter 1, compilers have to take into account the needs of users in deciding on the group of households and range of consumption goods and services covered by a CPI. As the prices of different goods and services do not all change at the same rate, or even all move in the same direction, changing the coverage of the index will change the value of the index. Thus, there can be no unique CPI and a range of possible CPIs could be defined.

2.3 While there may be interest in a CPI which is as broadly defined as possible, covering all the goods and services consumed by all households, there are many other options for defining CPIs covering particular sets of goods and services, which may be more useful for particular analytic or policy purposes. There is no necessity to have only a single CPI. When only a single CPI is compiled and published, there is a risk that it may be used for purposes for which it is not appropriate. More than one CPI could be published in order to meet different analytic or policy needs. It is important to recognize, however, that the publication of more than one CPI can be confusing to users who view consumer inflation as a pervasive phenomenon affecting all households equally. The coexistence of alternative measures could undermine their credibility for many users.

2.4 This chapter is intended not only to describe the most important uses for CPIs, but also to indicate how the coverage of a CPI can be affected by the use for which it is intended. The question of what is the most appropriate coverage of a CPI must be addressed before considering what is the most appropriate methodology to be used. Whether or not the CPI is intended to be a cost of living index (COLI), it is still necessary to determine exactly what kinds of good and services and what types of households are meant to be covered. This can only be decided on the basis of the main uses of the index.

Indexation

2.5 Indexation is a procedure whereby the monetary values of certain payments, or stocks, are increased or decreased in proportion to the change in the value of some price index. Indexation is most commonly applied to monetary flows such as wages, rents, interest or taxes, but it may also be applied to the capital values of certain monetary assets and liabilities. Under conditions of high inflation, the use of indexation may become widespread throughout the economy.

2.6 The objective of indexation of money incomes may be either to maintain the purchasing power of those incomes in respect of certain kinds of goods and services, or to preserve the standard of living or welfare of the recipients of the incomes. These two objectives are not quite the same, especially over the longer term. Maintaining purchasing power may be interpreted as changing money income in proportion to the change in the monetary value of a fixed basket of goods and services purchased out of that income. As explained further below and in detail in Chapter 3, maintaining the purchasing power of income over a fixed set of goods and services does not imply that the standard of living of the recipients of the income is necessarily unchanged.

2.7 When the indexation applies to monetary assets or liabilities, it may be designed to preserve the real value of the asset or liability relative to other assets or relative to the values of specified flows of goods and services.

Indexation of wages

2.8 As noted in Chapters 1 and 15, the indexation of wages seems to have been the main objective behind the original compilation of CPIs as the practice goes back over two centuries, although there has always been general interest in measuring inflation. If the indexation of wages is the main justification for a CPI, it has direct implications for the coverage of the index. First, it suggests that the index should be confined to expenditures made by households whose principal source of income is wages. Second, it may suggest excluding expenditures on certain types of goods and services which are considered to be luxurious or frivolous. If so, value judgements or political judgements may enter into the selection of goods and services covered. This point is elaborated further below.

Indexation of social security benefits

2.9 It has become common practice in many countries to index-link the rates at which social security benefits are payable. There are many kinds of benefits, such as retirement pensions, unemployment benefits, sickness benefits, child allowances, and so on. As in the case of wages, when index-linking to benefits of this kind is the main reason for compiling the CPI, it may suggest restricting the coverage of the index to certain types of households and goods and services. Many kinds of goods and services may then be excluded by political decision on the grounds that they are unnecessary or inappropriate. This type of thinking may lead to pressure to exclude expenditures on items such as holidays, gambling, tobacco or alcoholic drink.

2.10 An alternative procedure is to compile separate CPIs for different categories of households. For example, an index may be compiled covering the basket of goods and services purchased by households whose principal source of income is a social security pension. When this is done, it may be superfluous to decide to exclude certain types of luxury or inappropriate expenditures, as the actual expenditures on such items may be negligible anyway.

2.11 As already noted, publishing more than one CPI may be confusing if inflation is viewed as affecting everyone in the same way. Such confusion can be avoided by suitable publicity; it is not difficult to explain the fact that price changes are not the same for different categories of expenditures. In practice, some countries do publish more than one index.

2.12 The main reason why it may not be justifiable to publish more than one index is that the movements in the different indices may be virtually the same, especially in the short term. In such cases, the costs of compiling and publishing separate indices may not be worthwhile. In practice, it may need much bigger differences in patterns of expenditure than actually exist between different groups of households to yield significantly different CPIs.

2.13 Finally, it should be noted that the deliberate exclusion of certain types of goods and services by political decision on the grounds that the households towards whom the index is targeted ought not to be purchasing such goods, or ought not to be compensated for increases in the prices of such goods, cannot be recommended because it exposes the index to political manipulation. For example, suppose it is decided that certain products such as tobacco or alcoholic drink should be excluded from a CPI. There is then a possibility that when taxes on products have to be increased, these products may be deliberately selected in the knowledge that the resulting price increases do not increase the CPI. Such practices are not unknown.

The type of index used for indexation

2.14 When income flows such as wages or social security benefits are index-linked, it is necessary to consider the implications of choosing between a cost of living index and a price index that measures the changes in the cost of purchasing a fixed basket of goods and services, a type of index described here as a Lowe index. The widely used Laspeyres and Paasche indices are examples of Lowe indices. The Laspeyres index uses a typical basket purchased in the earlier of the two periods compared, while the Paasche uses a basket typical of the later period. This "fixed basket" method has a long history, as explained in Chapter 15. In contrast, a cost of living index (COLI) compares the cost of two baskets that may not be exactly the same but which bring the same satisfaction or utility to the consumer.

2.15 Indexation using a Laspeyres price index will tend to over-compensate the income recipients for changes in their cost of living. Increasing incomes in proportion to the change in the cost of purchasing a basket purchased in the past ensures that the income recipients have the opportunity to continue purchasing that same basket if they wish to do so. They would then be at least as well off as before. However, by adjusting their pattern of expenditures to take account of changes in the *relative* prices of the goods and services they buy, they will be able to improve their standard of living or welfare because they can substitute goods that have become relatively cheaper for ones that have become relatively dearer. In addition, they may be able to start to purchase completely new kinds of goods which provide new kinds of benefits that were not available in the earlier period. Such new goods tend to lower a cost of living index when they first appear even though no price can actually be observed to fall, as there was no previous price.

Indexation of interest, rents and other contractual payments

2.16 It is common for payments of both rents and interest to be index-linked. Governments may issue bonds with an interest rate specifically linked to the CPI. The interest payable in any given period may be equal to a fixed real rate of interest plus the percentage increase in the CPI. Payments of housing rents may also be linked to the CPI or possibly to some other index, such as an index of house prices.

2.17 Creditors receiving interest payments do not consist only of households, of course. In any case, the purpose of index-linking interest is not to maintain the standard of living of the creditors but rather to maintain their real wealth by compensating them for the real holding, or capital, losses on their loans incurred as a result of general inflation. A CPI may not be the ideal index for this purpose but may be used by default in the absence of any other convenient index, a point discussed further below.

2.18 Many other forms of contractual payments may be linked to the CPI. For example, legal obligations to pay alimony or for the support of children may be linked to the CPI. Payments of insurance premiums may be linked either to the index as a whole or to a sub-index relating to some specific types of expenditures, such the costs of repairs.

Taxation

2.19 Movements in a CPI may be used to affect the amounts payable in taxation in several ways. For example, liability for income tax may be affected by linking personal allowances that are deductible from taxable income to changes in the CPI. Under a system of progressive taxation, the various thresholds at which higher rates of personal income tax become operative may be changed in proportion to changes in the CPI. Liability for capital gains tax may be reduced by basing it on real rather than nominal capital gains through reducing the percentage increase in the value of the asset by the percentage change in the CPI over the same period, for taxation purposes. In general, there are various ways in which some form of indexation may be introduced into tax legislation.

Real consumption and real income

2.20 Price indices can be used to deflate expenditures at current prices or money incomes in order to derive measures of real consumption and real income. Real measures involve volume comparisons over time (or space). There are two different approaches to such comparisons which are analogous to the distinction between a Lowe, or basket, index and a cost of living index.

2.21 The first defines the change in real consumption as the change in the total value of the goods and services actually consumed measured at the fixed prices of some chosen period. This is equivalent to deflating the change in the current value of the goods and services consumed by an appropriately weighted Lowe price index. The change in real income can be measured by deflating the change in total money income by the same price index.

2.22 The alternative approach defines the change in real consumption as the change in welfare derived from the goods and services actually consumed. This may be estimated by deflating the change in the current value of consumption by using a COLI. Real income may be similarly obtained by deflating money income by the same COLI.

2.23 The two approaches cannot lead to the same results if the pure price index and the COLI diverge. The choice between the two approaches to the measurement of real consumption and real income will not pursued further here, as the issues involved are essentially the same as those already considered above in the parallel discussion of the choice between a Lowe, or basket, price index and a cost of living index.

Consistency between price indices and expenditure series

2.24 The data collected on prices and the data collected on household expenditures must be mutually consistent when measuring real consumption. This requires that both sets of data should cover the same set of goods and services and use the same concepts and classifications. Problems may arise in practice because the price indices and the expenditure series are often compiled independently of each other by different departments of a statistical agency or even by different agencies.

2.25 The coverage of a CPI need not be the same as that of total household consumption expenditures in the national accounts. The CPI may be targeted at selected households and expenditures for reasons given above. However, the difference in coverage between the CPI and the national accounts expenditures must be precisely identified so that it is possible to account for the differences between them. The price index used to deflate the expenditures ought to cover the additional goods and services not covered by the CPI. This may not be easy to achieve in practice because the relevant price data may not be easily available if the price collection procedures are geared to the CPI. Moreover, even if all the basic price data are available, the price index needed for deflation purposes is likely to be of a different type or formula from the CPI itself.

2.26 In principle, the deflation of national accounts estimates will normally require the compilation of appropriately defined price indices that differ from the CPI but may draw on the same price database. They may differ from the CPI not only in the range of the price and expenditure data they cover and the weighting and index number formula employed, but also in the frequency with which they are compiled and the length of the time periods they cover. The movements of the resulting indices will tend to differ somewhat from the CPI precisely because they measure different things. Although designed to be used to deflate expenditure data, they also provide useful additional information about movements in consumer prices. This information complements and supplements that provided by the CPI. The CPI itself is not designed to serve as a deflator. Its coverage and methodology should be designed to meet the needs of the CPI as described in other sections of this chapter.

2.27 When other types of consumer price indices are needed in addition to the CPI, this should be recognized at the data collection stage as it may be more efficient and cost effective to use a single collection process to meet the needs of more than one kind of price index. This may imply collecting rather more price data than are needed for the CPI itself if the coverage of the CPI has been deliberately restricted in some way.

Purchasing power parities

2.28 Many countries throughout the world, including all the member countries of the European Union (EU), participate in regular international programmes enabling purchasing power parities (PPPs) to be calculated for household consumption expenditures. The calculation of PPPs requires the prices of individual consumer goods and services to be compared directly between different countries. In effect, PPP programmes involve the compilation of international consumer price indices. Real expenditures and real incomes can then be compared between countries in much the same way as between different time periods in the same country.

2.29 It is not proposed to examine PPP methodology here but simply to note that PPPs create yet another demand for basic price data. When such data are being collected, therefore, it is important to recognize that they can be used for PPPs as well as CPIs. PPPs are essentially international deflators which are analogous to the inter-temporal deflators needed for the national accounts of a single country. Thus, while the processing and aggregation of the basic data for CPI purposes should be determined by the needs of the CPI itself, it is appropriate to take account of the requirements of these other kinds of price indices at the data collection stage. There may be important economies of scale to be realized by using a single collection process to meet the needs of several different types of indices.

2.30 Thus, operationally as well as conceptually, the CPI needs to be placed in the context of a wider set of interrelated price indices. The compilation of CPIs predates the compilation of national accounts by many years in some countries, so the CPI may have originated as a free-standing index. The CPI can, however, no longer be treated as an isolated index whose compilation and methodology can proceed quite independently of other interrelated statistics.

Use of the consumer price index for accounting under inflation

2.31 When there is inflation, both business and national accounts have to introduce adjustments which are not needed when the price level is stable. This is a complex subject which cannot be pursued in any depth here. Two methods of accounting are commonly used, and they are summarized below. Both require price indices for their implementation.

Current purchasing power accounts

2.32 Current purchasing power accounts are accounts in which the monetary values of the flows in earlier time periods are scaled up in proportion to the increase in some general index of inflation between the earlier period and the current period. In principle, the index used should be a general price index covering other flows in addition to household consumption expenditures, but in practice the CPI is often used by default in the absence of a suitable general index.

Current cost accounting

2.33 Current cost accounting is a method of accounting for the use of assets in which the cost of using the assets in production is calculated at the current prices of those assets as distinct from the prices at which the assets were purchased or otherwise acquired in the past (historic costs). The current cost of using an asset takes account not only of changes in the general price level but also of changes in the relative price of that type of asset since it was acquired. In principle, the price indices that are used to adjust the original prices paid for the assets should be specific price indices relating to that particular type of asset, and such indices are calculated and used in this way in some countries. However, when there are no such indices available there remains the possibility of using the CPI, or some sub-index of the CPI, by default, and CPIs have been used for this purpose.

Consumer price indices and general inflation

2.34 As already noted, measures of the general rate of inflation in the economy as a whole are needed for various purposes:

- Controlling inflation is usually one of the main objectives of government economic policy, although responsibility for controlling inflation may be delegated to the central bank. A measure of general inflation is needed in order to set targets and also to judge the degree of success achieved by the government or central bank in meeting anti-inflationary targets.
- As noted above, a measure of general inflation is also needed for both business and national accounting purposes, particularly for current purchasing power accounting.
- The concept of a relative price change is important in economics. It is convenient therefore to be able to measure the actual changes in the prices of individual goods or services relative to some measure of general inflation. There is also a need to be able to measure real holding (or capital) gains and losses on assets, including monetary assets and liabilities.

2.35 Suitable measures of general inflation are considered in Chapter 14, in which it shown that a hierarchy of price indices exists that includes the CPI. Clearly, a CPI is not a measure of general inflation, as it only measures changes in the prices of consumer goods and services purchased by households. A CPI does not cover capital goods, such as houses, or the goods and services consumed by enterprises or the government. Any attempt to analyse inflationary pressures in the economy must also take account of other price movements, such as changes in the prices of imports and exports, the prices of industrial inputs and outputs, and also asset prices.

Consumer price indices and inflation targets

2.36 Despite the obvious limitations of a CPI as a measure of general inflation, it is commonly used by governments and central banks to set inflation targets. Similarly, it is interpreted by the press and the public as the ultimate measure of inflation. Although governments and central banks are obviously well aware of the fact that the CPI is not a measure of general inflation, a number of factors help to explain the popularity of the CPI, and these are discussed below.

2.37 It may be noted, however, that even though the CPI does not measure general inflation its movements may be expected to be highly correlated with those of a more general measure, if only because consumption expenditures account for a large proportion of total final expenditures. In particular, the CPI should provide a reliable indicator of whether inflation is accelerating or decelerating and also of any turning points in the rate of inflation. This information is highly valuable even if the CPI may be systematically understating or overstating the general rate of inflation.

Consumer price indices and international comparisons of inflation

2.38 CPIs are also commonly used to make international comparisons of inflation rates. An important example of their use for this purpose is provided by the EU. In order to judge the extent to which rates of inflation in the different member countries were converging in the mid-1990s prior to the formation of the European Monetary Union, the member countries decided in the Maastricht Treaty that CPIs should be used. Although CPIs measure consumer inflation rather than general inflation, their use to evaluate the extent of convergence of inflation may be justified on similar grounds to those just mentioned. Presumably, the convergence in CPIs will be highly correlated with that in general inflation, so the use of a specific rather than a general measure of inflation may lead to the same conclusions about the extent of convergence and which countries diverge the most from the average.

Popularity of consumer price indices as economic statistics

2.39 CPIs seem to have acquired a unique status among economic statistics in most countries. There are several factors which help to explain this:

- First, all households have their own personal experience of the phenomenon the CPI is supposed to be measuring. The general public are very conscious of changes in the prices of consumer goods and services, and the direct impact those changes have on their living standards. Interest in CPIs is not confined to the press and politicians.
- Changes in the CPI tend to receive a lot of publicity. Their publication can make headline news. The CPI is a high-profile statistic.
- The CPI is published frequently, usually each month, so that the rate of consumer inflation can be closely monitored. The CPI is also a timely statistic that is released very soon after the end of the period to which it refers.
- The CPI is a statistic with a long history, as noted in Chapters 1 and 15. People have been familiar with it for a long time.
- Although price changes for certain kinds of consumer goods are difficult to measure because of quality changes, price changes for other kinds of goods and services such as capital goods and government services, especially public services, tend to be even more difficult to measure. The CPI may be a relatively reliable price index compared with the price indices for some other flows.
- The CPI is widely respected. Its accuracy and reliability are seldom seriously questioned.
- Most countries have deliberately adopted a policy of not revising the index once it has been published. This makes it more attractive for many purposes, especially those with financial consequences such as indexation. The lack of revisions may perhaps create a somewhat spurious impression of certainty, but it also seems to enhance the credibility and acceptability of the index.

2.40 The widespread use of the CPI for more purposes than it is designed for can be explained by the various factors listed above, together with the fact that no satisfactory alternative or more comprehensive measures of inflation are available monthly in most countries. For example, the CPI may be used as a proxy for a more general measure of inflation in business accounting, even though it may be clear that, conceptually, the CPI is not the ideal index for the purpose. Similarly, the fact that the CPI is not subject to revision, together with its frequency and timeliness, may explain its popularity for indexation purposes in business or legal contracts in contexts where it also may not be very appropriate conceptually. These practices may be defended on the grounds that the alternative to using the CPI may be to make no adjustment for inflation. Although the CPI may not be the ideal measure, it is much better to use it than to make no adjustment whatsoever.

2.41 Although the CPI is often used as a proxy for a general measure of inflation, this does not justify extending its coverage to include elements that go beyond household consumption. If broader indices of inflation are needed, they should be developed in addition to the CPI, leaving the CPI itself intact. Some countries are in fact developing additional and more comprehensive measures of inflation within the kind of conceptual framework outlined in Chapter 14 below.

The need for independence and integrity of consumer price indices

2.42 Because of the widespread use of CPIs for all kinds of indexation, movements in the CPI can have major financial ramifications throughout the economy. The implications for the government alone can be considerable, given that the CPI can affect interest payments and taxation receipts as well as the government's wage and social security outlays.

2.43 When financial interests are involved, there is always a risk that both political and non-political pressure groups may try to exert an influence on the methodology used to compile the CPI. The CPI, in common with other official statistics, must be protected from such pressures and be seen to be protected. Partly for this reason, many countries establish an advisory committee to ensure that the CPI is not subject to outside influence. The advisory committee may include representatives of a cross-section of interested parties as well as independent experts able to offer professional advice. Information about the methodology used to calculate CPIs should be publicly available.

3 CONCEPTS AND SCOPE

Introduction

3.1 The purpose of this chapter is to define and clarify the basic concepts of price and consumption used in a consumer price index (CPI) and to explain the scope of the index. While the general purpose of a *consumer* price index is to measure changes in the prices of consumption goods and services, the concept of "consumption" is an imprecise one that can be interpreted in several different ways, each of which may lead to a different CPI. The governmental agency or statistical office responsible for compiling a CPI also has to decide whether the index is meant to cover all consumers, i.e., all households, or to be restricted to a particular group of households. The precise scope of a CPI is inevitably influenced by what is intended, or believed, to be the main use of the index. Statistical offices should, however, bear in mind that CPIs are widely used as measures of general inflation, even though they may not have been designed for this purpose.

3.2 Consumption is an activity in which persons, acting either individually or collectively, use goods or services to satisfy their needs and wants. In economics, no attempt is made to observe and record such activities directly. Instead, consumption is measured either by the value of the goods and services wholly or partly used up in some period, or by the value of the goods and services that are purchased, or otherwise acquired, for purposes of consumption.

3.3 A consumer price index can have two different meanings, as "consumer" may refer either to a type of economic unit, typically a person or a household, or to a certain type of good or service. To avoid confusion, the term "consumer" will, so far as possible, be reserved here for persons or households, while so-called "consumer" goods will be described as "consumption" goods. *A consumption good or service is defined as one that members of households use, directly or indirectly, to satisfy their own personal needs and wants*. By definition, consumption goods or services provide *utility*. Utility is simply the generic, technical term preferred by economists for the satisfaction, benefit or welfare that people derive from consumption goods or services.

3.4 A CPI is generally understood to be a price index that measures changes in the prices of consumption goods and services acquired, or used, by households. As explained in Chapter 14, more broadly based price indices can be defined whose scope extends well beyond consumption goods and services, but a CPI is deliberately focused on household consumption. It is, however, possible to define a CPI that includes the prices of physical assets such as land or dwellings purchased by households. In the case of owner-occupied dwellings, a key issue is whether to include in the CPI the imputed rents for the flows of housing services provided by the dwellings, or alternatively whether to include the prices of the dwellings themselves in the index (notwithstanding the fact that they are treated as fixed assets and not consumption goods in the system of national accounts (SNA)). Views differ on this issue. In any case, purchases of financial assets, such as bonds or shares, are excluded because financial assets are not goods or services of any kind and are not used to satisfy the personal needs or wants of household members. Financial transactions do not change wealth as one type of financial asset is simply exchanged for another type of financial asset. For example, when securities are purchased, money is exchanged for a bond or share; or alternatively, when a debt is incurred, money is received in exchange for the creation of a liability.

3.5 Although, by definition, a CPI is confined to the prices of goods and services consumed by households, it does not necessarily follow that CPIs have to cover all households or all the goods and services they consume. For example, it might be decided to exclude publicly provided goods which households do not pay for. Many decisions have to be taken about the precise scope of a CPI even though the general purpose of the index may be determined. These issues are explored in this and the following chapter.

Alternative consumption aggregates

3.6 As already noted, the concept of consumption is not a precise one and may be interpreted in different ways. In this section, a hierarchy of different consumption concepts and aggregates is examined.

3.7 Households may acquire goods and services for purposes of consumption in four main ways:

– they may purchase them in monetary transactions;
– they may produce them themselves for their own consumption;
– they may receive them as payments in kind through barter transactions, particularly as remuneration in kind for work done;
– they may receive them as free gifts, or transfers, from other economic units.

3.8 The broadest concept of consumption for CPI purposes would be a price index embracing all four categories of consumption goods and services listed above. This set of consumption goods and services may be described as *total acquisitions*. Total acquisitions are equivalent to the total actual individual consumption of households as defined in the SNA (see Chapter 14). It should be noted that total acquisitions constitute a broader concept of consumption than total consumption expenditures.

Acquisitions and expenditures

3.9 Expenditures are made by the economic units who pay for the goods and services: in other words, who bear the costs. However, many of the goods and services consumed by households are financed and paid for by government units or non-profit institutions. They are mostly services such as education, health, housing and transport. Individual goods and services provided free of charge, or at nominal prices, to *individual* households by governments or non-profit institutions are described as *social transfers in kind*. They may make a substantial contribution to the welfare or standard of living of the individual households that receive them. (Social transfers in kind do not include *collective* services provided by governments to the community as whole, such as public administration and defence.)

3.10 The expenditures on social transfers in kind are incurred by the governments or non-profit institutions that pay for them and not by the households that consume them. It could be decided that the CPI should be confined to consumption expenditures incurred by households, in which case free social transfers in kind would be excluded from the scope of the index. Even if they were to be included, they can be ignored in practice when they are provided free, on the grounds that households incur zero expenditures on them. Of course, their prices are not zero from the perspective of the units that finance the social transfers, but the relevant prices for a CPI are those payable by the households.

3.11 Social transfers cannot be ignored, however, when governments and non-profit institutions decide to introduce charges for them, a practice that has become increasingly common in many countries. For example, if the CPI is intended to measure the change in the total value of a basket of consumption goods and services that includes social transfers, increases in their prices from zero to some positive amount increase the cost of the basket and ought to be captured by a CPI.

Monetary versus non-monetary expenditures

3.12 A distinction may also be drawn between monetary and non-monetary expenditures depending on the nature of the resources used to pay for the goods and services. A monetary expenditure occurs when a household pays in cash, by cheque or credit card, or otherwise incurs a financial liability to pay, in exchange for the acquisition of a good or service. Non-monetary expenditures occur when households do not incur a financial liability but bear the costs of acquiring the goods or services in some other way.

3.13 *Non-monetary expenditures*. Payments may be made in kind rather than cash, as in barter transactions. The goods and services offered as payment in barter transactions are equivalent to negative expenditures and their price changes should, in principle, carry negative weights in a CPI. If the price of goods sold increases, the household is better off. However, as the two sides of a barter transaction should in principle be equal in value, the net expenditure incurred by two households engaged in barter should be zero. Barter transactions between households may therefore be ignored in practice for CPI purposes.

3.14 Households also incur non-monetary expenditures when household members receive goods and services from their employers as remuneration in kind. The employees pay for the goods and services with their own labour rather than cash. Consumption goods and services received as remuneration in kind can, in principle, be included in a CPI using the estimated prices that would be payable for them on the market.

3.15 A third important category of non-monetary expenditure occurs when households consume goods and services that they have produced themselves. The households incur the costs, while the expenditures are deemed to occur when the goods and services are consumed. Own account expenditures of this kind include expenditures on housing services produced for their own consumption by owner-occupiers. The treatment of goods and services produced for own consumption raises important conceptual issues that are discussed in more detail below.

3.16 *Monetary expenditures*. The narrowest concept of consumption that could be used for CPI purposes is one based on monetary expenditures only. Such an aggregate would exclude many of the goods and services actually acquired and used by households for purposes of consumption. Only monetary expenditures generate the monetary prices needed for CPI purposes. The prices of the goods and services acquired through non-monetary expenditures can only be imputed on the basis of the prices observed in monetary transactions. Imputed prices do not generate more price information. Instead, they affect the weighting attached to monetary prices by increasing the weight of those monetary prices which are used to value non-monetary expenditures.

3.17 If the main reason for compiling a CPI is the measurement of inflation, it may be decided to restrict the scope of the index to monetary expenditures only, especially since non-monetary expenditures do not generate any demand for money. Harmonized Indices of Consumer Prices (HICPs), used to measure inflation within the European Union, are confined to monetary expenditures (see Annex 1).

Acquisitions and uses

3.18 It has been customary in the literature on CPIs to draw a distinction between acquisitions of consumption goods and services by households and their subsequent use to satisfy their households' needs or wants. Consumption goods are typically acquired at one point of time and used at some other point of time, often much later, or they may be used repeatedly, or even continuously, over an extended period of time. The times of acquisition and use nevertheless coincide for many services, although there are other kinds of services that provide lasting benefits and are not used up at the time they are provided.

3.19 The time at which a good is acquired is the moment at which ownership of the good is transferred to the consumer. In a market situation, it is the moment at which the consumer incurs a liability to pay, either in cash or in kind. The time at which a service is acquired is not so easy to determine precisely as the provision of a service does not involve any exchange of ownership. Instead, it typically leads to some improvement in the condition of the consumer. A service is acquired by the consumer at the same time that the producer provides it and the consumer accepts a liability to pay.

3.20 In a market situation, therefore, the time of acquisition for both goods and services is the time at which the liability to pay is incurred. When payments are not made immediately in cash, there may be a significant lapse of time before the consumer's bank account is debited for a purchase settled by cheque, by credit card or similar arrangement. The times at which these debits are eventually made depend on administrative convenience and on the particular financial and institutional arrangements in place. They have no relevance to the time of recording the transactions or the prices.

3.21 The distinction between time of acquisition and time of use is particularly important for durable goods and certain kinds of services.

Durables and non-durables

3.22 *Goods*. A "non-durable" good would be better described as a *single use* good. For example, food and drink are used once only to satisfy hunger or thirst. Heating oil, coal or firewood can be burnt once only, but they are nevertheless extremely durable physically and can be stored indefinitely. Households may hold substantial stocks of so-called non-durables, such as many foodstuffs and fuel, especially in periods of political or economic uncertainty.

3.23 Conversely, the distinguishing feature of consumer durables, such as furniture, household equipment or vehicles, is that they are durable under use. They can be used repeatedly or continuously to satisfy consumers' needs over a long period of time, possibly many years. For this reason, a durable is often described as providing a flow of "services"' to the consumer over the period it is used (see also Box 14.3 of Chapter 14). There is a close parallel between the definitions of consumer durables and fixed assets. Fixed assets are goods that are used repeatedly or continuously over long periods of time in processes of production: for example, buildings or other structures, machinery and equipment. A list of the different kinds of consumer durables distinguished in the Classification of Individual Consumption according to Purpose (COICOP) is given below. Some durables last much longer than others, the less durable ones being described as "semi-durables" in COICOP, for example clothing. Dwellings are not classified as consumer durables in COICOP. They are treated as fixed assets and not consumption goods and therefore fall outside the scope of COICOP. However, the housing services produced and consumed by owner-occupiers are included in COICOP and classified in the same way as the housing services consumed by tenants.

3.24 *Services*. Consumers may continue to benefit, and derive utility, from some services long after they were provided because they bring about substantial, long-lasting or even permanent improvements in the condition of the consumers. The quality of life of persons receiving medical treatments such as hip replacements or cataract surgery, for example, is substantially and permanently improved. Similarly, consumers of educational services can benefit from them over their entire lifetimes.

3.25 For some analytical purposes, it may be appropriate to treat certain kinds of services, such as education and health, as the service equivalents of durable goods. Expenditures on such services can be viewed as investments that augment the stock of human capital. Another characteristic that education and health services share with durable goods is that they are often so expensive that their purchase has to be financed by borrowing or by running down other assets.

Consumer price indices based on acquisitions and uses

3.26 The distinction between the *acquisition* and the *use* of a consumption good or service has led to two different concepts of a CPI being proposed.

- A CPI may be intended to measure the average change between two time periods in the prices of the consumption goods and services acquired by households.
- Alternatively, a CPI may be intended to measure the average change between two time periods in the prices of the consumption goods and services used by households to satisfy their needs and wants.

3.27 Flows of acquisitions and uses may be very different for durables. Acquisitions of durables, like producer capital goods, are liable to fluctuate, depending on the general state of the economy, whereas the using up of the stock of durables owned by households tends be a gradual and smooth process. A CPI based on the uses approach requires that the index should measure period-to-period changes in the *prices of the flows of services* provided by the durables. As explained in Chapter 23, the value of the flow of services from a durable may be estimated by its "user cost", which consists essentially of the depreciation on the asset (at current prices) *plus* the interest cost. The inclusion of the interest cost as well as the depreciation means that, over the long term, the weight given to durables is greater than when they are measured simply by acquisitions. In principle, the flows of services, or benefits, derived from major educational and medical expenditures might also be estimated on the basis of user costs.

3.28 When durables are rented on the market, the rentals have to cover not only the values of the service flows but additional costs such as administration and management, repairs and maintenance, and overheads. For example, the amount payable to use a washing machine in a launderette has to cover the costs of the room space in which the machine is housed, electricity, repairs and maintenance, the wages of supervisory staff, and so on, as well as the services provided by the machine itself. Similarly, the rentals payable for car hire may significantly exceed the cost of the service flow provided by the car on its own. In both cases, the customer is buying a bundle of services that includes more than just the use of the durable good.

3.29 Estimating the values and the prices of the flows of services provided by the stock of durables owned by households is difficult, whereas expenditures on durables are easily recorded, as are also the prices at which they are purchased. Partly because of these practical measurement difficulties, CPIs have, up to now, been based largely or entirely on the acquisitions approach. Similarly, national accounts tend to record expenditures on, or acquisitions of, durables rather than the flows of services they provide. As already noted, dwellings are treated as fixed assets and not consumer durables in the SNA. The treatment of owner-occupied housing is considered separately below.

Basket indices and cost of living indices

3.30 A fundamental conceptual distinction may be drawn between a *basket index* and a *cost of living index*. In a CPI context, a basket index is an index that measures the change between two time periods in the total expenditure needed to purchase a given set, or basket, of consumption goods and services. It is called a "Lowe index" in this manual. A cost of living index (COLI) is an index that measures the change in the minimum cost of maintaining a given standard of living. Both indices therefore have very similar objectives in that they aim to measure the change in the total expenditure needed to purchase *either* the same basket *or* two baskets whose composition may differ somewhat but between which the consumer is indifferent.

Lowe indices

3.31 CPIs are almost invariably calculated as Lowe indices in practice. Their properties and behaviour are described in detail in various chapters of this manual. The operational target for most CPIs is to measure the change over time in the total value of some specified basket of consumption goods and services purchased, or acquired, by some specified group of households in some specified period of time. The meaning of such an index is clear. It is, of course, necessary to ensure that the selected basket is relevant to the needs of users and also kept up to date. The basket may be changed at regular intervals and does not have to remain fixed over long periods of time. The determination of the basket is considered in more detail later in this chapter and in the following one.

Cost of living indices

3.32 The economic approach to index number theory treats the quantities consumed as being dependent on the prices. Households are treated as price takers who are assumed to react to changes in *relative* prices by adjusting the *relative* quantities they consume. A basket index that works with a fixed set of quantities fails to allow for the fact that there is a systematic tendency for consumers to substitute items that have become relatively cheaper for those that have become relatively dearer. A cost of living index based on the economic approach does take this substitution effect into account. It measures the change in the minimum expenditure needed to maintain a given standard when utility-maximizing consumers adjust their patterns of purchases in response to changes in relative prices. In contrast to a basket index, the baskets in the two periods in a cost of living index will generally not be quite the same in the two periods because of these substitutions.

3.33 The properties and behaviour of cost of living indices, or COLIs, are explained in some detail in Chapter 17. A summary explanation has already been given in Chapter 1. The maximum scope of a COLI would be the entire set of consumption goods and services consumed by the designated households from which they derive utility. It includes the goods and services received free as social transfers in kind from governments or non-profit institutions. Because COLIs measure the change in the cost of maintaining a given standard of living or level of utility, they lend themselves to a uses rather than an acquisitions approach, as utility is derived not by acquiring a consumer good or service but by using it to satisfy personal needs and wants.

3.34 Welfare may be interpreted to mean not only economic welfare, that is the utility that is linked to economic activities such as production, consumption and working, but also general well-being associated with other factors such as security from attack by others. It may not be possible to draw a clear distinction between economic and non-economic factors, but it is clear that total welfare is only partly dependent on the amount of goods and services consumed.

3.35 *Conditional and unconditional cost of living indices*. In principle, the scope of a COLI is influenced by whether or not it is intended to be a conditional and unconditional cost of living index. The *total welfare* of a household depends on a string of non-economic factors such as the climate, the state of the physical, social and political environment, the risk of being attacked either by criminals or from abroad, the incidence of diseases, and so on, as well as by the quantities of goods and services consumed. An *unconditional cost of living index* measures the change in the cost to a household of maintaining a given level of total welfare allowing the non-economic factors to vary as well as the prices of consumption goods and services. If changes in the non-economic factors lower welfare, then some compensating increase in the level of consumption will be needed in order to maintain the same level of total welfare. An adverse change in the weather, for example, requires more fuel to be consumed to maintain the same level of comfort as before. The cost of the *increased quantities* of fuel consumed drives up the unconditional cost of living index, irrespective of what has happened to prices. There are countless other events that can impact on an unconditional cost of living index, from natural disasters such as earthquakes to man-made disasters such as Chernobyl or acts of terrorism.

3.36 While there may be interest in an unconditional cost of living index for certain analytical and policy purposes, it is defined in such a way that it is deliberately intended to measure the effects of many other factors besides prices. If the objective is to measure the effects of price changes only, the non-price factors must be held constant. Given that a cost of living index is meant to serve as a consumer *price index*, its scope must be restricted to exclude the effects of events other than price changes. A *conditional* cost of living index is defined as the ratio of the minimum expenditures needed to maintain a given level of utility, or welfare, in response to price changes, assuming that all the other factors affecting welfare remain constant. It is conditional not only on a particular standard of living and set of preferences, but also on a particular state of the non-price factors affecting welfare. COLIs in this manual are to be understood as *conditional* cost of living indices.

3.37 A conditional COLI should not be viewed as second best. An unconditional COLI is a more comprehensive *cost of living* index than a conditional COLI, but it is not a more comprehensive *price* index than a conditional index. An unconditional index does not include more price information than a conditional index and it does not give more insight into the impact of price changes on households' welfare. On the contrary, the impact of the price changes is diluted and obscured as more variables impacting on welfare are included within the scope of the index.

3.38 Lowe indices, including Laspeyres and Paasche, are also conditional, being dependent on the choice of basket. The fact that the value of a basket index varies in predictable ways according to the choice of basket has generated much of the large literature on index number theory. Conceptually, Lowe indices and conditional COLIs have much in common. A Lowe index measures the change in the cost of a specified basket of goods and services, whereas a conditional COLI measures the change in the cost of maintaining the level of utility associated with some specified basket of goods and services, other things being equal.

Expenditures and other payments outside the scope of consumer price indices

3.39 Given that, conceptually, most CPIs are designed to measure changes in the prices of consumption goods and services, it follows that purchases of items that are not goods and services fall outside the intended scope of a CPI: for example, purchases of bonds, shares or other financial assets. Similarly, payments that are not even purchases because nothing is received in exchange fall outside the index: for example, payments of income taxes or social security contributions.

3.40 The implementation of these principles is not always straightforward, as the distinction between an expenditure on a good or service and other payments may not always be clear cut in practice. A number of conceptually difficult cases, including some borderline cases of a possibly controversial nature, are examined below.

Transfers

3.41 The definition of a transfer is a transaction in which one unit provides a good, service or asset to another without receiving any good, service or asset in return: i.e., transactions in which there is no counterpart. Transfers are unrequited. As no good or service of any kind is acquired by the household when it makes a transfer, the transfer must be outside the scope of a CPI. The problem is to determine whether or not certain kinds of transactions are in fact transfers, a problem common to both CPIs and national accounts.

3.42 *Social security contributions and taxes on income and wealth*. As households do not receive any specific, individual good or service in return for the payment of social security contributions, they are treated as transfers that are outside the scope of CPIs. Similarly, all payments of taxes based on income or wealth (the ownership of assets) are outside the scope of a CPI since they are unrequited compulsory transfers to government. Property taxes on dwellings (commonly levied as local authority taxes or rates) are outside the scope. It may be noted, however, that unrequited compulsory transfers could be incorporated within an unconditional COLI or within a more broadly defined conditional COLI that allows for changes in some other factors besides changes in the prices of consumption goods and services.

3.43 *Licences*. Households have to pay to obtain various kinds of licences and it is often not clear whether they are simply taxes under another name or whether the government agency providing the licence provides some kind of service in exchange, for example by exercising some supervisory, regulatory or control function. In the latter case, they could be regarded as purchases of services. Some cases are so borderline that they have been debated for years by taxation experts under the aegis of the International Monetary Fund (IMF) and other international agencies without reaching consensus. The experts therefore agreed to settle on a number of conventions based on practices followed in the majority of countries. It is appropriate to make use of these conventions for CPI, and also national accounts, purposes. These conventions are listed in the IMF's *Government Finance Statistics* (IMF, 2001) and have also been adopted in *SNA 1993*.

3.44 Payments by households for licences to own or use certain goods or facilities are, by convention, classified as consumption expenditures, not transfers, and are thus included within the scope of a CPI. For example, licence fees for radios, televisions, driving, firearms, and so on, as well as fees for passports, are included. On the other hand, licences for owning or using vehicles, boats and aircraft, and for hunting, shooting and fishing are conventionally classified as direct taxes and are therefore outside the scope of CPIs. Many countries, however, do include taxes for private vehicle use as they regard them as taxes on consumption for CPI purposes. As the actual circumstances under which licences are issued, and the conditions attaching to them, can vary significantly from country to country, statistical offices may wish to deviate from the proposed conventions in some instances. In general, however, it seems appropriate to make use of conventions internationally agreed by the relevant experts.

3.45 *Gifts and subscriptions*. Gifts are transfers, by definition, and thus outside the scope of a CPI. Payments of subscriptions or donations to charitable organizations for which no easily identifiable services are received in return are also transfers. On the other hand, payments of subscriptions to clubs and societies, including charities, which provide their members with some kind of service (e.g., regular meetings, magazines, etc.) can be regarded as final consumption expenditures and included in a CPI.

3.46 *Tips and gratuities*. Non-compulsory tips or gratuities are gifts that are outside the scope of a CPI. There may be cases, however, where, although tips are not compulsory, it can be very difficult to obtain a good or service without some form of additional payment, in which case this payment should be included in the expenditure on, and the price of, the good or service in question.

Insurance

3.47 There are two main types of insurance, life and non-life. In both cases the premiums have two components. One is a payment for the insurance itself, often described as the net premium, while the other is an implicit service charge payable to the insurance enterprise for arranging the insurance: i.e., a fee charged for calculating the risks, determining the premiums, administering the collection and investment of premiums, and the payment of claims.

3.48 The implicit service charge is not directly observable. It is an integral part of the gross premium that is not separately identified in practice. As a payment for a service it falls within the scope of a CPI, but it is difficult to estimate.

3.49 In the case of non-life insurance, the net premium is essentially a transfer that goes into a pool covering the collective risks of policy holders as a whole. As a transfer, it falls outside the scope of a CPI. In the case of life insurance, the net premium is essentially a form of financial investment. It constitutes the purchase of a financial asset, which is also outside the scope of a CPI.

3.50 Finally, it may be noted that when insurance is arranged through a broker or agent separate from the insurance enterprise, the fees charged by the brokers or agents for their services are included within the scope of the CPI, over and above the implicit service charges made by the insurers.

Gambling

3.51 The amounts paid for lottery tickets or placed in bets also consist of two elements that are usually not separately identified – the payment of an implicit service charge (part of consumption expenditures) and a current transfer that enters the pool out of which the winnings are paid. Only the implicit or explicit service charges payable to the organizers of the gambling fall within the scope of a CPI. The service charges are usually calculated at an aggregate level as the difference between payables (stakes) and receivables (winnings).

Transactions in financial assets

3.52 Financial assets are not consumption goods or services. The creation of financial assets/liabilities, or their extinction, e.g., by lending, borrowing and repayments, are financial transactions that are quite different from expenditures on goods and services and take place independently of them. The purchase of a financial asset is obviously not expenditure on consumption, being a form of financial investment.

3.53 Some financial assets, notably securities in the form of bills, bond and shares, are tradable and have market prices. They have their own separate price indices, such as stock market price indices.

3.54 Many of the financial assets owned by households are acquired indirectly through the medium of pension schemes and life insurance. Excluding the service charges, pension contributions by households are similar to payments for life insurance premiums. They are essentially forms of investment made out of saving, and are thus excluded from CPIs. In contrast, the explicit or implicit fees paid by households for the services rendered by financial auxiliaries such as brokers, banks, insurers (life and non-life), pension fund managers, financial advisers, accountants, and so on, are within the scope of a CPI. Payments of such fees are simply purchases of services.

Purchases and sales of foreign currency

3.55 Foreign currency is a financial asset. Purchases and sales of foreign currency are therefore outside the scope of CPIs. Changes in the prices payable, or receivable, for foreign currencies resulting from changes in exchange rates are not included in CPIs. In contrast, the service charges made by foreign-exchange dealers are included within the scope of CPIs when households acquire foreign currency for personal use. These charges include not only explicit commission charges but also the margins between the buying or selling rates offered by the dealers and the average of the two rates.

Payments, financing and credit

3.56 Conceptually, the time at which an expenditure is incurred is the time at which the purchaser incurs a liability to pay: that is, when the ownership of the good changes hands or the service is provided. The time of payment is the time at which the liability is extinguished. The two may be simultaneous when payment is made immediately in cash, i.e., notes or coin, but the use of cheques, credit cards and other forms of credit facilities means that it is increasingly common for the payment to take place some time after the expenditure occurs. A further complication is that payments may be made in stages, with a deposit payable in advance. Given the time lags and complexity of financial instruments and institutional arrangements, it may be difficult to determine exactly when payment takes place. The time may even be different from the standpoint of the purchaser and the seller.

3.57 For consistency with the expenditure data used as weights in CPIs, the prices should be recorded at the times at which the expenditures actually take place. This is consistent with an acquisitions approach.

Financial transactions and borrowing

3.58 Some individual expenditures may be very large: for example, the purchase of expensive medical treatment, a large durable good, or an expensive holiday. If the household does not have sufficient cash, or does not wish to pay the full amount immediately in cash, various options are open.

- The purchaser may borrow from a bank, moneylender or other financial institution.
- The purchaser may use a credit card.
- The seller may extend credit to the purchaser, or the seller may arrange for a third party, some kind of financial institution, to extend credit to the purchaser.

The creation of a financial asset/liability

3.59 When a consumer borrows to purchase a good or service, two distinct transactions are involved: the purchase of the good or service, and the borrowing of the requisite funds. The latter is a purely financial transaction between a creditor and a debtor in which a new financial asset/liability is created. This financial transaction is outside the scope of a CPI. As already noted, a financial transaction does not change wealth and there is no consumption involved. A financial transaction merely rearranges the individual's asset portfolio by exchanging one type of asset for another. For example, when a loan is made, the lender exchanges cash for a financial claim over the debtor. Similarly, the borrower acquires cash counterbalanced by the creation of an equal financial liability. Such transactions are irrelevant for CPI purposes.

3.60 In general, when a household borrows from financial institutions, including moneylenders, the borrowed funds may be used for a variety of purposes including the purchase of assets such as dwellings or financial assets (for example, bonds or shares), as well as the purchase of expensive goods and services. Similarly, the credit extended to the holder of a credit card can be used for a variety of purposes. In itself, the creation of a financial asset and liability by new borrowing has no impact on a CPI. There is no good or service acquired, no expenditure and no price.

3.61 It should be noted that interest payments are not themselves financial transactions. The payment of interest is quite different from the borrowing, lending or other financial transactions that give rise to it. Interest is considered separately below.

3.62 Hire purchase and mortgage loans must be treated consistently with other loans. The fact that certain loans are conditional on the borrower using the funds for a particular purpose does not affect the treatment of the loan itself. Moreover, conditional loans are by no means confined to the purchase of durable goods on "hire purchase". Conditional personal loans may be made for other purposes, such as large expenditures on education or health. In each case, the contracting of the loan is a separate transaction from the expenditure on the good or service and must be distinguished from the latter. The two transactions may involve different parties and may take place at quite different times.

3.63 Although the provision of finance is a separate transaction from the purchase of a good or service for which it is used, it may affect the price paid. Each case needs to be carefully considered. For example, suppose the seller agrees to defer payment for one year. The seller appears to make an interest free loan for a year, but this is not the economic reality. The seller makes a loan but it is not interest free. Nor is the amount lent equal to the "full" price. Implicitly, the purchaser issues a short-term bill to the seller to be redeemed one year later and uses the cash received from the seller to pay for the good. However, the present value of a bill at the time it is issued is its redemption value discounted by one year's interest. The amount payable by the purchaser at the time the purchase of the good actually takes place is the present discounted price of the bill and not the full redemption price to be paid one year later. It is this discounted price that should be recorded for CPI purposes. The difference between the discounted price and the redemption price is, of course, the interest that the purchaser implicitly pays on the bill over the course of the year. This way of recording corresponds to the way in which bills and bonds are actually valued on financial markets and also to the way in which they are recorded in both business and economic accounts. Deferring payments in the manner just described is equivalent to a price reduction and should be recognized as such in CPIs. The implicit interest payment is not part of the price. Instead, it reduces the price. This example shows that in certain circumstances the market rate of interest can affect the price payable, but it depends on the exact circumstances of the credit arrangement agreed between the seller and the purchaser. Each individual case needs to be carefully considered on its merits.

3.64 This case needs to be clearly distinguished from hire purchase, considered in the next section, when the purchaser actually pays the full price and borrows an amount equal to the full price while contracting to make explicit interest payments in addition to repaying the amount borrowed.

Hire purchase

3.65 In the case of a durable good bought on hire purchase, it is necessary to distinguish the de facto, or economic, ownership of the good from the legal ownership. The time of acquisition is the time the hire purchase contract is signed and the purchaser takes possession of the durable. From then onwards, it is the purchaser who uses it and derives the benefit from its use. The purchasing household becomes the de facto owner at the time the good is acquired, even though legal ownership may not pass to the household until the loan is fully repaid.

3.66 By convention, therefore, the purchasing household is treated as buying the good at the time possession is taken and paying the full amount in cash at that point. At the same time, the purchaser borrows, either from the seller or some financial institution specified by the seller, a sum sufficient to cover the purchase price and the subsequent interest payments. The difference between the cash price and the sum total of all the payments to be made is equal to the total interest payable. The relevant price for CPI purposes is the cash price payable at the time the purchase takes place, whether or not the purchase is facilitated by some form of borrowing. The treatment of hire purchase is the same as that of "financial leasing" whereby fixed assets, such as aircraft, used for purposes of production are purchased by a financial institution and leased to the producer for most or all of the service life of the asset. This is essentially a method of financing the acquisition of an asset by means of a loan and needs to be distinguished from operational leasing such as hiring out cars for short periods of time. The treatment of hire purchase and financial leasing outlined here is followed in both business and economic accounting.

Interest payments

3.67 The treatment of interest payments on the various kinds of debt that households may have incurred raises both conceptual and practical difficulties. Nominal interest is a composite payment covering four main elements whose mix may vary considerably:

- The first component is the pure interest charge: i.e., the interest that would be charged if there were perfect capital markets and perfect information.
- The second component is a risk premium that depends on the creditworthiness of the individual borrower. It can be regarded as a built-in insurance charge under uncertainty against the risk of the debtor defaulting.
- The third component is a service charge incurred when households borrow from financial institutions that make a business of lending money.
- Finally, when there is inflation, the real value of a loan fixed in monetary terms (that is, its purchasing power over real goods and services) declines with the rate of inflation. However, creditors are able to offset the real holding, or capital, losses they expect to incur by charging appropriately high rates of nominal interest. For this reason, nominal interest rates vary directly with the rate of general inflation, a universally familiar phenomenon under inflationary conditions. In these circumstances, the main component of nominal interest may therefore be the built-in payment of compensation from the debtor to the creditor to offset the latter's real holding loss. When there is very high inflation it may account for almost all of the nominal interest charged.

3.68 The treatment of the first component, pure interest, is somewhat controversial but this component may account for only a small part of the nominal interest charged. The treatment of the second component, insurance against the risk of default, is also somewhat controversial.

3.69 The fourth component, the payment of compensation for the creditor's real holding loss, is clearly outside the scope of a CPI. It is essentially a capital transaction. It may account for most of nominal interest under inflationary conditions.

3.70 The third component constitutes the purchase of a service from financial institutions whose business it is to make funds available to borrowers. It is known as the *implicit service charge* and clearly falls within the scope of a CPI. It is included in COICOP. The service charge is not confined to loans made by "financial intermediaries", institutions that borrow funds in order to lend them to others. Financial institutions that lend out of their own resources provide the same kind of services to borrowers as financial intermediaries. When sellers lend out of their own funds, they are treated as implicitly setting up their own financial institution that operates separately from their principal activity. The rates of interest of financial institutions also include implicit service charges. Because some capital markets tend to be very imperfect and most households may not have access to proper capital markets, many lenders are effectively monopolists who charge very high prices for the services they provide, for example village moneylenders in many countries.

3.71 It is clear that interest payments should not be treated as if they were just pure interest or even pure interest plus a risk premium. It is very difficult to disentangle the various components of interest. It may be practically impossible to make realistic and reliable estimates of the implicit service charges embodied in most interest payments. Moreover, for CPI purposes it is necessary to estimate not only the values of the service charges but changes in the prices of the services over time. Given the complexity of interest flows and the fact that the different flows need to be treated differently, there seems to be little justification for including payments of nominal interest in a CPI, especially in inflationary conditions.

Household production

3.72 Households can engage in various kinds of productive activities that may be either aimed at the market or intended to produce goods or services for own consumption.

Business activities

3.73 Households may engage in business or commercial activities such as farming, retail trading, construction, the provision of professional or financial services, and so on. Goods and services that are used up in the process of producing other goods and services for sale on the market constitute *intermediate* consumption. They are not part of the *final* consumption of households. The prices of intermediate goods and services purchased by households are not to be included in CPIs. In practice, it is often difficult to draw a clear distinction between intermediate and final consumption, as the same goods may be used for either purpose.

Consumption of own produce

3.74 Households do not in fact consume directly all of the goods and services they acquire for purposes of consumption. Instead, they use them as inputs into the production of other goods or services which are then used to satisfy their needs and wants. There are numerous examples. For example, basic foodstuffs such as flour, cooking oils, raw meat and vegetables may be processed into bread, cakes or meals with the assistance of other inputs including fuels, the services provided by consumer durables, such as fridges and cookers, and the labour services of members of the household. Inputs of materials, equipment and labour are used to clean, maintain and repair dwellings. Inputs of seeds, fertilizers, insecticides, equipment and labour are used to produce vegetables or flowers, and so on.

3.75 Some of the production activities taking place within households' activities, for example gardening or cooking, may perhaps provide satisfaction in themselves. Others, such as cleaning, may be regarded as chores that reduce utility. In any case, the goods or services used as inputs into these productive activities do not provide utility in themselves. Again, there are numerous examples of such inputs: raw foodstuffs that are unsuitable for eating without being cooked; cleaning materials; fuels such as coal, gas, electricity or petrol; fertilizers; the services of refrigerators and freezers; and so on.

3.76 Utility is derived from consuming the outputs from household production undertaken for own consumption. It is necessary, therefore, to decide whether a CPI should try to measure the changes in the prices of the outputs, rather than the inputs. In principle, it seems desirable to measure the output prices, but there are serious objections to this procedure.

3.77 On a conceptual level, it is difficult to decide what are the real final outputs from many of the more nebulous household production activities. It is particularly difficult to specify exactly what are the outputs from important service activities carried out within households, such as child care or care of the sick or elderly. Even if they could be satisfactorily identified, conceptually they would have to be measured and priced. There are no prices to be observed, as there are no sales transactions. Prices would have to be imputed for them and such prices would be not only hypothetical but inevitably very speculative. Their use in CPIs is not a realistic possibility in general and almost certainly would not be acceptable to most users who are primarily interested in the market prices paid by households.

3.78 The practical alternative is to treat the goods and services acquired by households on the market for use as inputs into the various kinds of household production activities as if they were themselves final consumer goods and services. They provide utility *indirectly*, assuming that they are used exclusively to produce goods and services that are directly consumed by households. This is the practical solution that is generally adopted not only in CPIs but also in national accounts, where household expenditures on such items are classified as final consumption. Although this seems a simple and conceptually acceptable solution to an otherwise intractable problem, exceptions may be made for one or two kinds of household production that are particularly important and whose outputs can readily be identified.

3.79 *Subsistence agriculture*. In the national accounts, an attempt is made to record the value of the agricultural output produced for own consumption. In some countries, subsistence agriculture may account for a large part of the production and consumption of agricultural produce. The national accounts require such outputs to be valued at their market prices. It is doubtful whether it is appropriate to try to follow this procedure for CPI purposes.

3.80 A CPI may record either the actual input prices or the imputed output prices, but not both. If the imputed output prices for subsistence agriculture are included in a CPI, the prices of the purchased inputs should be excluded. This could remove from the index most of the market transactions made by such households. Expenditures on inputs may constitute the principal contact that the households have with the market and through which they experience the effects of inflation. It therefore seems preferable to record the actual prices of the inputs and not the imputed prices of the outputs in CPIs.

3.81 *Housing services produced for own consumption*. The treatment of owner-occupied housing is difficult and somewhat controversial. There may no consensus on what is best practice. This is discussed in several chapters of this manual, especially in Chapters 10 and 23. Conceptually, the production of housing services for own consumption by owner-occupiers is no different from other types of own account production taking place within households. The distinctive feature of the production of housing services for own consumption, as compared with other kinds of household production, is that it requires the use of an extremely large fixed asset in the form of the dwelling itself. In economics, and also national accounting, a dwelling is usually regarded as a fixed asset so that the purchase of a dwelling is classified as gross fixed capital formation and not as the acquisition of a durable consumer good. Fixed assets are used for purposes of production, not consumption. The dwelling is not consumed directly. The dwelling provides a stream of capital services that are consumed as inputs into the production of housing services. This production requires other inputs, such as repairs, maintenance and insurance. Households consume the housing services produced as outputs from this production.

3.82 It is important to note that there are two quite distinct service flows involved:

- One consists of the flow of *capital services* provided by the dwelling which are consumed as *inputs* into the production of housing services.
- The other consists of the flow of *housing services* produced as *outputs* which are consumed by members of the household.

The two flows are not the same. The value of the output flow will be greater than that of the input flow. The capital services are defined and measured in exactly the same way as the capital services provided by other kinds of fixed assets, such as equipment or structures other than dwellings. As explained in detail in Chapter 23, the value of the capital services is equal to the user cost and consists primarily of two elements, depreciation and the interest, or capital, costs. Capital costs are incurred whether or not the dwelling is purchased by borrowing on a mortgage. When the dwelling is purchased out of own funds, the interest costs represent the opportunity cost of the capital tied up in the dwelling; that is, the foregone interest that could have been earned by investing elsewhere.

3.83 There are two main options for the own-account production and consumption of housing services in CPIs. One is to price the output of housing services consumed. The other is to price the inputs, including the inputs of capital services. If housing services are to be treated consistently with other forms of production for own consumption within households, the input approach must be adopted. The production and consumption of housing services by owner-occupiers may, however, be considered to be so important as to merit special treatment.

3.84 If it is decided to price the outputs, the prices may be estimated using the market rents payable on rented accommodation of the same type. This is described as the rental equivalence approach. One practical problem is that there may be no accommodation of the same type that is rented on the market. For example, there may be no rental market for rural dwellings in developing countries where most of the housing may actually be constructed by the households themselves. Another problem is to ensure that the market rents do not include other services, such as heating, that are additional to the housing services proper. A further problem is that market rents, like the rentals charged when durables are leased, have to cover the operating expenses of the renting agencies as well as the costs of the housing services themselves, and also provide some profit to the owners. Finally, rented accommodation is inherently different from owner-occupied housing in that it may provide the tenants with more flexibility and mobility. The transaction costs involved in moving house may be much less for tenants.

3.85 In principle, if the output, or rental equivalence, approach is adopted then the prices of the inputs into the production of housing services for own consumption, such as expenditures on repairs, maintenance and insurance, should not be included as well. Otherwise, there would be double counting.

3.86 The alternative is to price the inputs into the production of housing services for own consumption in the same way that other forms of production for own consumption within households are treated. In addition to intermediate expenditures such as repairs, maintenance and insurance, the costs of the capital services must be estimated and their prices included in the CPI. The technicalities of estimating the values of the flow of capital services are dealt with in Chapter 23. As in the case of other types of production for own consumption within households, it is not appropriate to include the estimated costs of the labour services provided by the owners themselves.

3.87 Whether the input or the output approach is adopted, it is difficult to estimate the relevant prices. The practical difficulties experienced may sometimes be so great as to lead compilers and users to query the reliability of the results. There is also some reluctance to use imputed prices in CPIs, whether the prices refer to the inputs or the outputs. It has therefore been suggested that the attempt to measure the prices of housing service flows should be abandoned. Instead, it may be preferred to include the prices of the dwellings themselves in the CPI. In most cases these are observable market prices, although many dwellings, especially in rural areas in developing countries, are also built by their owners, in which case their prices still have to be estimated on the basis of their costs of production.

3.88 Including the prices of dwellings in CPIs involves a significant change in the scope of the index. A dwelling is clearly an asset and its acquisition is capital formation and not consumption. While the same argument applies to durables, there is a substantial difference of degree between a household durable and a dwelling, as reflected by the considerable differences in their prices and their service lives. In principle, therefore, extending the scope of a CPI to include dwellings implies extending the scope of the index to include household gross fixed capital formation.

3.89 The advantage of this solution is that it does not require estimates of either the input or output service flows, but conceptually it deviates significantly from the concept of a CPI as traditionally understood. In the case of both consumer durables and dwellings, the options are either to record the acquisitions of the assets in the CPIs at their market prices or to record the estimated prices of the service flows, but not both. Just as no service flows from durables are included in CPIs at present because their acquisitions are included, similarly if the prices of dwellings are included in CPIs the service flows would have to be excluded. As explained in Chapter 23, the acquisitions approach may give insufficient weight to durables and dwellings over the long run because it does not take account of the capital costs incurred by the owners of the assets.

Coverage of households and outlets

3.90 The group of households included in the scope of a CPI is often referred to as the "reference households", or the "reference population".

Definition of household

3.91 For CPI purposes, households may be defined in the same way as in population censuses. The following definition is recommended for use in population censuses (United Nations, 1998a):

> A household is classified as either (a) a one person household defined as an arrangement in which one person makes provision for his or her food or other essentials for living without combining with any other person to form part of a multi-person household; or (b) a multi-person

household, defined as a group of two or more persons living together who make common provision for food or other essentials for living. The persons in the group may pool their incomes and have a common budget to a greater or lesser extent; they may be related or unrelated persons or a combination of persons both related and unrelated.

3.92 This definition is essentially the same as that used in household budget surveys and in the SNA. The scope of a CPI is usually confined to private households, and excludes institutional households such as groups of persons living together indefinitely in religious institutions, residential hospitals, prisons or retirement homes. Nevertheless, convalescent homes, schools and colleges, the military, and so on are not treated as institutional households; their members are treated as belonging to their private households. The HICP coverage of households, however, is consistent with the *SNA 1993* definition and thus includes institutional households.

Types of household

3.93 In almost all countries, the CPI scope is designed to include as many private households as possible, and is not confined to those belonging to a specific socio-economic group. The HICP regulations require that coverage should be of households independent of their income level.

3.94 In some countries, however, extremely wealthy households are excluded for various reasons. Their expenditures may be considered to be very atypical, while their expenditure data, as collected in household budget surveys, may be unreliable. The response rates for wealthy households in household budget surveys are usually quite low. In addition, it may be too costly to collect prices for some of the consumer goods and services purchased exclusively by the wealthy. Some countries may decide to exclude other kinds of households. For example, the United Kingdom CPI excludes not only the top 4 per cent of households by income but also households mainly dependent on state pensions, with the net result that roughly 15 per cent of households, and 15 per cent of expenditure, is excluded. Japan and the Republic of Korea exclude households mainly engaged in agriculture, forestry and fishing, and all one-person households. Such exclusions affect the expenditure weights to the extent that the patterns of expenditures of the excluded groups differ from those of the rest of the population.

3.95 In addition to a single wide-ranging official (headline) CPI relevant to the country as a whole, many countries publish a range of subsidiary indices relating to sub-sectors of the population. For example, the Czech Republic compiles separate indices for:

– all households;
– all employees;
– employees with children;
– low-income employees;
– employees, incomplete families;
– pensioners;
– low-income pensioners;
– households in Prague;
– households in communities with populations of over 5,000.

3.96 In India, CPI compilation originated from a need to maintain the purchasing power of workers' incomes, and so four different CPIs are compiled at the national level for reference households headed by the following kinds of workers:

– agricultural labourers;
– industrial workers;
– rural labourers;
– urban non-manual employees.

Geographical coverage

3.97 *Urban and rural*. Geographical coverage may refer either to the geographical coverage of expenditures or the coverage of price collection. Ideally these two should coincide, whether the CPI is intended to be a national or a regional index. In most countries, prices are collected in urban areas only since their movements are considered to be representative of the price movements in rural areas. In these cases national weights are applied and the resulting index can be considered a national CPI. If price movements in urban and rural areas are felt to be sufficiently different – although price collection is restricted to urban areas because of resource constraints – then urban weights should be applied and the resulting index must be considered as purely an urban and not a national CPI. For example, the following countries cover urban households only (expenditure weights and prices): Australia, Mexico, Republic of Korea, Turkey, United States. Most other developed countries tend to use weights covering urban and rural households, although in nearly every case price collection takes place in urban areas only. Of course, the borderline between urban and rural is inevitably arbitrary and may vary from country to country. For example, in France urban price collection is interpreted to include villages with as few as 2,000 residents.

3.98 Decisions about geographical coverage in terms of urban versus rural coverage will depend on population distribution and the extent to which expenditure patterns and the movements of prices tend to differ between urban and rural areas.

3.99 *Foreign purchases of residents and domestic purchases of non-residents*. Problems arise when households make expenditures outside the boundaries of the area or country in which they are resident. Decisions about the treatment of such expenditures depend on the main use of a CPI. For inflation analysis, it is the price change within a country which is of interest. An index of inflation is needed that covers all so-called "domestic" consumption expenditures that take place within the geographical boundaries of the country, whether made by residents or non-residents. HICPs (see Annex 1) are defined in this way as indices of domestic inflation. Thus they exclude consumption expenditures made by residents when they are outside the country (which belong to the inflation indices of the countries where the purchases are made), and they include expenditures within the country made by residents of other countries. In practice, expenditures by visitors from abroad may be difficult to estimate, since household budget surveys do not cover non-resident households, although estimates might be possible for some commodities using retail sales data or special surveys of visitors. These issues become more important when there is significant cross-border shopping as well as tourism.

3.100 When CPIs are used for escalating the incomes of residents, it may be appropriate to adopt the so-called "national" concept of expenditure which covers all the expenditures of residents, whether inside or outside the country, including remote purchase from non-resident outlets, for example by the Internet, telephone or mail. Household budget surveys can cover all these types of expenditure, although it may be difficult to identify the country from which remote (Internet, mail, etc.) purchases are being made. The prices paid for airline tickets and package holidays bought within the domestic territory should also be covered. It can be difficult, however, to obtain price data for the goods and services purchased by residents when abroad, although in some cases sub-indices of the partner countries' CPIs might be used.

3.101 *Regional indices*. When compiling regional indices, the concept of residence applies to the region in which a household is resident. It is then possible to draw a distinction between the expenditures within a region and the expenditures of the residents of that region, analogous to the distinction between the "domestic" and "national" concepts of expenditure at the national level. The same issues arise for regional indices as were discussed in paragraph 3.97. The principles applying to cross-border shopping between regions are the same as for international cross-border shopping, but data availability is generally different. If the scope of the regional index is defined to include the purchases by regional residents when in other regions (abroad), then, although price data for the other regions should be readily available, it is unlikely that expenditure data will be available with the necessary split between expenditure within and expenditure outside the region of residence.

3.102 Care must be taken to treat cross-border shopping in the same way in all regions. Otherwise double counting, or omission, of expenditures may occur when regional data are aggregated. Where regional indices are aggregated to give a national index, the weights should be based on regional expenditure data rather than on population data alone.

3.103 Many countries try to satisfy the differing needs of their many CPI users by deriving a family of indices with differing coverage, headed by a single wide-ranging official (headline) CPI which is relevant to the country as a whole. In some large countries, regional indices are more widely used than the national CPI, particularly where the indices are used for escalating incomes. Thus, in addition to the headline CPI, which has the widest coverage possible, subsidiary indices are published which may relate to:

– sub-sectors of the population;
– geographical regions;
– specific commodity groups; sub-indices of the overall (official all-items) CPI should be published at as detailed a level as possible, since many users are interested in the price change of specific commodity groups.

3.104 In effect, many statistical offices are moving towards a situation in which a database of prices and weights is maintained from which a variety of subsidiary indices is derived.

Outlet coverage

3.105 The coverage of outlets is dictated by the purchasing behaviour of the reference households. As already stated, in principle, the prices relevant to CPIs are the prices paid by households. In practice, however, it is usually not feasible to collect price information directly from households, although as more sales are made through electronic points of sale which record and print out both the items purchased and their prices, it may become increasingly practical to collect information on the actual transaction prices paid by households. In the meantime, it is necessary to rely mainly on the prices at which products are offered for sale in retail shops or other outlets. All the outlets from which the reference population makes purchases are within the scope of the CPI, and should be included in the sampling frame from which the outlets are selected.

3.106 Examples of outlets are:

– retail shops – from very small permanent stalls to multinational chains of stores;
– market stalls and street vendors;
– establishments providing household services – electricians, plumbers, window cleaners, and so on;
– leisure and entertainment providers;
– health and education services providers;
– mail or telephone order agencies;
– the Internet;
– public utilities;
– government agencies and departments.

3.107 The principles governing the selection of a sample of outlets from which to collect prices are discussed in some detail in Chapters 5 and 6.

Price variation

3.108 Price variation occurs when exactly the same good or service is sold at different prices at the same moment of time. Different outlets may sell exactly the same product at different prices, or the same product may be sold from a single outlet to different categories of purchasers at different prices.

3.109 If markets were "perfect" in an economic sense, identical products would all sell at the same price. If more than one price were quoted, all purchases would be made at the lowest price. This suggests that products sold at different prices cannot be identical but must be qualitatively different in some way. When the price differences are, in fact, attributable to quality differences, the price differences are only apparent, not genuine. In such cases, a change in the average price resulting from a shift in the pattern of quantities sold at different prices would reflect a change in the average quality of the products sold. This would affect the volume and not the price index.

3.110 If statistical offices do not have enough information about the characteristics of goods and services selling at different prices, they have to decide whether to assume that the observed price differences are genuine or only apparent. The default procedure most commonly adopted in these circumstances is to assume that the price differences are apparent. This assumption is typically made for both CPI and national accounts purposes.

3.111 However, markets are seldom perfect. One reason for the co-existence of different prices for identical products may be that the sellers are able to practise price discrimination. Another reason may simply be that consumers lack information and may buy at higher prices out of ignorance. Also, markets may be temporarily out of equilibrium as a result of shocks or the appearance of new products. It must be recognized, therefore, that genuine price differences do occur.

Price discrimination

3.112 Economic theory shows that price discrimination tends to increase profits. It may not be feasible to practise price discrimination for goods because they can be retraded. Purchasers discriminated against would not buy directly but would try to persuade those who could purchase at the lowest prices to buy on their behalf. Services, however, cannot be retraded, as no exchange of ownership takes place.

3.113 Price discrimination appears to be extremely common, almost the norm, for many kinds of services including health, education and transport. For example, senior citizens may be charged less than others for exactly the same kinds of health or transportation services. Universities may charge foreign students higher fees than domestic students. As it is also easy to vary the qualities of the services provided to different consumers, it can be difficult to determine to what extent observed price differences are a result of quality differences or pure price discrimination. Sellers may even attach trivial or spurious differences in terms or conditions of sale to the services sold to different categories of purchasers in order to disguise the price discrimination.

3.114 Price discrimination can cause problems with regard to price indices. Suppose, for example, that a service supplier discriminates by age by charging senior citizens aged 60 years or over price p_2 and others price p_1, where $p_1 > p_2$. Suppose, further, that the supplier then decides to redefine senior citizens as those aged 70 years or over while otherwise keeping prices unchanged. In this case, although neither p_1 nor p_2 changes, the price paid by individuals aged 60 to 70 years changes and the average price paid by all households increases.

3.115 This example illustrates a point of principle. Although neither of the stated prices, p_1 and p_2, at which the services are on offer changes, the prices paid by certain households do change if they are obliged to switch from p_2 to p_1. From the perspective of the households, price changes have occurred and a CPI should, in principle, record a change. When prices are collected from sellers and not from households, such price changes are unlikely to be recorded.

Price variation between outlets

3.116 The existence of different prices in different outlets raises similar issues. Pure price differences are almost bound to occur when there are market imperfections, if only because households are not perfectly informed. When new outlets open selling at lower prices than existing ones, there may be a time lag during which exactly the same item sells at different prices in different outlets because of consumer ignorance or inertia.

3.117 Households may choose to switch their purchases from one outlet to another or even be obliged to switch because the universe of outlets is continually changing, some outlets closing down while new outlets open up. When households switch, the effect on the CPI depends on whether the price differences are pure or apparent. When the price differences are genuine, a switch between outlets changes the average prices paid by households. Such price changes ought to be captured by CPIs. On the other hand, if the price differences reflected quality differences, a switch would change the average quality of the products purchased, and hence affect volume, not price.

3.118 Most of the prices collected for CPI purposes are offer prices and not the actual transactions prices paid by households. In these circumstances, the effects of switches in the pattern of households' purchases between outlets may remain unobserved in practice. When the price differences reflect quality differences, the failure to detect such switches does not introduce any bias into the CPI. Buying at a lower price means buying a lower-quality product, which does not affect the price index. However, when the price differences are genuine, the failure to detect switches will tend to introduce an upward bias in the index, assuming households tend to switch towards outlets selling at lower prices. This potential bias is described as *outlet substitution bias*.

Outlet rotation

3.119 A further complication is that, in practice, prices are collected from only a sample of outlets and the samples may change, either because outlets open and close or because there is a deliberate rotation of the sample periodically. When the prices in the outlets newly included in the sample are different from those in the previous outlets, it is again necessary to decide whether the price differences are apparent or genuine. If they are assumed to be apparent, the difference between the price recorded previously in an old outlet and the new price in the new outlet is not treated as a price change for CPI purposes, the difference being treated as attributable to quality difference. As explained in more detail in Chapter 7, if this assumption is correct, the price changes recorded in the new outlets can simply be linked to those previously recorded in the old outlets without introducing any bias into the index. The switch from the old to the new outlets does not have any impact on the CPI.

3.120 If the price differences between the old and the new outlets are deemed to be genuine, however, the simple linking just described can lead to bias. When households change the price they pay for a product by changing outlets, the price changes should be captured by the CPI. As explained in more detail in Chapter 7, it seems that most statistical offices tend to assume that the price differences are not genuine and simply link the new price series on to the old. Given that it is unrealistic to assume that markets are always perfect and that pure price variation never occurs, this procedure, although widely used, is questionable and may lead to upward bias. Such bias is described as *outlet rotation bias*. One possible strategy that has been suggested is to assume that half of any observed price difference between old and new outlets is genuine and half is a result of quality difference, on the grounds that, although inevitably somewhat arbitrary, it is likely to be closer to the truth than assuming that the difference is either entirely genuine or entirely attributable to quality differences (see McCracken, Tobin et al., 1999).

Treatment of some specific household expenditures

3.121 Some of the expenditures made by households may not be on goods and services for household consumption and may therefore fall outside the scope of a CPI. One major category consists of the business expenditures made by households.

Fees of agents and brokers

3.122 When a house is purchased for own use by an owner-occupier, it can be argued that the transfer costs associated with purchase (and sale) should be treated as consumption expenditures in the same way as the brokers' fees incurred when financial assets are bought or sold. The fees paid to an agent to buy or sell houses are included in many national CPIs, provided that the house is to be occupied by the owner and not rented to a third party.

Undesirable or illegal goods and services

3.123 All the goods and services that households willingly purchase in order to satisfy their personal needs or wants constitute consumers' expenditures and therefore fall within the scope of a CPI, irrespective of whether their production, distribution or consumption is illegal or carried out in the underground economy or on the black market. Particular kinds of goods or services must not be excluded because they are considered to be undesirable, harmful or objectionable. Such exclusions could be quite arbitrary and undermine the objectivity and credibility of the CPI:

- First, it should be noted that some goods and services might be deemed to be undesirable at some times and desirable at others, or vice versa. People's attitudes change as they acquire more information, especially as a result of scientific advances. Similarly, some goods or services may be deemed to be undesirable in some countries but not in others at the same point of time. The concept of an undesirable good is inherently subjective and somewhat arbitrary and volatile.
- Second, if it is accepted that some goods and services may be excluded on the grounds that they are undesirable, the index is thereby exposed to actual or attempted manipulation by pressure groups.
- Third, attempts to exclude certain goods or services by pressure groups may be based on a misunderstanding of the implications of so doing. For example, if the CPI is used for escalating incomes, it may be felt that households ought not be compensated for increases in the prices of certain undesirable products. However, excluding them does not imply lowering the index. A priori, excluding some item is just as likely to increase the CPI as reduce it, depending on whether the price increase for the item in question is below or above the average for other goods and services. For example, if it is decided to exclude smoking from a CPI and the price increase for smoking products is below average, excluding smoking actually increases the income of smokers (just as it does for non-smokers).

3.124 While goods and services that households willingly choose to consume should not, in principle, be excluded from a CPI because they are acquired in the underground economy or even illegally, it may be impossible to obtain the requisite data on the expenditures or the prices, especially on illegal goods and services. They may well be excluded in practice.

Luxury goods and services

3.125 When a CPI is used as an index of general inflation, it ought to include all households regardless of their socio-economic group and also all consumer goods and services regardless of how expensive they are. Similarly, the scope of an index used for purposes of escalating incomes should include all the goods and services purchased by the reference households, irrespective of whether any of these goods and services are considered to be luxuries or otherwise unnecessary or undesirable.

3.126 Of course, if the reference households are confined to a select group of households, the index will effectively exclude all those items that are purchased exclusively by households that are not in the group. For example, excluding the wealthiest 5 per cent of households will, in practice, exclude many luxury items from the scope of the index. As already noted, such households may be excluded for various reasons, including the unreliability of their expenditure data and the fact that collecting prices for some items purchased exclusively by a tiny minority of households may not be cost-effective. Once the group of reference households has been decided and defined, however, judgements should not be made about whether to exclude certain of their expenditures that are considered to be non-essential or on luxuries.

Second-hand goods

3.127 Markets for used or second-hand goods exist for most durable goods. Household expenditures include expenditures on second-hand goods and are therefore within the scope of a CPI. Households' sales of durables constitute negative expenditures, however, so that the weights for second-hand goods are based on households' net expenditures: i.e., total purchases less sales. The total expenditure on a particular type of second-hand good is a function of the rate at which it is bought and sold, i.e., a higher turnover rate (number of transactions) gives a higher total expenditure. A higher turnover does not, however, increase the rate at which any individual good can be used for purposes of consumption or the flow of services that may be obtained from the good.

3.128 Households may buy second-hand goods through any of the following routes:

- *Directly from another household* – the selling household will record the proceeds of the sale as receipts. Net expenditures, i.e., expenditures *less* receipts, are zero so no weight is attached to purchases and sales from one household to another.
- *From another household via a dealer* – in principle, households' expenditures on the services of the dealers are given by the values of their margins (the difference between their buying and selling prices). These intermediation services should be included in CPIs. They should be treated in the same way as the fees charged by agents such as financial auxiliaries. The margins may be extremely difficult to estimate in practice. Care should be taken to include trade-ins either as purchases by the dealers or receipts of households.
- *Directly from another sector, i.e., from an enterprise or from abroad* – the weight would be household purchases of the second-hand goods from other sectors *less* sales to other sectors.
- *From an enterprise or from abroad via a dealer* – the appropriate weight is given by household purchases from dealers *less* any household sales to dealers *plus* the aggregate of dealers' margins on the products that they buy from and resell to households. Trade-ins should count as part of sales by households (in the case of cars, the weight given to new cars should not include any deduction for the value of trade-ins).

3.129 In some countries, many of the durables purchased by households, especially vehicles, may be imports of second-hand goods from other countries. The prices and expenditures on these goods enter the CPI in the same way as those for newly produced goods. Similarly, in some countries there may be significant net purchases of second-hand vehicles by households from the business sector, these vehicles possibly carrying more weight in the index than new vehicles purchased by households.

Imputed expenditures on goods and services

3.130 As explained in earlier sections, many of the goods and services acquired and used by households for purposes of their own final consumption are not purchased in monetary transactions but are acquired through barter or as remuneration in kind or are produced by households themselves. It is possible to estimate what households would have paid if they had purchased these goods and services in monetary transactions or, alternatively, what it cost to produce them. In other words, values may be imputed for these non-monetary expenditures.

3.131 The extent to which it is desirable to include imputed expenditures within the scope of a CPI depends partly on the main purpose of the index. If the CPI is intended to be a measure of consumer inflation, it can be argued that only monetary expenditures should be included. Inflation is a monetary phenomenon measured by changes in monetary prices recorded in monetary transactions. Even when the main use of a CPI is for indexation purposes, it can be argued that it should only reflect changes in the monetary prices actually paid by the reference population. Consistent with the objective of monitoring inflation in the European Union, the aim of the Harmonized Index of Consumer Prices (HICP) compiled by Eurostat is to measure inflation faced by consumers. The concept of "household final monetary consumption expenditure" (HFMCE) used in the HICP defines both the goods and services to be covered, and the price concept to be used, i.e., prices net of reimbursements, subsidies and discounts. HFMCE refers only to monetary transactions and includes neither consumption of own production (e.g., agricultural goods or owner-occupied housing services) nor consumption of goods and services received as income in kind.

3.132 When the CPI is intended to be a cost of living index, some imputed expenditures would normally be included within the scope of the CPI on the grounds that the goods and services acquired in non-monetary transactions affect households' living standards. As already noted, most countries include households' imputed expenditures on housing services produced by owner-occupiers but not imputed expenditures on goods such as agricultural goods produced for own consumption.

Price coverage

3.133 A CPI should reflect the experience of the consumers to whom it relates, and should therefore record what consumers actually pay for the goods and services which are included in the scope of the index. The expenditures and prices recorded should be those paid by consumers, including any taxes on the products, and taking account of all discounts, subsidies and most rebates, even if discriminatory or conditional. It may be virtually impossible, however, to take account of all discounts and rebates in practice. Sensible practical compromises are needed, for which recommendations and examples are given in Chapter 6.

3.134 When households pay the full market prices for products and are then subsequently reimbursed by governments or social security schemes for some of the amounts paid, CPIs should record the market prices *less* the amounts reimbursed. This kind of arrangement is common for educational and medical expenditures.

Taxes and subsidies

3.135 All taxes on products, such as sales taxes, excise taxes and value added tax (VAT), are part of the purchasers' prices paid by consumers that should be used for CPI purposes. Similarly, subsidies should be taken into account, being treated as negative taxes on products.

3.136 For some analytical and policy purposes, it may be useful to estimate a CPI that measures price movements excluding the effects of changes in taxes and subsidies. For monetary policy-makers, the price increases resulting from changes in indirect taxes or subsidies are not part of an underlying inflationary process but are attributable to their own manipulation of these economic levers. Similarly, when a CPI is used for escalation purposes, any increase in a CPI resulting from increases in indirect taxes leads to an increase in wages and benefits linked to the CPI, despite the fact that the aim of the tax increase might have been to reduce consumers' purchasing power. Alternatively, an increase in subsidies might be intended to stimulate consumption, but the resulting lower prices could be offset by a smaller increase in indexed wages and benefits.

3.137 *Net price indices*. Net price indices may be compiled in which taxes on consumer goods or services are deducted from the purchasers' prices, and subsidies are added back on. Such indices do not, however, necessarily show how prices would have moved if there were no taxes or changes in taxes. It is notoriously difficult to estimate the true incidence of taxes on products: that is, the extent to which taxes or subsidies, or changes therein, are passed on to consumers. It is also difficult to take account of the secondary effects of changes in taxes. In order to estimate the secondary effects, input–output analysis can be used to work out the cumulative impact of taxes and subsidies through all the various stages of production. For example, some of the taxes on vehicle fuel will enter the price of transport services which in turn will enter the prices of transported goods, some of which will enter the prices paid for consumer goods by retailers and hence the prices which they charge to consumers. To track all these impacts would demand a much more detailed and up-to-date input–output table than is available in most countries. A more practicable alternative is therefore simply to confine the taxes and subsidies for which correction is made to those levied at the final stage of sale at retail; that is, primarily to VAT, sales and excise taxes. Estimating prices less these taxes only, or corrected for changes in these taxes only, is more feasible. In the case of a percentage sales tax or VAT, the calculation is simple, but in the case of excise taxes, it is necessary to ascertain the percentage mark-up by the retailer, since the excise tax will also be marked up by this percentage.

Discounts, rebates, loyalty schemes and "free" products

3.138 CPIs should take into account the effects of rebates, loyalty schemes, and money-off vouchers. Given that a CPI is meant to cover all the reference households, whether in the country as a whole or in a particular region, discounts should be included even if they are available only to certain households or to consumers satisfying certain payment criteria.

3.139 It may be difficult to record discriminatory or conditional discounts for practical reasons. When only one selected group of households can enjoy a certain discount on a specific product, the original stratum for that product is split into two new strata, each experiencing different price changes and each requiring a weight. So, unless base period expenditures for all possible strata are known, it is not possible to record discriminatory discounts correctly. Similarly, with conditional discounts, e.g. discounts on utility bills for prompt payment, it can be difficult to record the effect of the introduction of such offers unless data are available on the proportion of customers taking advantage of the offer. These kinds of practical problems also arise when there is price discrimination and the sellers change the criteria that define the groups to whom different prices are charged, thereby obliging some households to pay more or less than before without changing the prices themselves. These cases are discussed further in Chapter 7.

3.140 Although it is desirable to record all price changes, it is also important to ensure that the qualities of the goods or services for which prices are collected do not change in the process. While discounted prices may be collected during general sales seasons, care should be taken to ensure that the quality of the products being priced has not deteriorated.

3.141 The borderline between discounts and rebates can be hazy and is perhaps best drawn according to timing. In other words, a discount takes effect at the time of purchase, whereas a rebate takes effect some time later. Under this classification, money-off vouchers are discounts, and as with the conditional discounts mentioned above, can only be taken into account in a CPI if they relate to a single product and if the take-up rate is known at the time of CPI compilation. Since this is highly unlikely, the effect of money-off vouchers is usually excluded from a CPI. It should be noted that the discount is recorded only when the voucher is used, not when the voucher is first made available to the consumer.

3.142 Rebates may be made in respect of a single product, e.g. air miles, or may be more general, e.g. supermarket loyalty schemes where a \$10 voucher is awarded for every \$200 spent. As with discounts discussed above, such rebates can only be recorded as price falls if they relate to single products and can be weighted according to take-up. Bonus products provided "free" to the consumer, either by larger pack sizes or offers such as "two packs for the price of one", should be treated as price reductions, although they may be ignored in practice when the offers are only temporary and quickly reversed. When permanent changes to pack sizes occur, quality adjustments should be made (see Chapter 7).

3.143 Given the practical difficulties in correctly recording all these types of price falls, it is usual to reflect discounts and rebates only if unconditional, whereas loyalty schemes, money-off coupons, and other incentives are ignored. Discounts during seasonal sales may be recorded provided that the quality of the goods does not change.

Classification

3.144 The classification system upon which any CPI is built provides the structure essential for many stages of CPI compilation. Most obviously, it provides the weighting and aggregation structure, but it also provides the scheme for stratification of products in the sampling frame, at least down to a certain level of detail, and it dictates the range of sub-indices available for publication. Several factors must be taken into account when a CPI classification system is being developed.

- First, the classification must reflect economic reality. For example, it must be possible to accommodate new goods and services in a manner that minimizes the need for later restructuring of the higher-level categories. Restructuring is undesirable because many users require long time series, and restructuring of the classification will produce breaks in the series.
- Second, the needs of users for sub-indices should be given a high priority when constructing aggregate groups, so that if, for example, some users are particularly interested in price change in food products, then the classification should provide sufficient detail in that area.
- Third, it is a requirement of any classification that its categories are unambiguously mutually exclusive, and at the same time provide complete coverage of all products considered to be within its scope. In practice this means that it should be a straightforward task to assign any particular expenditure, or price, to a single category of the classification system.

3.145 The availability and nature of the data themselves will also affect the design of a classification system. The availability of expenditure and price data will dictate the lowest level of detail that might be possible. Obviously it is not possible to produce a separate index for a product for which either weights or prices are not available. At the most detailed level, a high variance of the price changes, or relatives, will suggest where additional categories are needed. In line with standard sampling procedures, the stratification scheme should minimize the within-stratum variance while at the same time maximizing the between-stratum variance. The classification should reflect this requirement.

Criteria for classifying consumption expenditure

3.146 Although a classification may be conceived according to economic theory or user requirements using a top-down approach, in practice the statistical compiler collects data about individual products and then aggregates them according to the classification scheme (a bottom-up application). For example, the units of classification for the Classification of Individual Consumption according to Purpose (COICOP) are expenditures for the acquisition of consumer goods and services, not expenditures on purposes as such. Divisions 01 to 12 of COICOP convert these basic statistics into a purpose classification by grouping together the various goods and services which are deemed to fulfil particular purposes, such as nourishing the body, protecting it against inclement weather, preventing and curing illness, acquiring knowledge, travelling from one place to another, and so on.

3.147 Classifications of expenditure data are schemes for aggregating expenditures on products according to certain theoretical or user-defined criteria, such as:

- *Product type* – products may be aggregated by:
  - physical characteristics of goods and the nature of services; for example, biscuits are divided into those with and without a chocolate coat. This criterion can be meaningfully implemented down to the most detailed level, and is the basis of the Central Product Classification 1.0 (United Nations, 1998b);
  - economic activity from which the product originated. The International Standard Industrial Classification of All Economic Activities (ISIC), Revision 3.1 (United Nations, 2002) is the international standard classification;
  - production process from which the product originated;
  - retail outlet type from which the product was purchased;
  - geographical origin of the product.
- *Purpose* to which the products are put, e.g. to provide food, shelter, transport, etc. COICOP is the international standard.
- The *economic environment*, where products could be aggregated according to criteria such as:
  - substitutability of products;
  - complementarity of products;
  - application of sales taxes, consumer subsidies, excise taxes, customs duties, etc.;
  - imports from different countries (and in some cases, a classification of exportable products may be of interest).

Classification by product type

3.148 Where indices of price change for specific products groups are required, a product-based classification would be appropriate. Product classifications may combine several of the criteria listed above; for example, the Classification of Products by Activity (CPA) in the European Economic Community (Eurostat, 1993), which is linked to the CPC at the detailed level and the ISIC at the aggregate level.

3.149 Inevitably, price collectors and index compilers will encounter products for which no detailed class or sub-class exists, for example, entirely new products, or mixed products which are bundles of existing products. This is a problem frequently encountered with high technology goods, telecommunications goods and services, and food items in the form of "ready meals". Initially, the expenditure on these products may be recorded in an "other" or n.e.c. (not elsewhere classified) class, but once expenditure on these products becomes significant, a separate class should be created.

Classification by purpose

3.150 For a CPI compiler aiming to produce a measure of the change in the cost of satisfying particular needs, a purpose-based classification is appropriate. The COICOP breakdown at the highest level is by purpose such that the 12 divisions of COICOP are categories of purpose, and below this level the groups and classes are product types. In other words, products are allocated to purpose headings. The allocation of products is complicated by the existence of multi-purpose products (single products that can be used for a variety of purposes), such as electricity, and mixed purpose (bundled) products, such as package holidays comprising transport, accommodation, meals, and so on.

3.151 *Multi-purpose goods and services*. The majority of goods and services can be unambiguously assigned to a single purpose, but some goods and services could plausibly be assigned to more than one purpose. Examples include motor fuel, which may be used to power vehicles classified as transport as well as vehicles classified as recreational, and snowmobiles and bicycles which may be bought for transport or for recreation.

3.152 In drawing up COICOP, the general rule followed has been to assign multi-purpose goods and services to the division that represents the predominant purpose. Hence, motor fuel is shown under "Transport". Where the predominant purpose varies between countries, multi-purpose items have been assigned to the division that represents the main purpose in the countries where the item concerned is particularly important. As a result, snowmobiles and bicycles are both assigned to "Transport" because this is their usual function in the regions where most of these devices are purchased – that is, North America and the Nordic countries in the case of snowmobiles, and Africa, South-East Asia, China and the low countries of Northern Europe in the case of bicycles.

3.153 Examples of other multi-purpose items in COICOP include: food consumed outside the home, which is shown under "Hotels and restaurants", not "Food and non-alcoholic beverages"; camper vans, which are shown under "Recreation and culture", not "Transport"; and basket-ball shoes and other sports footwear suitable for everyday or leisure wear, which are shown under "Clothing and footwear", not "Recreation and culture".

3.154 National statisticians may wish to reclassify multi-purpose items if they consider that an alternative purpose is more appropriate in their country. Such reclassifications should be footnoted.

3.155 *Mixed purpose goods and services*. Single outlays may sometimes comprise a bundle of goods and services which serve two or more different purposes. For example, the purchase of an all-inclusive package tour will include payments for transport, accommodation and catering services, while the purchase of educational services may include payments for health care, transport, accommodation, board, educational materials, and so on.

3.156 Outlays covering two or more purposes are dealt with case by case with the aim of obtaining a breakdown by purpose that is as precise as possible and consistent with practical considerations of data availability. Hence, purchases for package holidays are shown under "Package holidays" with no attempt to isolate separate purposes such as transport, accommodation and catering. Payments for educational services, in contrast, should as far as possible be allocated to "Education", "Health", "Transport", "Hotels and restaurants" and "Recreation and culture".

3.157 Two other examples of mixed purpose items are: the purchase of in-patient hospital services which include payments for medical treatment, accommodation and catering; and the purchase of transport services which include meals and accommodation in the ticket price. In both cases, there is no attempt to isolate separate purposes. Purchases of in-patient hospital services are shown under "Hospital services" and purchases of transport services with accommodation and catering are shown under "Transport services".

Classifications for consumer price indices

3.158 In practice, most countries use a hybrid classification system for their CPI in the sense that the breakdown of expenditure at the highest level is by purpose, with breakdowns by product at the lower levels. In some countries the higher-level purpose classifications were developed many years ago for CPIs that were originally devised as measures of the changing cost of a basket of goods and services that were, at the time, considered necessary for survival or maintaining some "basic" standard of living. Thus, the classifications were based on consumer needs, where "need" may have had a somewhat subjective interpretation depending on political requirements.

3.159 The recommended practice today is still to use a purpose classification at the highest level, with product breakdowns below, but to use the recently developed international standard classifications as far as possible, with adaptations to national requirements where necessary. In other words, divisions 01 to 12 of COICOP, with Central Product Classification (CPC) product classes and sub-classes mapped onto them to provide the next two levels of detail.

Publication level

3.160 As mentioned above, any restructuring of the classification of published indices will inconvenience users and should be avoided so far as possible by careful planning and development of the classification scheme in the first place. There is a trade-off between providing users with as much detail as they would like in terms of product indices and weights, and preserving some freedom to restructure the lower levels (unpublished) without apparently affecting the published series.

3.161 Item samples below the level at which weights are published can be revised between major weight revisions. As explained in Chapter 9, new and replacement items and varieties can also be introduced provided they can be included within an existing published weight. A major new product, such as a personal computer, could only be introduced at the time of a major weight revision, whereas it might be possible to introduce mobile phones at any time if the lowest-level weight published in the telecommunications category is for telephone services.

Classification of Individual Consumption according to Purpose (COICOP)

3.162 *COICOP structure*. The international standard classification of individual consumption expenditures is the Classification of Individual Consumption according to Purpose (COICOP). COICOP is a functional classification that is also used in *SNA 1993* and covers the individual consumption expenditures incurred by three institutional sectors, i.e. households, non-profit institutions serving households (NPISHs), and general government. Individual consumption expenditures are those which benefit individual persons or households.

3.163 COICOP has 14 divisions:

– divisions 01 to 12 covering the final consumption expenditure of households;
– division 13 covering the final consumption expenditure of NPISHs;
– division 14 covering the individual consumption expenditure of general government.

The classification has three levels of detail:

– division or two-digit level, e.g. 01. Food and non-alcoholic beverages;
– group or three-digit level, e.g. 01.1 Food;
– class or four-digit level, e.g. 01.1.1 Bread and cereals.

3.164 The 12 divisions covering households consist of 47 groups and 117 classes and are listed in Annex 2. Below the level of class, CPI compilers have to create additional detail by further subdividing the classes according to their national needs. Of course, there are clear advantages, in terms of comparability between countries, and between the different uses of COICOP (CPIs, household expenditure statistics, national accounts aggregates), if the basic, higher-level structure of COICOP is maintained.

3.165 There are some COICOP classes which may, or may not, be included in most CPIs, or for which expenditure data cannot be collected directly from households. For example, COICOP has a class for the imputed rentals of owner-occupiers, which may be outside the scope of some CPIs. COICOP also has a class for financial intermediation services indirectly measured, which may be outside the scope of some CPIs because of practical measurement difficulties. In any case, the expenditures on these services cannot be collected in household budget surveys. Similarly, COICOP has a group for expenditure on insurance service charges, which may be within the scope of CPIs but cannot be measured using household surveys.

3.166 *Type of product*. COICOP classes are divided into: services (S), non-durables (ND), semi-durables (SD) and durables (D). This supplementary classification provides for other analytical applications. For example, an estimate may be required of the stock of consumer durables held by households, in which case the goods in COICOP classes that are identified as "durables" provide the basic elements for such estimates.

3.167 As explained above, the distinction between non-durable goods and durable goods is based on whether the goods can be used only once or whether they can be used repeatedly or continuously over a period of considerably more than one year. Moreover, durables, such as motor cars, refrigerators, washing machines and televisions, have a relatively high purchasers' value. Semi-durable goods differ from durable goods in that their expected lifetime of use, though more than one year, is often significantly shorter and their purchasers' value is substantially less. Because of the importance attached to durables, the categories of goods defined as durables in COICOP are listed below:

– furniture, furnishings, carpets and other floor coverings;
– major household appliances;
– tools and equipment for house and garden;
– therapeutic appliances and equipment;
– vehicles;
– telephone and fax equipment;
– audiovisual, photographic and information processing equipment (except recording media);
– major durables for recreation;
– electrical appliances for personal care;
– jewellery, clocks and watches.

The following goods are listed as semi-durables:

– clothing and footwear;
– household textiles;
– small electrical household appliances;
– glassware, tableware and household utensils;
– spare parts for vehicles;
– recording media;
– games, toys, hobbies, equipment for sport, camping, etc.

3.168 Some COICOP classes contain both goods and services because it is difficult for practical reasons to break them down into goods and services. Such classes are usually assigned an (S) when the service component is considered to be predominant. Similarly, there are classes which contain either both non-durable and semi-durable goods or both semi-durable and durable goods. Again, such classes are assigned a (ND), (SD) or (D) according to which type of good is considered to be the most important.

Appendix 3.1 Consumer price indices and national accounts price deflators

1. The purpose of this appendix is to explain why and how consumer price indices (CPIs) differ from the price indices used to deflate household consumption expenditures in national accounts. The differences between the two kinds of price index are often not well understood.

Coverage of households

2. The sets of households covered by CPIs and the national accounts are not intended to be the same, CPIs typically covering a smaller set of households. Household consumption expenditures in national accounts cover the expenditures made by all households, including institutional households resident in the country or region, whether those expenditures are made inside or outside the country or region of residence. CPIs tend to cover the expenditures and prices paid by households within the geographical boundaries of a country or region, whether the households are residents or visitors. More importantly, most CPIs are purposely defined to cover only selected groups of non-residential households. For example, CPIs may exclude very wealthy households or be confined to households in urban areas or headed by wage-earners.

Coverage of consumption expenditures

3. The sets of expenditures covered by CPIs and national accounts are not intended to be the same, CPIs typically covering a smaller set of expenditures. Most CPIs do not cover most of the imputed non-monetary consumption expenditures included in national accounts, either on principle or in practice because of lack of data. Many CPIs include the imputed rents on owner-occupied housing, but CPIs are not intended to cover the imputed expenditures and prices of agricultural products or other goods produced for own consumption that are included in national accounts.

Timing

4. Most CPIs measure price changes between two points of time or very short intervals of time such as a week. The price indices in national accounts are intended to deflate expenditures aggregated over long periods of time, generally a year. The ways in which monthly or quarterly CPIs are averaged to obtain annual CPI indices are unlikely to be conceptually consistent with the annual price indices in national accounts.

Index number formulae

5. The index number formulae used by CPIs and national accounts are not intended to be the same. In practice, most CPIs tend to use some kind of Lowe price index that uses the quantities of an earlier period, whereas the price indices, or price deflators, in national accounts are usually meant to be Paasche indices. Paasche indices are used in order to obtain Laspeyres volume indices. These differences, arising from the use of different index formulae, would tend to be reduced if both CPIs and national accounts adopted annual chaining.

Conclusions

6. It is clear that, in general, CPIs and the price deflators for national accounts can differ for a variety of reasons, such as major differences in the coverage of households and expenditures, differences in timing and differences in the underlying index number formulae. These differences are intentional and justified. Of course, the price data collected for CPI purposes may also be used to build up the detailed price deflators used for national accounts purposes, but at an aggregate level CPIs and national accounts deflators may be quite different for the reasons just given.

SOME SPECIAL CASES

10

Introduction

10.1 This chapter focuses on a number of expenditure areas that pose particular problems for price index compilers, both in terms of identifying an agreed conceptual approach and also overcoming practical measurement difficulties. Six areas have been selected for discussion, mainly from the service sector. They are:

– owner-occupied housing;
– clothing;
– telecommunication services;
– financial services;
– real estate agency services;
– property insurance services.

10.2 This chapter is therefore structured into six sections, in turn dealing with the problem areas listed above. Under each section, any necessary theoretical considerations are discussed and relevant measurement issues explored. Where appropriate, illustrative examples of alternative approaches to the measurement of weights or price changes are provided, and the advantages and disadvantages are outlined.

10.3 It is important to note that the examples shown are neither definitive nor prescriptive, but rather provide broad guidance as to how the problem areas can be approached. User requirements, data availability and the statistical resources available are important factors that need to be taken into consideration in choosing an appropriate methodology. Market conditions and product market regulations, which can differ widely between countries, also have a critical impact on the choice of method.

Owner-occupied housing

10.4 The treatment of owner-occupied housing in consumer price indices (CPIs) is arguably the most difficult issue faced by CPI compilers. Depending on the proportion of the reference population that are owner-occupiers, the alternative conceptual treatments can have a significant impact on the CPI, affecting both weights and, at least, short-term measures of price change.

10.5 Ideally, the approach chosen should align with the conceptual basis that best satisfies the principal purpose of the CPI. However, the data requirements for some (or even all) of these options may be such that it is not feasible to adopt the preferred treatment. Equally important, it may be difficult to identify a single principal purpose for the CPI. In particular, the dual use of CPIs as both macroeconomic indicators and also for indexation purposes can lead to clear tensions in designing an appropriate treatment for owner-occupied housing costs. In these circumstances, it may be necessary to adopt a treatment that is not entirely consistent with the approach adopted for other items in the CPI. In some countries, the difficulties in resolving such tensions have led to the omission of owner-occupied housing from the CPI altogether or the publication of more than one index.

10.6 The remainder of this section discusses the conceptual basis and data requirements for the *use*, *payments* and *acquisitions* approaches in turn.

Use

10.7 The general objective of this approach is to measure the change over time in the value of the flow of shelter services consumed by owner-occupiers. Detailed approaches fall under one of two broader headings: user cost or rental equivalence.

10.8 The *user cost* approach attempts to measure the changes in the cost to owner-occupiers of using the dwelling. In the weighting base period, these costs comprise two elements: recurring actual costs, such as those for repairs and maintenance, and property taxes; and the opportunity cost of having money tied up in the dwelling rather than being used for some other purpose. At its simplest, and where houses are purchased outright, this latter element is represented by the rate of return available on alternative assets. More usually, house purchase will be at least part financed through mortgage borrowing. In this case, opportunity cost can be viewed as an average of interest rates on mortgages and alternative assets, weighted by the proportion of the purchase price borrowed and paid outright, respectively.

10.9 Estimation of the base period weight for recurring actual costs such as expenditures on repairs and maintenance is relatively straightforward and generally obtainable from household expenditure surveys. Similarly, the construction of price measures for these items presents few difficulties.

10.10 Estimation of the base period weight for opportunity costs is more complicated and will require modelling. One approach is to assume that all owner-occupiers purchased their dwellings outright at the beginning of the period and sold them at the end. During the period their opportunity costs comprise the amount of interest forgone (i.e. the amount of interest they might have earned by investing this money elsewhere) and depreciation. Offsetting these costs would be any capital gains earned on the sale of the dwellings. Construction of the required measures of price change is likewise quite complicated (see Chapter 23 for a more complete discussion) and, particularly for the depreciation element, a good deal of imputation is required. Allowing for house purchases part financed by mortgage borrowing, a typical formula for user cost (UC) is:

$$UC = rM + iE + D + RC - K$$

where M and E represent mortgage debt and equity in the home, and r and i represent mortgage interest rates and the rate of return available on alternative assets, respectively. D is depreciation, RC other recurring costs and K capital gains.

10.11 No national statistical office is currently using the full user cost approach. This partly reflects the conceptual and methodological complexity of the measure, which may also make it difficult to obtain widespread public support for the approach. For this reason, the methodology is not discussed in detail here. It is, however, worth noting that both the weights and the ongoing measures of price change are significantly influenced by the relative rate of change in house prices. Since the user cost formula is typically dominated by capital gains and interest rates, where house price inflation exceeds nominal interest rates the user cost weight is likely to be negative (implying a negative price for user cost).

10.12 In practice, it is possible to avoid some of these difficulties by adopting a variant or a narrower definition of user cost. For example, some countries have adopted a variant of the user cost approach focusing on gross mortgage interest payments and depreciation, in part because these items are readily recognizable as key costs by home owners. The former may be viewed as the cost of retaining housing shelter today, while the depreciation element represents current expenditure that would be required to offset the deterioration and obsolescence in dwellings that would otherwise occur over time. Methodologies for calculating actual average mortgage interest payments for index households are described in the section on the payments approach to owner-occupied housing costs, below.

10.13 Depreciation is a gradual process and so is best represented by the amount that needs to be put aside year by year as opposed to actual expenditures (which will typically be large but infrequent). The base period weight for depreciation may be estimated from the current market value of the owner-occupied housing stock excluding land values, multiplied by an average rate of depreciation. The latter may be derived from national accounts estimates of housing capital consumption. Imputed this way, the appropriate price indicator should ideally be an index of house prices excluding land rather than an index of the costs of renovation work.

10.14 The *rental equivalence* approach attempts to measure the change in the price of the housing service consumed by owner-occupiers by estimating the market value of those services. In other words, it is based on estimating how much owner-occupiers would have to pay to rent their dwelling. Under this approach, it would be inappropriate also to include those input costs normally borne by landlords such as dwelling insurance, major repair and maintenance, and property taxes as this would involve an element of double counting. The rental equivalence approach is recommended in *SNA 1993* for measuring household consumption and is also used in constructing international comparisons of living standards.

10.15 Deriving the weight for rental equivalence requires estimating how much owner-occupiers would have paid in the weighting base period to rent their dwellings. This is not something that owner-occupiers can normally be expected to estimate reliably in a household expenditure survey. In principle, however, it can be estimated by matching the dwellings of owner-occupiers with comparable dwellings that are being rented and applying those rents to the owner-occupied dwellings.

10.16 In practice, this raises a number of problems, particularly in countries where the overall size of the private rental market is small or if rented housing is of a different type from owner-occupied housing in terms of general quality, age, size and location. Direct imputation from actual rents may also be inappropriate if the rental market is subject to price control. In addition, owner-occupiers may be considered to derive significant additional utility from features such as security of tenure and the ability to modify the dwelling, implying a need to make additional adjustments to the initial imputations.

10.17 In those countries where the reference population for the CPI corresponds to all resident households, the estimation problem is identical to that faced by the national accountants and a collaborative approach would be beneficial.

10.18 The corresponding price series for owner-occupiers' rent can be derived from an actual rent index, except where such rents are subject to price control. Depending on both the relative significance of owner-occupiers to renters and the composition of the two markets in terms of dwelling characteristics, any existing rent surveys may need to be modified to meet the particular requirements of an owners' equivalent rent series. If the total value of owners' equivalent rent is significantly larger than actual rents, the absolute size of the existing price sample may be deemed insufficient. If the characteristics of owner-occupied dwellings differ significantly from the overall rental market, the existing rent survey may also require stratifying more finely (e.g. by type and size of dwelling, and by location). The price measures for the different strata can then be given different weights when calculating the actual rents and the owners' equivalent rent series, respectively.

10.19 While it may be acceptable to include subsidized and controlled prices in the actual rent series, these should not be used in calculating the owners' equivalent rent series. Given the increased significance of rent prices in the overall index, it may also be necessary to pay greater attention to the measurement of price change for individual properties when tenancies change. As this often presents landlords with an opportunity to refurbish properties and increase rents, the practice of regarding the whole of all such price changes as arising from quality change should be avoided. Furthermore, the rent series may need to be quality-adjusted to take account of ongoing depreciation to housing structures.

This question is discussed in Chapter 23, paragraphs 23.69 to 23.78.

Payments

10.20 The item domain for a payments index is defined by reference to actual outlays made by households to gain access to consumer goods and services. The set of outlays peculiar to owner-occupiers in the weighting base period includes:

- down payments or deposits on newly purchased dwellings;
- legal and real estate agency fees payable on property transfers;
- repayments of mortgage principal;
- mortgage interest payments;
- alterations and additions to the dwelling;
- insurance of the dwelling;
- repair and maintenance of the dwelling;
- property rates and taxes.

10.21 While it is conceivable to include all of these items in the index, it is generally agreed that at least some represent capital transactions that ought to be excluded from a CPI. For example, while down payments and repayments of mortgage principal result in a running down of household cash reserves, they also result in the creation of a real asset (at least part of a dwelling) or in the reduction of a liability (the amount of mortgage debt outstanding). Similarly, any cash expenditures on alterations and additions result in a running down of cash reserves offset by increases in dwelling values. In other words, those transactions which result in no net change to household balance sheets should be excluded.

10.22 The remaining items can be regarded as current expenditures which do not result in any offsetting adjustments to household balance sheets. It is therefore considered appropriate that these items be included in a payments-based CPI. By defining a payments index in this way, it is clear that the aggregate payments equal a household's source of funds which comprise income after tax (wages, transfers, property income, insurance claims, etc.) and net savings (as a balancing item). It is for this reason that a payments-based CPI is commonly considered to be the best construct for assessing changes in net money incomes over time.

10.23 Estimation of gross expenditures on these items in the weighting base period is readily achievable via a household expenditure survey, as the items are generally reportable by households. The construction of price indices for real estate agency fees and insurance is discussed later in this chapter. Indices for repair and maintenance, and property rates and taxes are not considered particularly problematic so are not discussed here. The remainder of this section is therefore devoted to the construction of price measures for mortgage interest charges.

10.24 The construction of price indices for mortgage interest charges is not altogether straightforward. The degree of complexity will vary from country to country depending on the operation of domestic financial markets and the existence (or otherwise) of any income tax provisions applying to mortgage interest payments. What follows therefore is a description of an overall objective and an illustrative methodology for producing the required index in the most straightforward of cases. The methodology will require modifying to account for additional complexities that may be encountered in some countries.

10.25 The general approach may be summarized briefly as follows. Under a fixed basket approach, the objective of the index is to measure the change over time in the interest that would be payable on a set of mortgages equivalent to those existing in the weighting base period. This base stock of mortgages will, of course, vary widely in age, from those taken up in the base period itself to those taken up many years previously. In compiling a fixed base index, the distribution of mortgages by age is required to be held constant.

10.26 The amount of interest payable on a mortgage is determined by applying some rate of interest, expressed as a percentage, to the monetary value of debt. Changes in mortgage interest charges over time therefore can, in principle, be measured by periodically collecting information on a representative selection of mortgage interest rates, using these to derive an average interest rate, and then applying this to an appropriate debt figure. At least for standard variable rate mortgages, interest due on the revalued stock of base period mortgages may be derived simply with reference to current mortgage interest rates.

10.27 The main problem then is in determining the appropriate debt figure in each of the comparison periods. Since the real value of any monetary amount of debt varies over time according to changes in the purchasing power of money, it is not appropriate to use the actual base period monetary value of debt in calculations for subsequent periods. Rather, it is necessary first to update that monetary value in each comparison period so that it remains constant in real terms (i.e. so that the quantities underpinning the base period amount are held constant).

10.28 In order to do this, it is necessary to form at least a theoretical view of the quantities underpinning the amount of debt in the base period. The amount of mortgage debt outstanding for a single household in the base period depends on the original house purchase price and loan-to-value ratio, and also the rate of repayment of principal since the house was purchased. An equivalent value of debt can be calculated in subsequent comparison periods by holding constant the age of the debt, the original value of the debt (as some fixed proportion of the total value of the dwelling when the mortgage was initially entered into) and the rate of repayment of the principal (as some proportion of the original debt), and applying these factors to house prices for periods corresponding to the age of the debt.

10.29 To illustrate, suppose a base period household purchased a dwelling five years earlier for $100,000 and financed 50 per cent by mortgage. If, between the time of purchase and the base period, the household repaid 20 per cent of this debt, then the outstanding debt on which base period interest charges were calculated would have been $40,000. Now move to some subsequent comparison period and suppose that it is known that house prices doubled between the period when the household originally purchased and the period five years prior to the comparison period. The equivalent amount of outstanding debt in the comparison period would be calculated by first taking 50 per cent of the revalued house price (of $200,000) to give $100,000, and then reducing this by the principal repayment rate (of 20 per cent) to give $80,000.

10.30 Under these assumptions, it is clear that the comparison period value of outstanding debt may be estimated directly from the base period value of outstanding debt solely on the basis of house price movements between five years prior to the base period and five years prior to the comparison period. In other words, while preservation of original debt/equity ratios and rates of repayment of principal help in understanding the approach, estimates of these variables are not strictly required to calculate the required comparison period debt. All that is required is the value of the outstanding debt in the base period, the age of that debt and a suitable measure of changes in dwelling prices.

10.31 Now suppose that all mortgages are of the variable rate type, and that average nominal interest rates rose from 5 per cent in the base period to 7.5 per cent in the comparison period. Interest payments in the two periods can be calculated as $2,000 and $6,000 respectively, and so the mortgage interest payments index for the comparison period is 300.0. An identical result may of course be found directly from index number series for debt and nominal interest rates. The mortgage interest charges index equals the debt index multiplied by the nominal interest rate index divided by 100. In this example, the debt index equals 200.0 and the nominal interest rate index equals 150.0. Therefore the mortgage interest rate index equals $(200.0 \times 150.0)/100$ or 300.0. This simple example also serves to illustrate the very important point that percentages (interest rates, taxes, etc.) are not prices and cannot be used as if they were. Percentages must be applied to some monetary value in order to determine a monetary price.

10.32 While the single-household example shown above is useful in explaining the basic concepts, it is necessary to devise a methodology that can be employed to calculate a mortgage interest charges index for the reference population as a whole. The main complication when moving from the single-household to the many-household case is the fact that the age of the debt will vary across households. Given the importance of revaluing base period debt to maintain a constant age, this is no trivial matter. While it is conceivable that information on the age of mortgage debt could be collected in household expenditure surveys, the additional respondent burden and the generally small number of households reporting mortgages often serve to make estimates from this source unreliable. Another option is to approach a sample of providers of mortgages (banks, building societies, etc.) for an age profile of their current mortgage portfolio. This type of data is normally available and is generally reliable.

10.33 Table 10.1 illustrates how an aggregate debt price index can be constructed. For the purpose of illustrating the methodology, some simplifying assumptions have been made:

- The index is assumed to be quarterly rather than monthly.
- The oldest age of mortgage debt is assumed to be between three and four years (in practice, it is normally the case that debt older than eight years is insignificant).
- Each annual cohort of debt is assumed to be distributed evenly across the year.
- A quarterly index of dwelling prices (new and second-hand dwellings, including land) is available.

10.34 Column (1) of Table 10.1(a) contains index numbers for dwelling prices extending back four years prior to the base period for the debt series (quarter 1 of year 0). Column (2) contains a four-quarter moving average of the first series – this is required to reflect "yearly" prices to correspond with the debt cohorts, which are only available in yearly age groups in this example (if quarterly cohorts were available it would not be necessary to calculate the moving average series).

10.35 Columns (1) to (4) of Table 10.1(b) contain the calculated debt indices for each cohort re-referenced to Y0 Q1 = 100. These series are simple transformations of the series in column (2) of Table 10.1(a), each with a different starting point. For example, the debt series for that cohort contracted for between three and four years ago has as a starting point the index number from Y−4 Q4 (i.e. 113.9) in column (2), and the series for debt aged between two and three years starts from Y−3 Q4 (i.e. 118.7) and so on. Column (5) of Table 10.1(b) contains the aggregate debt index which is derived by weighting together the indices for the four age cohorts. The weights are derived from data from financial institutions on debt outstanding by age, revalued to period Y0 Q1 prices.

10.36 A nominal mortgage interest rate index number series is obtained by calculating average quarterly interest rates on variable rate mortgages from a sample of lending institutions (starting in period Y0 Q1) and presenting them in index number form. The nominal interest rate series can then be combined with the debt series to calculate the final mortgage interest rate charges series, as illustrated in Table 10.2.

10.37 The construction of equivalent indices for fixed interest mortgages is more complicated in so far as an interest charges index has to be calculated separately for each age cohort of debt to reflect the fact that interest payable today, on a loan four years old, depends on the interest rate prevailing four years ago. This requires the compilation of a nominal fixed interest rate index extending back as far as the dwelling price series. To the extent that the interest rates charged on fixed interest loans also depend on the duration of the loan, calculation of the nominal fixed interest rate series is also more complex. The additional complexity of these indices may make the construction of a mortgage interest charges index impractical for countries where fixed interest rate mortgages predominate.

10.38 The construction of the index for mortgage interest payments is predicated on the assumption that

Table 10.1 Calculation of a mortgage debt series
(a) Dwelling price index

Year	Quarter	Original house price index (1)	Four-quarter moving average of (1) (2)
Y−4	Q1	111.9	
	Q2	112.8	
	Q3	114.7	
	Q4	116.2	113.9
Y−3	Q1	117.6	115.3
	Q2	118.5	116.8
	Q3	119.0	117.8
	Q4	119.8	118.7
Y−2	Q1	120.1	119.4
	Q2	120.3	119.8
	Q3	120.5	120.2
	Q4	122.0	120.7
Y−1	Q1	122.3	121.3
	Q2	123.8	122.2
	Q3	124.5	123.2
	Q4	125.2	124.0
Y0	Q1	125.9	124.9
	Q2	126.1	125.4
	Q3	127.3	126.1
	Q4	129.2	127.1

(b) Debt index

Year	Quarter	Age of debt				
		3–4 years Wt = 10% (1)	2–3 years Wt = 20% (2)	1–2 years Wt = 30% (3)	0–1 year Wt = 40% (4)	Weighted average (5)
Y0	Q1	100.0	100.0	100.0	100.0	100.0
	Q2	101.2	100.6	100.7	100.7	100.7
	Q3	102.5	100.9	101.6	101.1	101.4
	Q4	103.4	101.3	102.2	101.7	101.9

the purpose of the mortgage is to finance the purchase of the dwelling (hence revaluation of debt by changes in dwelling prices). However, it is increasingly common, particularly in developed countries, for households to draw down on the equity they have in their home. That is, households may take new or additional mortgages, or redraw part of the principal already paid to finance other activities, for example to purchase a large consumer durable such as a car or a boat, to go on holiday or even to purchase stocks and bonds. If these alternative uses of the funds made available by way of mortgages are significant, it may be appropriate to regard at least some proportion of mortgage interest charges as the cost of a general financial service rather than a housing cost. For that proportion of the debt deemed to be used for other purposes, it would be more appropriate to use a general index of price inflation for debt revaluation purposes.

Table 10.2 Calculation of a mortgage interest charges series

Year	Quarter	Debt index (1)	Nominal interest rates index (2)	Mortgage interest charges index $(1) \times (2)/100$ (3)
Y0	Q1	100.0	100.0	100.0
	Q2	100.7	98.5	99.2
	Q3	101.4	100.8	102.2
	Q4	101.9	101.5	103.4

Acquisitions

10.39 The item domain for an acquisitions index is defined as all those consumer goods and services acquired by households. Those countries which compile their CPIs on an acquisitions basis have generally concluded that the principal purpose of their CPI is to provide a measure of price inflation for the household sector as a whole. Based on the view that price inflation is a phenomenon peculiar to the operation of markets, the domain is also normally restricted to those consumer goods and services acquired in monetary transactions. That is, consumer goods and services provided at no cost to households by governments and non-profit institutions serving households are excluded.

10.40 The expenditures of owner-occupiers that could be included in an acquisitions index are:

– net purchases of dwellings (i.e. purchases less sales by the reference population);
– direct construction of new dwellings;
– alterations and additions to existing dwellings;

– legal and real estate agency fees payable on property transfers;
– repair and maintenance of dwellings;
– insurance of dwellings;
– property rates and taxes.

10.41 The construction of price indices for real estate agency fees and insurance is discussed later in this chapter. Indices for repair and maintenance, and property rates and taxes are not considered particularly problematic so are not discussed here. The remainder of this section is therefore devoted to a discussion of the issues involved in constructing measures for dwelling purchase, construction, and alterations and additions. An advantage of the acquisitions approach is that, consistent with the treatment of most other goods and services in the CPI, the owner-occupied housing index will reflect the full price paid for housing. Moreover, it is not affected by methods of financing for house purchase.

10.42 As CPIs are constructed to measure price change for a group of households in aggregate (the reference or target population), the index should not include any transactions that take place between those households. In the case of an index covering all private households, the weight should only reflect net additions to the household sector owner-occupied housing stock. In practice, net additions will mainly comprise those dwellings purchased from businesses (newly constructed dwellings, company houses, or rental dwellings) and those purchased from or transferred from the government sector plus any purchases, for owner-occupation, of rental dwellings from reference population households. If the CPI is constructed for some subgroup of the population (e.g. wage and salary earners), the weight should also include purchases from other household types.

10.43 Economists regard all housing as fixed capital and hence would exclude purchases of dwellings from household consumption. While this is unambiguously the case for housing purchased for rental, the case is less clear-cut when it comes to housing for owner-occupation. Although households recognize the likelihood of making capital gains when they purchase housing and invariably regard their dwelling as an asset, they also commonly cite the primary motivation for the purchase of a dwelling as being to gain access to a service (i.e. shelter and security of tenure). From the households' perspective, therefore, the costs borne by owner-occupiers in respect of their principal dwelling represent a mix of investment and consumption expenditure, and the total exclusion of these costs from an acquisitions-based CPI can lead to a loss of confidence in the CPI by the population at large. Particularly in those countries where rental sectors are relatively small, with limited opportunities for substitution between owner-occupation and renting, it might be argued that the consumption element dominates.

10.44 The problem confronting compilers of CPIs is how to separate the two elements so as to include only the consumption element in the CPI. Although there is no single agreed technique, one approach is to regard the cost of the land as representing the investment element and the cost of the structure as representing the consumption element. The rationale for this is that while the structure may deteriorate over time and hence be "consumed", the land remains at constant quality for all time (except under extremely unusual circumstances). As the land (or location element) accounts for most of the variation in observable prices for otherwise identical dwellings sold at the same point in time, the exclusion of land values may also be seen as an attempt to exclude asset price inflation from the CPI. (Measures of asset price inflation are, of course, useful in their own right.)

10.45 Derivation of weighting base period expenditures on the net acquisition of dwellings (excluding land), the construction of new dwellings, and alterations and additions to existing dwellings poses some problems. Although household expenditure surveys may yield reliable estimates of the amounts households spend on alterations and additions, and construction of dwellings, it is unlikely that they will provide reliable estimates of net expenditures on existing dwellings exclusive of the value of the land.

10.46 An alternative approach is to combine data from censuses of population and housing and building activity surveys. Population censuses normally collect information on housing tenure, from which average annual growth in the number of owner-occupier households represents a good proxy for net additions to the housing stock. Building activity surveys are also conducted in most countries, providing data on the total value of dwellings constructed. These data can be used to estimate the average value of new dwellings, which can then be applied to the estimated volumes derived from the population census. Of course, the suitability of this approach would need to be assessed by each country and may be complicated if the CPI relates only to some subset of the total population.

10.47 The price index is required to measure the change over time in existing dwelling structures, newly constructed dwellings, and alterations and additions. As the appropriate price for existing dwelling structures is current replacement cost, an index measuring changes in prices of newly constructed dwellings is also appropriate for this purpose. Given that the prices for both newly constructed dwellings and alterations and additions are, in principle, determined by costs of building materials, labour costs and producers' profits, it may also be satisfactory to construct a single price sample for all elements. The requirement for a separate price sample for alterations and additions will depend on the relative significance of this activity and whether the material and labour components differ significantly from those for a complete dwelling (e.g. if alterations and additions are predominantly to kitchens and bathrooms). In all cases, it is important that the price indices are mix-adjusted to eliminate price variations that reflect changes in the characteristics of newly constructed dwellings.

10.48 The type of dwelling constructed in individual countries will significantly influence the complexity and cost of constructing appropriate price measures. If each newly constructed dwelling is essentially unique (i.e. designed to meet site or other requirements) it will be necessary to adopt ‘‘model pricing’’. This requires selection of a sample of building firms, identifying samples of recently constructed dwellings and collecting prices for constructing identical dwellings in subsequent periods (exclusive of site preparation costs, which will vary from site to site). This approach is likely to entail significant costs for the respondents. Moreover, care needs to be taken to ensure that the supplied prices truly reflect all prevailing market conditions. That is, prices need to reflect the amount builders could realistically expect to be able to charge in the current market rather than the prices they would like to be able to charge based on conditions prevailing in some prior period.

10.49 In a number of countries, a significant proportion of newly constructed dwellings are of the type referred to as ‘‘project homes’’. These are homes that builders construct on a regular basis from a suite of standard designs maintained for this purpose. This practice is most feasible in countries where a significant proportion of new dwelling construction takes place in new developments (i.e. land recently developed or redeveloped specifically for residential housing). Where project home construction is significant in scale, then it is possible to select a sample of these project homes for pricing over time, safe in the knowledge that the prices provided will be actual transaction prices (again, priced net of any site preparation costs). Even if project homes do not account for the majority of new dwellings constructed, they may still provide a representative measure of overall price change.

10.50 In pricing project homes, it is necessary to monitor the selected sample to ensure that the selected plans remain representative and to detect changes in quality arising from modifications in design and changes to basic inclusions. Whenever a change is made to the plans, the change in overall quality has to be estimated. For physically measurable characteristics, such as a small increase in the overall size of the dwelling, it may be assumed that the change in quality is proportional to the change in the relevant quantity. Other changes, such as the addition of insulation, inclusion of a free driveway and so on, will need to be valued, preferably in terms of current value to the consumer. These could be estimated by obtaining information on the amounts that consumers would have to pay if they were to have the items provided separately (the option cost method). An alternative is to ask the builder if a cash rebate is available in lieu of the additional features. Where plans are modified to meet changed legal requirements, the consumer has no choice in purchase and so it is acceptable to classify the full change in price as pure price movement (even though there may be some discernible change in quality).

Clothing

10.51 Clothing is a semi-durable good and its treatment is not affected by the conceptual basis chosen for the CPI (acquisitions, use or payments). Particular features of the clothing market do, however, create problems for price index compilers. Although clothing is purchased throughout the year, many types of clothing are only available in particular seasons and, unlike seasonal fruit and vegetables, the specific items on sale in one season (say summer) may not return the following year. In addition to seasonal availability, the physical characteristics of some items of clothing can also change as a result of changing fashions.

10.52 The remainder of this section seeks to provide a general description of the clothing market applicable to most countries, discusses the most significant problems faced by index compilers and looks at some options for overcoming or at least minimizing these.

The clothing market

10.53 Most countries experience at least some climatic variation throughout the year. The number of discrete ‘‘seasons’’ may range from two (‘‘wet’’ and ‘‘dry’’, summer and winter) up to the four experienced in most regions (winter, spring, summer and autumn). Items of clothing tend to fall into two categories: those that are available in one season only, and those that are available all year round.

10.54 Clothing (whether seasonal or not) is also subject to changes in fashion. The fashion for trousers can change from straight legged to flared; jackets from single-breasted to double-breasted; shirts from button-down collar to not; skirts from long length to short length, and so on.

10.55 Even within categories of garments which are not unduly affected by seasonal influences or general changes in fashions, the garments that are available for pricing from one period to the next can vary greatly. Retailers change suppliers in order to seek the best prices or to maintain an image of a constantly changing range in order to attract shoppers. Many producers will also frequently change product lines in order to maintain buyer appeal. The practice of single producers using different and changing brands as a marketing tool is also common. Isolated countries that rely predominantly on imported clothing also face the additional problem of discontinuities in supply because of shipping failures or even the whim of importers.

10.56 The often short life cycles of specific items, and whole categories of items in the case of seasonal items, mean that retailers have to pay particular attention to inventory control, since they cannot afford to be left with large volumes of stock that they cannot sell. This is most commonly handled by progressively discounting or marking down prices throughout the estimated life cycle of an item.

10.57 The fragmented and changing nature of the clothing market invariably means that price index compilers have to strike a balance between the ideal requirements for index purposes and the cost of data collection (of both prices and characteristics that may be required to make quality adjustments).

Approaches to constructing indices for non-seasonal clothing

10.58 Even where seasonality is not a problem, the construction of a price index for clothing is not a simple task. The range of available items can differ significantly across outlets, making central determination and detailed specification of items to be priced ineffective. The brands and styles of particular garment types can also vary significantly over time in individual outlets, requiring close attention to procedures for replacing items and making quality adjustments.

10.59 Although it is virtually impossible to set out specific procedures that will be applicable in all countries, it is possible to develop a set of guidelines to help avoid the most significant pitfalls. In developing these guidelines, the key objective is to maximize the number of usable price quotations (for a given collection cost) in any month, and to minimize the incidence of measures of price change being affected by changes in quality.

10.60 In some circumstances, it may be possible to identify "national" specifications to be priced at each outlet (e.g. brand X, model Y jeans). The use of these types of specifications can help minimize the effort that needs to be put into quality adjustment, and movements in prices of these items can provide a useful benchmark against which to assess the movements of other items. Reliable identification of such items necessitates ongoing relationships with the buyers for large chains, or large domestic producers or importers. These sources need to be contacted on a regular basis to identify the current range of items, the extent of their availability across the country and any planned changes (including changes in style and quality as well as deletions from and additions to the range). This information may be used proactively to update specifications or descriptions of items to be priced in the field, so minimizing the incidence of price collectors attempting to price items that are no longer available. It can also be used to assist in the quantification of any quality changes.

10.61 For some items where availability by brand varies, it may be possible to identify a number of brands which are assessed as being of equal quality (e.g. different brands of T-shirts). In these cases, price collectors could be provided with the list of equivalent brands and instructed to price the cheapest one of these available at each outlet without having to ensure that the same brand is priced this time as on the last visit. The argument for this practice is that, if the brands are truly equivalent, discerning shoppers will purchase the cheapest at the time of purchase, and to reflect this in the CPI will result in an index that more closely follows the experience of households. Clearly, the success or otherwise of this technique depends vitally on the assessment of the "equality" of brands which, while largely a matter of judgement, may be assisted by an analysis of past price behaviour. In general, brand equality might be indicated by narrow longer-term price dispersion and a tendency for brands to swap prices over time or outlets.

10.62 In other cases it might be appropriate to restrict sampled items to a subset of brands without regarding the brands as equivalent. For example, a number of brands of jeans might together dominate the market but with the availability of the individual brands varying by outlet. In these cases, price collectors could be provided with a list of acceptable brands and instructed to price the most representative of these brands at each outlet. Once the initial selection has been made, price collectors should be instructed to record the specific brand and model priced at each outlet, and should continue to price that specification on subsequent visits until such time as it ceases to be stocked (or it becomes clear that it is no longer representative of the sales of that particular outlet).

10.63 The clothing market has become so diverse that it is not always possible to specify centrally either the item to be priced or even the brand (or brands). In these cases, it is necessary to give price collectors much greater discretion when it comes to selecting the individual items for pricing. To avoid the selection of inappropriate items, it is important for price collectors to be provided with guidelines to assist in this process. At the very least, they should be instructed to select the brand and model that the retailer advises is both representative and is expected to be stocked for some time (little advantage is to be gained from selecting an item which, while popular, has been purchased by the retailer on a one-off basis and is thus unlikely to be available for pricing in subsequent periods).

10.64 More sophisticated guidelines can incorporate a checklist of features that the selected item should match as closely as possible. These features should be ranked from most to least important, and it should be clear which features the selected item possesses and which it does not (either from the detailed description recorded by the price collector or through the completion of a separate feature pro-forma). In addition to brand (or acceptable brands), where possible, the list might include features such as:

- fabric type (e.g. cotton, wool, linen);
- weight of the fabric (e.g. heavy, medium, light);
- existence of a lining;
- number of buttons;
- type of stitching (e.g. single, double).

10.65 It is recognized that high fashion items pose particular difficulties in terms of quality adjustment. There is certainly clear potential for such items to bias the CPI towards the end of their life cycle when prices may be heavily discounted and sales volumes are low. For example, compilers need to guard against the danger that items leave the index at a heavily discounted price to be replaced by items that are on sale at the full price (which for a highly fashionable item may be at a premium). More generally, any decision on the inclusion of high fashion items ought certainly to reflect the intended reference population of the index, for example where this excludes households at the upper end of the income distribution.

Replacement of items and quality change

10.66 Even for garment types that are available all year round, there remains a strong need to replace items or to otherwise recognize changes in item characteristics. It is therefore important to ensure that procedures are established to minimize any bias resulting from changes in the quality of items priced.

10.67 The appropriate conceptual basis for assessing changes in the quality of garments is from the perspective of value to the consumer. In other words, a garment can be said to be of different quality to another garment if it is valued differently by the consumer. The difficulty confronted by index compilers is that quality differences are only observable in terms of changes in the physical characteristics of garments (including brand), some of which will have an impact on customer value and some of which will not. The problem is how to distinguish between them.

10.68 To assist in this task it is important to develop guidelines for selecting replacement items, with the general objective of minimizing the quality difference between the old and new items. For most items, research has shown that brand is an important price- and quality-determining characteristic (particularly for items that have a significant fashion element) and so, in the first instance, an effort should be made to select a replacement from the same brand (but noting the danger that as brands go out of fashion they become less representative). As this will not always be possible, it is useful to enlist experts in the trade to assist in drawing up a list that classifies brands into quality groups along the following lines:

– exclusive brands, usually international brands, mostly sold in exclusive stores;
– higher-quality brands, well-known brands at the national level (which may also include international brands);
– average quality brands;
– other or unknown brands.

10.69 If it is not possible to select a replacement from the same brand, the fallback should be to select a replacement from a brand in the same quality group. Similarity of price should never be the guiding objective when a substitute variety has to be chosen.

10.70 Once a replacement item has been selected, a detailed description of the new item needs to be recorded. The physical differences between the old and new items should be described in as much detail as possible to enable the index compiler to assess whether the replacement item is comparable (i.e. of equal quality) to the old item or not. As a general guide, changes such as single rows of stitching replacing double rows, of lighter-weight fabrics replacing heavier-weight ones, reductions in the number of buttons on shirts, reductions in the length of shirt tails, disappearance of linings and so on should be regarded as changes in quality. Changes in physical characteristics attributable solely to changes in fashion (e.g. straight leg to flared leg trousers) should not be regarded as quality changes.

10.71 Where an item is assessed as not being comparable, action will need to be taken to remove the impact of the quality change from the index. There are a number of approaches that may be taken to value the quality difference:

- Industry experts may be asked to place a cash value on the differences.
- The statistical office may arrange for some index compilers to receive additional training to become commodity experts able to estimate the value of such changes themselves.
- Hedonic methods may be employed if resources permit. Descriptions of hedonic techniques for clothing can be found in Liegey (1992) and Norberg (1999).

10.72 Each of these methods requires that the changes in the quality-determining characteristics (such as quality of material and standard of manufacture) are quantifiable. If such information is not available, implicit quality adjustment methods may have to be used. In this case, it is important that the price for the outgoing specification is returned to a normal price before it is removed from index calculation.

Approaches to including seasonal clothing in the consumer price index

10.73 The practices adopted by statistical agencies for handling seasonal clothing in CPIs vary widely, ranging from complete exclusion of such items to various methods of imputation of prices of items that are unavailable at a particular time of year, or to systems of weights that vary throughout the year. In some respects, the treatment of seasonal clothing raises similar issues to those found in dealing with fashion items, in particular reflecting the short life cycles of products and the likelihood of price-discounting during those cycles.

10.74 This section describes some practical alternatives for indices constructed using the traditional annual basket approach to produce a monthly CPI (i.e. systems of explicitly changing weights are not explored, nor is the use of year-on-year changes as proposed in Chapter 22). Further, the examples will be restricted to the so-called multiple basket approach because of the inherent difficulty of making quality adjustments between seasons in the so-called single basket approach. (The single basket approach takes the view that, say, summer and winter seasonal items are different varieties of the same article, whereas the multiple basket approach takes the view that they are completely different articles.)

10.75 CPI compilers may choose to exclude seasonal clothing from the CPI altogether. While this might simplify the job of compiling the index, it clearly reduces the representativeness of the basket. This might be considered as the option of last resort and will cause presentational difficulties from the point of view of external users, particularly where relative expenditure on seasonal clothing is high. Including seasonal items makes the basket more representative of consumption patterns but complicates the process of compiling the index. In reaching a decision, it will be necessary to strike a balance between representativeness and complexity (cost). Where seasonal items are excluded, their expenditure weight should be distributed among non-seasonal counterparts.

10.76 Six possible approaches to constructing aggregate clothing price indices in the presence of seasonal items are described below. A synthetic set of prices is used (see Table 10.3) to illustrate the various options. For simplicity, it is assumed that there are only three categories of clothing: those available all year (non-seasonal); and two seasonal categories (labelled summer and winter here). The two seasons are assumed to be

Table 10.3 Synthetic price data to illustrate approaches to constructing clothing price indices

Month	Year Y − 1			Year Y			Year Y + 1		
	Non-seasonal	Summer seasonal	Winter seasonal	Non-seasonal	Summer seasonal	Winter seasonal	Non-seasonal	Summer seasonal	Winter seasonal
1	100	100		113	110		127	125	
2	101	80		114	90		128	100	
3	102	60		115	70		130	80	
4	103			116			131		
5	104			117			132		
6	105			118			133		
7	106		100	120		110	135		125
8	107		80	121		90	136		100
9	108		60	122		70	137		80
10	109			123			139		
11	110			124			140		
12	112			126			142		

non-overlapping and the prices of the seasonal varieties are contrived to show progressive discounting over the course of each season. The prices of the non-seasonal items show a steady rate of growth. Within each category, prices are assumed to be for items of identical physical characteristics (or alternatively, to have been adjusted to remove the effects of changes in physical characteristics).

10.77 The price indices have been compiled with a base period of month 1 in year 0 and extend for 24 months (prices are provided for year Y−1 in order to impute base period prices for the winter seasonal item). For the purpose of weighting, it is assumed that each of the seasonal categories accounts for 25 per cent of expenditure, while non-seasonal items account for the remaining 50 per cent. For ease of computation, imputation is based on the simple arithmetic average of the price movements of the available series (including movements from imputed to real prices), though in practice these imputations would be based on weighted averages. Tables 10.4 to 10.6 present the calculated indices and monthly percentage changes for summer seasonal, winter seasonal and total clothing, respectively, based on the alternative methodologies described below.

10.78 *Exclude seasonal items*. This is the simplest option from an index construction point of view, but suffers from a lack of representativeness, which may be a cause of concern to some users. In this example, only 50 per cent of expenditures would be directly represented in the index. Clearly, the greater the relative expenditure on seasonal items, the more users are likely to be concerned about the lack of representativeness of the index. The results for this index are shown in column (1) of Table 10.6 and may be used as a benchmark against which the following options can be assessed.

10.79 *Impute only on items available all year*. This approach is one of the targeted imputation approaches. In this case, the out of season prices for both summer and winter items are imputed based only on the movement in the prices of those items available all year round. The results for the summer and winter items are shown in column (1) of Tables 10.4 and 10.5, respectively, while the total clothing index is shown in column (2) of Table 10.6.

10.80 *Impute on all available items*. This approach imputes all missing prices based on the movements in all available prices of related or similar items. This approach is similar in principle to the approach that would be taken in the case of a missing price observation. Prices for seasonal items are collected while they are observable, and when out of season are imputed based on items available all year round together with other seasonal items if available. The results are shown in column (2) in Tables 10.4 and 10.5, and in column (3) of Table 10.6.

10.81 *Carry forward of last observed price*. This simpler variant of the methods described above involves the carry forward of the last observed prices for seasonal items during the months when such prices are unavailable. This approach would not normally be recommended in the general case where prices are not available for non-seasonal items, on the grounds that the likely downward bias imparted could easily be avoided by observing the price of some similar item that is available. But where a whole class of goods is unavailable and hence unobservable, and particularly where price movements are not strongly correlated with other items, carry forward of prices may be seen as an acceptable approach. The results are shown in column (3) in Tables 10.4 and 10.5, and in column (4) of Table 10.6.

10.82 Under this approach, it is preferable to determine in advance during which months seasonal prices will be collected. This helps prevent distortion of the index through collection of possibly atypical prices for seasonal items unexpectedly available outside those periods when they would normally be available. Such decisions should be subject to regular review on the basis of market developments.

10.83 *Return to normal, then impute*. This approach requires the index compiler to estimate the “normal” price for the item during the first month when it is unavailable (out of season). This estimated normal price is then imputed forward until such time as the item becomes available again. Compared to the methods discussed so far, this approach is designed to avoid artificial depression of the aggregate index beyond the end of season, following progressive discounts over the item’s short life cycle.

Table 10.4 Alternative price indices for summer seasonal clothing

Month	Impute only on items available all year (1)	Impute on all available items (2)	Carry forward of last observed price (3)	Return to normal, then impute (4)	Include first seasonal observation, then impute (5)
			Index numbers		
1	100.0	100.0	100.0	100.0	100.0
2	81.8	81.8	81.8	81.8	100.9
3	63.6	63.6	63.6	63.6	101.8
4	64.2	64.2	63.6	100.0	102.7
5	64.7	64.7	63.6	100.9	103.5
6	65.3	65.3	63.6	101.7	104.4
7	66.4	77.0	63.6	102.9	105.4
8	67.0	70.3	63.6	94.0	106.3
9	67.5	62.8	63.6	83.9	107.1
10	68.1	63.3	63.6	108.3	108.0
11	68.6	63.8	63.6	109.2	108.9
12	69.7	64.9	63.6	110.9	110.7
13	113.6	113.6	113.6	113.6	113.6
14	90.9	90.9	90.9	90.9	114.5
15	72.7	72.7	72.7	72.7	116.3
16	73.3	73.3	72.7	113.6	117.2
17	73.8	73.8	72.7	114.5	118.1
18	74.4	74.4	72.7	115.4	119.0
19	75.5	93.3	72.7	117.4	120.8
20	76.1	84.3	72.7	106.1	121.7
21	76.6	76.2	72.7	95.8	122.6
22	77.8	77.3	72.7	123.5	124.4
23	78.3	77.9	72.7	124.4	125.3
24	79.4	79.0	72.7	126.2	127.1
			Monthly percentage changes		
2	−18.2	−18.2	−18.2	−18.2	0.9
3	−22.2	−22.2	−22.2	−22.2	0.9
4	0.9	0.9	0.0	57.2	0.9
5	0.8	0.8	0.0	0.9	0.8
6	0.9	0.9	0.0	0.8	0.9
7	1.7	17.9	0.0	1.2	1.0
8	0.9	−8.7	0.0	−8.6	0.9
9	0.7	−10.7	0.0	−10.7	0.8
10	0.9	0.8	0.0	29.1	0.8
11	0.7	0.8	0.0	0.8	0.8
12	1.6	1.7	0.0	1.6	1.7
13	63.0	75.0	78.6	2.4	2.6
14	−20.0	−20.0	−20.0	−20.0	0.8
15	−20.0	−20.0	−20.0	−20.0	1.6
16	0.8	0.8	0.0	56.3	0.8
17	0.7	0.7	0.0	0.8	0.8
18	0.8	0.8	0.0	0.8	0.8
19	1.5	25.4	0.0	1.7	1.5
20	0.8	−9.6	0.0	−9.6	0.7
21	0.7	−9.6	0.0	−9.7	0.7
22	1.6	1.4	0.0	28.9	1.5
23	0.6	0.8	0.0	0.7	0.7
24	1.4	1.4	0.0	1.4	1.4

Table 10.5 Alternative price indices for winter seasonal clothing

Month	Impute only on items available all year (1)	Impute on all available items (2)	Carry forward of last observed price (3)	Return to normal, then impute (4)	Include first seasonal observation, then impute (5)
			Index numbers		
1	100.0	100.0	100.0	100.0	100.0
2	100.9	91.4	100.0	91.4	100.9
3	101.8	81.6	100.0	81.6	101.8
4	102.7	82.3	100.0	105.3	102.7
5	103.5	83.0	100.0	106.2	103.5
6	104.4	83.7	100.0	107.1	104.4
7	175.2	112.4	183.3	107.8	104.6
8	143.4	91.9	150.0	88.2	105.4
9	111.5	71.5	116.7	68.6	106.3
10	112.4	72.1	116.7	107.8	107.2
11	113.3	72.7	116.7	108.7	108.1
12	115.2	73.9	116.7	110.4	109.8
13	116.1	101.9	116.7	112.2	111.7
14	117.0	92.1	116.7	101.5	112.6
15	118.8	83.6	116.7	92.1	114.4
16	119.7	84.3	116.7	118.4	115.2
17	120.6	84.9	116.7	119.3	116.1
18	121.6	85.6	116.7	120.2	117.0
19	199.1	127.7	208.3	122.5	118.8
20	159.3	102.2	166.7	98.0	119.7
21	127.4	81.7	133.3	78.4	120.6
22	129.3	82.9	133.3	122.5	122.4
23	130.2	83.5	133.3	123.4	123.2
24	132.1	84.7	133.3	125.2	125.0
			Monthly percentage changes		
2	0.9	−8.6	0.0	−8.6	0.9
3	0.9	−10.7	0.0	−10.7	0.9
4	0.9	0.9	0.0	29.0	0.9
5	0.8	0.9	0.0	0.9	0.8
6	0.9	0.8	0.0	0.8	0.9
7	67.8	34.3	83.3	0.7	0.2
8	−18.2	−18.2	−18.2	−18.2	0.8
9	−22.2	−22.2	−22.2	−22.2	0.9
10	0.8	0.8	0.0	57.1	0.8
11	0.8	0.8	0.0	0.8	0.8
12	1.7	1.7	0.0	1.6	1.6
13	0.8	37.9	0.0	1.6	1.7
14	0.8	−9.6	0.0	−9.5	0.8
15	1.5	−9.2	0.0	−9.3	1.6
16	0.8	0.8	0.0	28.6	0.7
17	0.8	0.7	0.0	0.8	0.8
18	0.8	0.8	0.0	0.8	0.8
19	63.7	49.2	78.6	1.9	1.5
20	−20.0	−20.0	−20.0	−20.0	0.8
21	−20.0	−20.1	−20.0	−20.0	0.8
22	1.5	1.5	0.0	56.3	1.5
23	0.7	0.7	0.0	0.7	0.7
24	1.5	1.4	0.0	1.5	1.5

10.84 There are some problems with this procedure. Particularly during periods of high inflation, it will be difficult to determine what the normal price is. More generally, it can be argued that the procedure reduces the objectivity of the index. In the illustrative examples presented here, the normal price to which the item is returned is the price observed at the start of the season. Compared with the previous three approaches, it can be seen that this has the effect of shifting the price increase from the commencement of the next season to immediately after the current season, i.e. the index records a sharp price change when none is observable. The results are shown in column (4) in Tables 10.4 and 10.5, and in column (5) of Table 10.6.

10.85 *Include only the first seasonal observation, then impute*. This approach requires that seasonal items be priced only once per season, when they first appear in the marketplace. This first observed price is then imputed forward until the item is priced again at the commencement of the next season. The rationale for this technique

Table 10.6 Alternative price indices for total clothing

Month	Only items available all year round (1)	Impute only on items available all year (2)	Impute on all available items (3)	Carry forward of last observed price (4)	Return to normal, then impute (5)	Include first seasonal observation, then imput (6)
			Index numbers			
1	100.0	100.0	100.0	100.0	100.0	100.0
2	100.9	96.1	93.8	95.9	93.8	100.9
3	101.8	92.3	87.2	91.8	87.2	101.8
4	102.7	93.1	88.0	92.2	102.7	102.7
5	103.5	93.8	88.7	92.7	103.5	103.5
6	104.4	94.6	89.5	93.1	104.4	104.4
7	106.2	113.5	100.5	114.8	105.8	105.6
8	107.1	106.2	94.1	106.9	99.1	106.5
9	108.0	98.8	87.6	99.1	92.1	107.4
10	108.8	99.5	88.3	99.5	108.4	108.2
11	109.7	100.3	89.0	99.9	109.3	109.1
12	111.5	102.0	90.5	100.8	111.1	110.9
13	112.4	113.6	110.1	113.8	112.7	112.5
14	113.3	108.6	102.4	108.5	104.8	113.4
15	115.0	105.4	96.6	104.9	98.7	115.2
16	115.9	106.2	97.4	105.3	116.0	116.1
17	116.8	107.0	98.1	105.8	116.9	117.0
18	117.7	107.9	98.9	106.2	117.8	117.9
19	119.5	128.4	115.0	130.0	119.7	119.7
20	120.4	119.1	106.8	120.0	111.2	120.6
21	121.2	111.6	100.1	112.1	104.2	121.4
22	123.0	113.3	101.6	113.0	123.0	123.2
23	123.9	114.1	102.3	113.5	123.9	124.1
24	125.7	115.7	103.8	114.3	125.7	125.9
			Monthly percentage changes			
2	0.9	−3.9	−6.2	−4.1	−6.2	0.9
3	0.9	−4.0	−7.0	−4.3	−7.0	0.9
4	0.9	0.9	0.9	0.5	17.8	0.9
5	0.8	0.8	0.8	0.5	0.8	0.8
6	0.9	0.9	0.9	0.5	0.9	0.9
7	1.7	20.0	12.3	23.3	1.3	1.1
8	0.8	−6.4	−6.4	−6.9	−6.3	0.9
9	0.8	−7.0	−6.9	−7.4	−7.1	0.8
10	0.7	0.7	0.8	0.4	17.7	0.7
11	0.8	0.8	0.8	0.4	0.8	0.8
12	1.6	1.7	1.7	0.9	1.6	1.6
13	0.8	11.4	21.7	12.8	1.4	1.4
14	0.8	−4.4	−7.0	−4.6	−7.0	0.8
15	1.5	−2.9	−5.7	−3.4	−5.8	1.6
16	0.8	0.8	0.8	0.4	17.5	0.8
17	0.8	0.8	0.7	0.4	0.8	0.8
18	0.8	0.8	0.8	0.4	0.8	0.8
19	1.5	19.0	16.3	22.4	1.6	1.5
20	0.8	−7.2	−7.1	−7.7	−7.1	0.8
21	0.7	−6.3	−6.3	−6.6	−6.3	0.7
22	1.5	1.5	1.5	0.8	18.0	1.5
23	0.7	0.7	0.7	0.4	0.7	0.7
24	1.5	1.4	1.5	0.8	1.5	1.5

is that it is a means of adjusting for the quality degradation of seasonal items associated with the commonly observed feature of falling prices throughout the season. Further, if it is desirable that the index behave as if it were constructed as a moving year index (see Chapter 22), then this approach provides a cost-effective alternative that also accommodates changing seasons (e.g. when the items that were in season last March do not appear until April this year).

10.86 On the downside, in fully discounting observable price movements through a seasonal item's life cycle, an implicit assumption is made that all such movements reflect quality changes with no change in underlying price. This is not likely to fully accord with user perceptions of price evolution and, unless similar techniques are employed for fashion items, it can be argued that the approach is inconsistent. The results are shown in column (5) in Tables 10.4 and 10.5, and column (6) in Table 10.6.

Summary comments

10.87 First, it is worth noting that the consequences of imputing price changes for baskets of seasonal items based on the price movements for other items of clothing is equivalent to allocating the weight for seasonal items to other items when they are out of season, so avoiding the complexity involved in systems of explicitly changing weights. In these circumstances, some care needs to be taken in the presentation of estimates of the contribution of both seasonal and non-seasonal items to the change in the aggregate CPI. The standard practice of determining an item's contribution to the total change in the CPI is to multiply the item's previous period (price-updated) weight by its percentage change. Only those seasonal items for which prices are actually measured in the current period will contribute to the change in the aggregate index. Similarly, though only non-seasonal items will contribute to the change in the aggregate index when seasonal items are out of season, the standard measure of their contribution will be understated. This is mainly an issue of presentation, although some compilers might prefer to present assessments of contributions only down to the level that includes both the seasonal and non-seasonal baskets.

10.88 There is likely to be a range of views across countries, and indeed users, concerning the appropriate treatment of seasonal items within a CPI. There is likely to be a particular diversity of views about whether the quality of seasonal items should be regarded as diminishing over the life of the season or not and, if so, whether a similar approach should (or can) be taken in respect of fashion items. The example data set was contrived so that each category displayed broadly constant growth in prices on a year-on-year basis. Those users primarily interested in measures that best capture persistent or underlying price pressures in the economy are likely to prefer those approaches which do not yield significant variations in the rate of price change that are solely attributable to how the statistical agency treats seasonal items. Such users may prefer that seasonal items be excluded altogether or that only the first seasonal observation be included with prices for other months being imputed.

10.89 What is clear is that national statistical offices need to carefully weigh up user requirements, theoretical issues, costs and the implications of alternative approaches before settling on the methodology to be adopted.

Telecommunication services

10.90 The global telecommunications sector has undergone rapid change in recent years. Technological innovation has resulted in a proliferation of new services while deregulation has led to sharp growth in the number of providers in many countries. Taken together, these factors have resulted in suppliers adopting a range of new strategies to differentiate their services in order to attract and retain customers.

10.91 Characteristics of particular significance to compilers of price indices are:

- fewer linear pricing schedules and the adoption of different pricing structures across providers;
- the increasing tendency to offer contracts that bundle services together in different ways to appeal to different types of consumers;
- rapid changes in the contracts offered to consumers as an effective means of encouraging the take-up of the ever-increasing range of services.

10.92 Increasingly, telecommunication companies offer services via plans that require customers to enter into longer-term contractual arrangements with the providers. This also poses problems for index compilation. Two broad types of plan are typically offered. The first has no fixed duration and makes allowance for the provider to change pricing structures with advance notice to the consumer. The second and increasingly more popular type provides a fixed term contract (generally of one to two years) with prices fixed for the duration of the contract. These plans are differentiated by charging different prices for different services. For example, a simple plan may be differentiated by charging more for monthly line rental but less for local calls, so appealing to users who make a higher volume of local calls. The emergence of new tailored plans designed to maximize customer demand overall is continuous.

10.93 If statistical agencies follow traditional sampling approaches and select price schedules according to some base period set of plans, and follow them until they expire, no price changes will be observed (likewise if plans expire and replacements are linked to show no change). The marketplace reality, by contrast, is that unit values for telecommunication services have been declining significantly in many countries.

10.94 All statistical agencies are struggling to develop methodologies capable of coping with the complexities of this sector. In particular, it is recognized that current best practice approaches have difficulty in accounting for substitutions across providers and in adequately accounting for changes in the quality of the services provided.

10.95 With the telecommunications sector under continual change, statistical practices need to be kept under constant review. Statistical agencies that are considering the construction of telecommunications indices for the first time, or considering reviewing their current practices, are advised to seek out the most recent research in this field. Nevertheless, this section seeks to provide a general description of four approaches that are currently used by national statistical agencies to measure changes in the prices of telecommunication services. The approaches, in increasing order of cost, are:

- representative items – matched samples;
- representative items – unit values;
- customer profiles;
- sample of bills.

10.96 Each approach is briefly described and potential deficiencies noted. There is no firm recommendation on the best approach as the choice will depend largely on the market conditions prevailing in individual countries, the sophistication of the index compilation system in use, and the extent of access to accurate and timely telecommunication services data. Depending on these factors, it may be appropriate to use different approaches for different telecommunication services, or even for the different services of specific providers.

Representative items – matched samples

10.97 This approach mirrors traditional techniques adopted elsewhere in the CPI. Total expenditure of reference group households on telecommunication services in the weighting base period is derived from sources such as household expenditure surveys. A sample of service providers is approached to obtain information on revenue by types of services (such as line rental, local calls, international calls, handset sales or rentals, connection fees, voicemail services, Internet charges and so on) and a number of these are selected as *representative items* with weights derived from the revenue data.

10.98 For each representative item, a sample of detailed specifications (such as a telephone call from location A to location B, at time X, of duration Y minutes) is drawn up sufficient to represent the range of specific services purchased by consumers within each representative item. This sample of specifications is held constant from period to period, and movements in the indices for representative items are computed, based on the movements in the prices of this *matched sample* of specifications. Table 10.7 illustrates the approach.

10.99 The list of representative items (the lowest level in the structure) generally does not need to cover all telecommunication services, but those selected should be sufficient to be representative of price behaviour as a whole, in particular taking account of published tariffs. Expenditures on those services not selected for pricing should be distributed over the other services within that general class for the purpose of deriving weights. For example, the expenditures on any fixed line services not selected for pricing should be distributed over those fixed line services selected.

10.100 Compared to suppliers of goods, service providers have an almost infinite capacity to tailor both the services and the prices they charge, for example based on the time at which the service is provided. A telephone call of five minutes' duration at 8 a.m. can be regarded as a different product to an equivalent call made at 8 p.m., and service providers are able to charge different prices for these calls. Representative items therefore need to be described in sufficient detail to capture all the price-determining characteristics.

10.101 Furthermore, given the ease with which providers can adjust the differential aspects of their pricing schedules (such as the time span designated as peak and the duration of a call before a different rate applies), it is necessary to use a sufficient number of varied specifications to capture these aspects reliably. It is not sufficient to simply describe a call as peak or off-peak, or from zone 1 to zone 2. Illustrative examples of the types of specifications that may be applicable for two representative items – international calls (fixed line) and usage fees (Internet services) – are provided in Table 10.8.

10.102 It is assumed that the origin of both the telephone calls and Internet access is also identified. All times are domestic. It should also be noted that the nature of Internet access generally precludes pricing on the basis of access, and hence the timing of access cannot be as tightly defined as for international telephone calls; instead, all specifications are for total monthly use.

10.103 The most costly aspect of this approach therefore is obtaining the data required to establish the representative items and to identify suitable specifications, as this will require detailed information from service providers. Once implemented, most price information should be readily available from published fee schedules, so minimizing the burden on respondents between reviews of the specifications.

10.104 The dynamic nature of the telecommunication sector and the common use of the pricing mechanism to change consumer behaviour are likely to require that the specifications be updated relatively frequently. When a specification disappears (i.e. a particular plan is no longer offered), all efforts must be made to find a suitable comparison specification. Where specifications are replaced, it is possible to argue that because different plans involve different conditions of sale they are fundamentally different products. It is equally reasonable

Table 10.7 An illustrative index structure for telecommunication services (representative item approach)

Fixed line services
Telephone connection costs
Telephone line rental
Local calls
Long-distance national calls
International calls
Mobile telephones
Connection costs
Handset purchase or rental
National calls
International calls
Payphones
Local calls
Internet services
Connection fees
Usage fees

Table 10.8 Examples of specifications of telecommunication services

Representative item	Examples of specifications
International calls (fixed line)	Plan A: Call to Athens at 8 a.m. on a Friday, duration 10 minutes
	Plan B: Call to London at 9 p.m. on a Saturday, duration 5 minutes
	Plan A: Call to New York at 11 a.m. on a Wednesday, duration 20 minutes
	Plan B: Call to Paris at 7 p.m. on a Sunday, duration 15 minutes
	Plan A: Call to Durban at 8 p.m. on a Monday, duration 30 minutes
Usage fees (Internet)	Plan A: 10 hours dial-up connect time between 4 p.m. and 7 p.m. weekends, total download 20 Mb
	Plan B: 20 hours dial-up connect time between 6 p.m. and midnight weekdays, total download 50 Mb
	Plan C: Permanent broadband connection, total download 100 Mb

to question whether all of the price difference between plans is due to quality differences, particularly in light of the evidence of ever-increasing volumes and reductions in unit values. The difficulty lies in quantifying the quality differences. Although hedonic techniques offer some prospects for resolving this dilemma, they are costly to implement.

Representative items – unit values

10.105 The unit value approach is similar to the previous approach, with the exception that specifications are not priced. The price for each representative item is calculated from revenue and quantity data collected from the service provider. For example, the price for national long-distance calls can be derived as the total revenue received from such calls divided by the number of call-minutes. Similarly, in the case of monthly line rental fees, the price can be calculated as the total revenue from line rental divided by the total number of subscribers.

10.106 Compared to the matched sample approach, the unit value approach attributes all of the difference between plans, and time and duration of calls to price (i.e. the quality difference is assumed to be zero). The unit value approach is also seen as providing a method for accounting for price change when the items are subject to a proliferation of discount schemes or promotions (e.g. $2 to call anywhere for as long as you like for the next week). While the approach avoids some of the customer sampling choices inherent in other methodologies, compilation does rest on analysis of aggregate company data and so is likely to be less timely than methodologies based on pre-published prices. Moreover, care needs to be exercised with this approach to ensure that the measure is not affected by undesirable compositional changes (see Chapter 9, where unit value indices are discussed in more detail). A unit value index should only be constructed for truly homogeneous items. This points to a requirement for defining the representative items at a relatively fine level of disaggregation. For example, international calls may need to be further subdivided by destination to avoid changes in unit values arising purely from shifts in the numbers of calls made to different destinations.

10.107 Although this approach appears to address at least some of the known deficiencies of the matched sample approach, it is likely to have a medium- to long-term downward bias and, unless implemented carefully, it is likely to exhibit period-to-period volatility because of compositional shifts, if only as a result of seasonal variations in usage patterns. There are also a number of respondent and data quality aspects that need to be considered. The unit value approach imposes a greater data burden on service providers, who often regard revenue and quantity data as highly commercially sensitive. To be effective, the service providers also need to be able to furnish data relating only to households (i.e. they have to be able to separate out revenue and quantities relating to businesses) and the revenue information needs to conform to the requirements of the index. For example, some service providers may record certain discounts as a marketing expense, rather than a reduction in revenue as is required for the unit value index.

Customer profiles

10.108 For marketing purposes, telecommunication companies often classify their customers according to their volume of service use. Although the number of categories can vary, a common approach is to use a three-way classification: low-volume, medium-volume and high-volume customers. Service providers analyse customer usage patterns by category when developing new plans targeted specifically at each group. National regulatory authorities may also be in a position to provide detailed customer use profiles on a confidential basis.

10.109 Statistical agencies can take a similar approach for the construction of price indices by devising profiles which reflect the average usage patterns for each category of consumer. Costs faced by these average consumers in each period can then be estimated by reference to the rates set out in that plan that is currently most commonly applicable to each customer category. Variations on this general theme include estimation of costs based on the plan that would deliver the cheapest overall cost to the consumer (based on the simplifying assumption of cost-minimizing consumer behaviour with perfect knowledge). This has the advantage of providing a clear basis for choosing a comparable replacement should an existing package cease to be available. Alternatively, costs to each customer group may be estimated with reference to several plans, where sales information indicates that this is a closer approximation to reality. The overall index is derived by weighting together the results from these user profiles according to information about the relative importance of each category of consumer.

10.110 In constructing the aggregate index, these calculations are likely to be made for a representative sample of service providers, exploiting information on their overall market share for sampling or weighting purposes if available. This opens up the possibility of fully exploiting all the possible relevant permutations of profiles and companies. Information on the distribution of customer profiles by service provider may, however, not be available or at least very costly to obtain. Table 10.9 gives an example of a profile for mobile telephone services, taken from Beuerlein (2001), which describes the current approach used in the German CPI.

10.111 Consistent with the fixed basket approach, the activity of consumers (in terms of numbers and types of calls) is held constant between comparison periods. Prices may, of course, change when not fixed by contract or when plans are replaced. Index compilers may also allow rates to change in response to a changing mix of plans within customer categories. This approach assumes that plan changes, as such, fundamentally represent price change rather than quality change, but it eliminates the cruder compositional effects associated with the unit value approach, which does not take account of customer profiles.

10.112 The success of this approach is determined by the degree to which the profiles truly reflect consumer

Table 10.9 Example of a user profile for mobile phone services

Specification	Unit	Rare callers	Low-volume callers	Average callers
Total length of calls	Minutes	16	42	96
Length of individual call				
Type A	Seconds	35	45	45
Type B	Seconds	65	95	115
Calls[1]	Number	20	36	72
Within the same network	Number	8	12	24
Beyond the network	Number	12	24	48

[1]The calls are distributed over times of the day and days of the week so that it is possible to take account of changes in the delimitation of between peak and off-peak, weekday and weekend tariffs.
Source: Beuerlein (2001).

behaviour and therefore a great deal of thought needs to be put into their development. The construction of the customer profiles will require a high degree of co-operation from service providers and, given the known volume changes, they will require updating at reasonably regular intervals, possibly more frequently than other items in the CPI basket. Data on plan usage by customer category for each index compilation period (month or quarter) may also be required if compilers decide to allow for such effects.

Sample of bills

10.113 This method can be seen as a more refined application of the customer profile approach. A fixed level of service activity from an actual sample of customers is priced each month rather than defining profiles representative of the average monthly activities of customers. A sample of customers should be selected from each category of customer (low-, medium- and high-volume customers) and, ideally, the bills (or activity statements) should cover a full year's activity.

10.114 The advantages of this approach compared to the customer profile approach are:

- It is able to take account of any within-year variations in customer behaviour (e.g. a higher incidence of international calls associated with religious or cultural events of significance).
- It better reflects the diversity of consumer behaviour by identifying actual activities (i.e. calls actually made by a sample of consumers).
- It accommodates within each bill any instances of annual charges.
- It allows for the detection and recording of other sources of price change associated with customers' overall relationship with the service provider (e.g. where overall discounts are provided when aggregate monthly spending exceeds certain values, or where an aggregate discount is provided if customers acquire bundles of services from a single provider, such as fixed line phone plus Internet).

10.115 Calculation of the index still requires monthly information on the relative significance of various plans by customer category (which can then be randomly allocated across the sampled bills). With the bill sample repriced each period, the resulting index measures the cost of a full year's consumption at the prices prevailing in each index period compared to the same cost at base prices. This assumes that the quality difference between old and new plans is zero for households' changing plans. Because of the generally larger number of bills (compared with the number of available profiles), price changes can be reflected more gradually, as the proportion of bills priced using each plan can better mirror the changing population distribution.

10.116 As with the profile approach, it is important that the sample of bills is updated regularly to reflect changes in consumption patterns and the take-up of new services such as call-waiting, voicemail and text messaging. Although, with adequate sampling, the bill approach is likely to provide a better measure of the aggregate rate of price change for telecommunication services as a whole, it may not be best suited to the calculation of separate indices for the components of those services (depending on whether overall or bottom-line discounts are offered). The approach is also data intensive, requiring a large number of calculations each period and thus a sophisticated data processing system.

Financial services

10.117 The construction of reliable, comprehensive price indices for financial services in CPIs is in its infancy. Given the increasing use of financial services by households, however, national statistical agencies are coming under pressure to account for at least some financial services in their CPIs. There is a particularly strong demand for CPIs to include those fees and charges faced by households in respect of deposit and loan accounts held with financial institutions.

10.118 The construction of price indices for financial services is inherently difficult, as there is no unanimous view about which financial services ought to be included in the CPI, or indeed about precisely how they should be measured. The discussion in this section attempts to present what might be regarded as the majority view based on what is practically feasible. Much of the material is based on Fixler and Zieshang (2001), Frost (2001) and Woolford (2001).

10.119 Common examples of financial services acquired by households include financial advice, currency exchange, services associated with deposit and loan facilities, services provided by fund managers, life insurance offices and superannuation funds, stockbroking services, and real estate agency services. The range of items explicitly regarded as financial services for inclusion in a CPI, and also the way in which they are measured, will depend on the principal purpose of the CPI and hence on whether an acquisitions, use or payments approach is employed.

10.120 Where a *payments* approach is used, the gross interest payable on mortgages is often included as a cost of owner-occupied housing (see paragraphs 10.4 to 10.50 above). In the interests of strict consistency, this might imply that the CPI should also include consumer credit charges (measured in a similar way to mortgage interest charges), as well as gross outlays on direct fees and charges paid in respect of other financial services. In practice, and as noted in the earlier section on housing costs, the treatment of housing sometimes differs in concept from other interest charges in national CPIs, partly reflecting mixed objectives for the overall index combined with public perceptions of the importance of this item within overall budgets. The specific requirements for a payments approach will not be discussed further here as the principles are either described elsewhere (e.g. under owner-occupied housing) or are relatively straightforward.

10.121 Assuming that households acquire all of their financial services from the private sector (i.e. they are not generally subsidized by governments or provided by nonprofit institutions serving households), the *acquisitions* and *use* approaches take an identical view of the measurement of financial services. In terms of coverage, however, some proponents of the use approach take a more restrictive view of which services should be included by limiting the scope to only those financial services which are acquired to directly facilitate current household consumption.

10.122 Under the more restrictive view of coverage, it is argued that the use of some financial services is inextricably linked with capital (or investment) activity. This suggests that such activities should be considered outside the scope of CPIs intended to provide measures of changes in consumption prices. Proponents of this view often draw upon national accounts practices as the starting point. For example, *SNA 1993* classifies expenses associated with the transfer of real estate (real estate agents' commissions, legal fees, and government taxes and charges) as part of gross fixed capital formation. It is important to note, however, that the CPI is not constrained to follow the practices adopted for national accounting. Rather, individual countries will need to make decisions on the item coverage of the CPI which best meets the domestic requirements of the price index itself.

10.123 One broad definition that could be adopted for the coverage of financial services within the CPI is: *all those services acquired by households in relation to the acquisition, holding and disposal of financial and real assets, including advisory services, except those acquired for business purposes*. This definition serves two purposes. First, it distinguishes between the services facilitating the transfer and holding of assets and the assets themselves. Second, it makes no distinction between whether the underlying asset is a real asset or a financial asset.

10.124 The degree of complexity involved in placing a value on financial services acquired by households and constructing the companion price indices varies markedly by service. Three specific examples reflecting current Australian research are used to illustrate the issues: currency exchange, stockbroking, and deposit and loan facilities. Real estate agency services are discussed separately in this chapter (see paragraphs 10.149 to 10.155) because they may be classified as either a housing expense or a financial service.

Currency exchange

10.125 For weighting purposes, the estimation of the base period expenditures incurred by households in exchanging domestic currency for currencies of other countries is, in principle, relatively straightforward and should be reportable in household expenditure surveys.

10.126 Construction of the companion price index is more complex. The service for which a price is required is that of facilitating the exchange of domestic currency for that of another country (the acquisition of an asset – foreign currency). The price for the service is usually specified in terms of some percentage of the domestic currency value of the transaction. These percentage margins may change only rarely, with service providers relying on the nominal value of the transactions increasing over time to deliver increases in fee receipts. The price required for index construction purposes is the monetary value of the margin (i.e. the amount determined by applying the percentage rate to the value of the currency transaction). To measure price change over time, the index compiler has to form a view about the quantity underpinning the original transaction.

10.127 The purchase of foreign currency can be seen as facilitating the purchase of some desired quantity of foreign goods and services (e.g. expenditure on foreign travel, or direct import of a commodity). The service price in comparison periods would be expressed as the amount payable on the conversion of a sum of domestic currency corresponding to that sum of foreign currency required to purchase the same quantum of foreign goods and services purchased in the base period.

10.128 A practical translation implies that the original foreign currency amount is indexed forward using changes in foreign prices, and then converted to domestic currency at the prevailing exchange rate, with the prevailing percentage margin applied to this new amount to deliver the current price. This current price would be compared to the base price to derive the measure of price change. Although the ideal measure for indexing forward the foreign currency amount would be an index specifically targeting those foreign goods and services purchased by resident households, this is unlikely to be feasible. A practical alternative is to use the published aggregate CPI for the foreign countries.

10.129 If a single margin (percentage rate) does not apply to all transactions (e.g. different rates apply to different size transactions), then the price measure should be constructed by reference to a representative sample of base period transactions. The value margin for each transaction in the current period in the domestic currency would be determined by the current domestic currency value of each transaction and the current period percentage margin applying to each. This captures any price change resulting from the value of an underlying transaction moving from one price band to another.

Stockbroking services

10.130 Consider the case of the purchase of a parcel of shares in a publicly listed company. In most countries, the purchase has to be arranged through a licensed broker (stockbroker). The total amount paid by the purchaser generally comprises three elements: an amount for the shares (the asset); a fee for the brokerage service; and some form of transaction tax (stamp duty).

10.131 The tax should be considered part of the cost of acquiring the shares, as opposed to being part of the price of the security. The tax should be included along with the brokerage cost in the CPI. This is consistent with both the intention of the tax and the more commonly accepted basis for the valuation of the shares. (It also proves convenient to adopt this principle here, as it allows for the – perhaps less contentious – comparable treatment of taxes on banking services.) Allowing for current tax schedules poses no difficulty in that they will be widely available in all countries.

10.132 Working from the premise that stockbrokers' fees are more likely to follow a step function than a linear function, a price measure would be constructed as follows. First, select a representative sample of transactions (domestic currency values) and calculate the tax payable and the fees payable by reference to the respective schedules. The taxes and fees payable in subsequent periods are calculated by first indexing forward the values of the sample transactions and then applying current fee and tax schedules to the revalued transactions. This methodology raises two main issues. First, what is the most appropriate index for revaluing the transactions and, second, how should the current schedule of fees be determined?

10.133 The quantum underlying share transactions can be regarded as forgone consumption, i.e. the quantity of goods and services that could have been purchased instead. The value of a constant quantum of consumption forgone in successive comparison periods therefore will vary with consumer prices. In this case, the obvious choice for an escalator would be the CPI itself, based on current period preliminary estimates, or the previous period's result. However, the use of a single period's movement in the CPI (either previous or current) has the potential to result in the prices of stockbroking services moving in a way that is unlikely to reflect reality. This would be particularly evident where, for example, the current or previous period's CPI was influenced significantly by some one-off, temporary or unusual price change (e.g. an oil price shock, or change to health care arrangements). Any "echoing" of abnormal shorter-term price changes through the precise treatment of stockbrokers' or similar fees is likely to stretch public credibility in the CPI. As an alternative, a 12-month moving average CPI might be employed, itself consistent with a base period comprising a full year's activity.

10.134 Alternatively, it might be argued that the quantum of shares could be revalued in subsequent periods in line with movements in equity prices themselves. According to this view, the price of equities may be seen as an important influence on the actual costs of storing forgone consumption in much the same way as tax and fee schedules specific to equity purchases are allowed to enter the calculations described above. The strong argument against this treatment is that it assumes that households have a desire to own equities per se, rather than using them simply as an appropriate vehicle to store forgone consumption. Moreover, the introduction of equity prices within the price indicator is likely to impart additional short-term volatility to the CPI.

10.135 Competition in the stockbroking industry means that there is unlikely to be a common fee schedule. If individual brokers adhere reasonably closely to an in-house fee schedule, obtaining copies of these schedules should be a relatively simple matter. On the other hand, if no such fee schedules exist, then a survey of stockbrokers may be required to collect information on a sample of trades (value of trade and fee charged), and this information used to derive a current period fee schedule.

10.136 In the case of sales of shares, the underlying transaction represents the exchange of one asset for another (shares for cash). Quantities underlying sales can be viewed similarly to share purchases (i.e. some current period basket of consumption goods and services). In reality, households review their investment strategies regularly in order to "store" their deferred consumption in whatever asset class they believe offers the greatest security or prospect for growth. A symmetrical treatment of the purchase and sale of shares is particularly appealing. Unless different fees or taxes apply to sales, there is no need to distinguish between the two in constructing the index.

Deposit and loan facilities

10.137 Accounting for the costs of services provided by financial intermediaries represents a significant step up in complexity. Even where a prior decision has been made to include such facilities within the scope of the CPI, the service being provided is difficult to visualize comprehensively, and the prices comprise significant elements that are not directly observable.

10.138 *SNA 1993* recommends (6.125 and Annex III) that the value of financial intermediation services output produced by an enterprise should be valued as the following sum:

- for financial assets involved in financial intermediation, such as loans, the value of services provided by the enterprise to the borrower per monetary unit on account is the margin between the rate payable by the borrower and a reference rate; plus
- for financial liabilities involved in financial intermediation, such as deposits, the value of services provided by the enterprise to the lender or depositor per monetary unit on account is the margin between the reference rate and the rate payable by the enterprise to the lender; plus
- the value of actual or explicit financial intermediation service charges levied.

10.139 For a summary of the developments in national accounts treatment in this area, and a discussion of the notion of a reference rate, see OECD (1998). In concept, *SNA 1993* describes the reference rate as the risk-free or pure interest rate. The value of the service provided to a borrower is the difference between the actual amount of interest paid by the borrower and the lower amount that would have been paid had the reference rate applied. The converse applies for depositors. In practice, it is very difficult to identify the reference rate, and in particular to avoid either volatility in or even negative measures of the value of such services (as would occur if the reference rate lay above the lending rate or below the deposit rate). As a matter of practical expediency, an average of borrowing and lending rates may be used (with the mid-point being favoured).[1] Given the complexities involved, expenditures on financial intermediation required for index weighting purposes cannot be collected from households in expenditure surveys and so must be estimated by collecting data from financial institutions.

10.140 In thinking about the construction of the index number, it is useful to start by considering the case of a traditional bank providing a single loan product and a single deposit product; the example will then be extended to a typical bank. In some countries, the traditional bank does not charge direct fees, but all income is derived through an interest margin on lending rates over deposit rates.

10.141 The base period weighting value of the financial service (and so household consumption of such services) therefore is estimated by applying a margin (the absolute difference between the reference rate and the rate of interest charged to borrowers or paid to depositors) to an aggregate balance (loan or deposit). In line with the suggested treatment of other financial transactions, the construction of accompanying price measures should allow for the indexation forward of base period balances, applying comparison period margins to calculate a money value. The price index is then calculated as the ratio of comparison period and base period money values.

10.142 Again, the issue of an appropriate escalator needs to be addressed. While the base period flows of deposits and withdrawals can readily be conceptualized as forgone consumption at base period prices, how should the balances (stocks) reflecting an accumulation of flows over a number of years be viewed? If an age profile for balances were available, accumulated consumption forgone could be computed as a moving average of the CPI. The more practical alternative is to view base period balances as representing some quantum of consumption goods and services at base period prices, in which case the 12-month moving average CPI can be used. This is consistent with the idea that households review temporal consumption or investment decisions (and so accumulated financial balances) on a regular basis, in this case annually.

10.143 The traditional bank has all but disappeared in some countries and most financial institutions now derive income from a combination of indirect fees (margins) and direct fees and charges, with the trend being for a move from margins towards direct fees. In this case, the challenge is to construct measures of price change that reflect the total price of the service and therefore capture any offsets between margins and direct fees. As with stockbroking services, there may also be taxes levied on financial transactions or balances and these should also be included in the "price". Frost (2001), for example, provides a description of the more practical aspects of constructing price indices for deposit and loan facilities based on recent Australian experience.

10.144 Given the clear scope for financial intermediaries to shift charges between the direct (fee) and indirect (margin) elements, there are clear dangers in constructing broad measures of margins – known by national accountants as financial intermediation services indirectly measured (FISIM) – independent of direct fees and taxes. Rather, the approach should be to construct price measures for specific (relatively homogeneous) products that can then be weighted together to provide a measure for deposit and loan facilities in aggregate, and taking account of both the direct and indirect elements in total price. This represents a similar strategy to that adopted throughout the CPI. For example, the index for motor vehicles is constructed by pricing a sample of individual vehicles and weighting these price measures to derive an aggregate, instead of, for example, attempting to directly construct an index for the supplier or producer of a range of vehicles.

10.145 The basic process is: first, to select a sample of representative products from each sampled institution; second, to select a sample of customers for each product, and third, to estimate the total base period value of the service associated with each product by element (margin, direct fees and taxes). These value aggregates can be viewed as being equivalent to prices for some quantum. Comparison period prices are derived by moving forward the base period value aggregates as follows:

- Margin – index forward the base period balance and apply the comparison period margin (the difference between the comparison period reference rate and the product yield). In practice, the "price" movement is given as the product of the indexation factor and the ratio of margins.
- Fees – index forward the transaction values for each sampled account (or profile) and apply the comparison period fee structure. The ratio of new aggregate fees to base fees is used to move the fee value aggregate. The aggregate fees in the base and comparison periods can be constructed as either arithmetic or geometric averages of the fees calculated for the individual customers.
- Taxes – as for fees, but use tax schedules instead of fee schedules.

10.146 Appendix 10.1 contains a worked example of the calculation of a price index for a single deposit product.

10.147 Since step function pricing and taxing schedules (for example, fees that are only payable after some number of transactions or if balances fall below some level) are prevalent in financial services, samples of detailed customer accounts with all the necessary charging variables identified will be required. These samples

[1]OECD (1998) expresses some concerns about the use of a mid-point reference rate as a measure of the risk-free rate of interest. There are, however, some doubts about whether the conceptual ideal is for some "risk-free" interest rate, or whether a more appropriate concept might be the interest rate that would have been struck in the absence of financial intermediaries (i.e. the rate that would have been struck by depositors dealing directly with borrowers). Such a rate would have incorporated the lenders' knowledge of risk. Taking the mid-point of the borrowing and lending rates would appear to be a good means of estimating this market-clearing rate.

should cover a full year's activity. If it is not possible to sample actual accounts, customer profiles may be developed as a fallback option.

10.148 To minimize problems associated with non-response and changing industry structures, a separate reference rate should be constructed for each sampled service provider. The reference rate should be calculated in respect of all loans and deposits (including those to businesses). Further, to avoid problems that may arise in the timing of accounting entries (e.g. revisions, or interest income on credit cards), monthly yields, reference rates and margins should be constructed by reference to three-month moving averages of the reported underlying balances and interest flows.

Real estate agency services

10.149 The services provided by real estate agencies in the acquisition and disposal of properties can be treated in a number of ways. If the CPI is constructed as an economic cost of *use* index, these services are out of scope as they form part of the input costs of the notional landlords (*SNA 1993* also assigns all transfer costs on dwellings to gross fixed capital formation). The transfer costs associated with the acquisition of a dwelling (legal fees, real estate agency fees and taxes) can be included in both a *payments* and an *acquisitions* CPI. They can be classified as either a cost of home ownership or as a distinctly separate financial service. Although all transfer costs should be included in such measures, the discussion below focuses on real estate agents' fees for simplicity. Price measures for the other elements are calculated using similar procedures. In all cases, the general approach is to estimate the current cost of the various services relative to, and as they would apply to, some fixed basket of activity in the base period. Consistent with some of the areas already discussed, this involves indexing forward the base period expenditures on which the fees are charged (to preserve the underlying quantum) via some appropriate price index, and then estimating the fees payable in the comparison period.

10.150 Real estate agents typically quote their fees as some percentage of the price received for the dwelling. In common with other items where charges are determined as a margin, this needs to be converted to a domestic currency price. If the percentage margin is known, the agents' price for any given transaction (sale/purchase of a dwelling for a known price) can be computed by multiplying the value of the dwelling by the percentage margin, and the index can be constructed on the basis of estimates of both components.

10.151 The methodology chosen for estimating the percentage margin will depend upon an assessment of the variation in margins across and within individual agencies. In the most straightforward case, firms may operate with a single percentage margin applicable to all transactions regardless of value. In other words, at any point in time the percentage margins charged may vary by agency, but not by value of transaction within agency. In this case, what is required is an estimate, in each comparison period, of the average percentage margin charged by agencies. This can be achieved by collecting the percentage margins, exclusive of any taxes levied on agents' fees such as value added tax (VAT) or goods and services tax (GST), from a sample of agencies and deriving an average.

10.152 Percentage margins charged by individual agencies sometimes vary with transaction price (typically declining with increasing prices of dwellings). Where tariffs do vary within agencies, a more sophisticated estimation procedure may be required. Using data from a sample of transactions from a sample of agents, the relationship between the value of transaction and the percentage margin can be derived through econometric analysis. Empirical analysis will be required to determine the precise functional form for this relationship. For example, in the Australian case research has shown that ordinary least squares regression can be used to estimate this relationship and that the following functional form is adequate:

$$R = a + b_1(1/p) + b_2(1/p)^2$$

where: R = the commission rate, p = the house price, a = a constant, and b_1 and b_2 are parameters to be estimated.

10.153 Estimation of the current period value of transactions to which the percentage margin applies depends on whether real estate agency fees are classified as a cost of housing or as a separate financial service. If the former, the value of the current period transaction, relative to the value of the base period transaction, would reflect changes in house prices. If the latter, where the purchase of a dwelling is regarded as forgone consumption, the current period value would reflect changes in the CPI itself.

10.154 If a single percentage margin is assumed to operate, then only a single current period transaction is required, i.e. an estimate of the average value of base period transactions at comparison period prices. For example, if real estate agency fees are classified as a housing cost, then the base period price is calculated by applying the average base period percentage margin to the average house price in the base period, with any VAT or GST then added. The comparison period price is calculated by indexing forward the average base period house price, applying the average comparison period percentage margin and adding GST or VAT.

10.155 If a single percentage margin is not assumed to operate, then a sample of representative base period transactions is required. The monetary value of the margin on each representative transaction is then calculated from published tariffs or from an estimated functional relationship, such as that described above. Comparison period prices are likewise estimated by first indexing forward each of the base period representative transactions and then applying the same model. Note that, in this case, there is no need to exclude any GST or VAT from the initial margins data.

Property insurance services

10.156 The construction of reliable price indices for insurance can be difficult to achieve in practice. This section is restricted to a discussion of property insurance, as this type of insurance can be assumed to operate in similar ways across countries. It nevertheless provides only an illustration of the issues that index compilers face, with each sector raising specific conceptual and measurement difficulties. For example, in the case of life insurance, insurance policies are often bundled with a long-term investment service yielding a financial payout when insured persons survive the policy term. Separation of the service charges relating to the insurance and investment elements within a single premium poses significant problems for index compilers.

10.157 For the purposes of the discussion below, property insurance is defined to include:

– dwelling insurance;
– household contents insurance;
– motor vehicle insurance.

10.158 The common feature of these policies is that for a fee (premium), households receive financial compensation if a nominated event results in the loss of, or damage to, designated property. The alternative to purchasing insurance is for the household to self-insure. For households as a group, the service received is represented by the elimination of the risk of a financial loss. The appropriate treatment of property insurance in the CPI depends on whether the CPI is constructed using the acquisitions, use or payments approach.

Payments

10.159 Under the *payments* approach, each of the above policy types is in scope. In thinking about how this property insurance should be included in the CPI, it is necessary to consider both the gross premiums payable and the claims receivable by households. The definitions of gross premiums payable and claims receivable are straightforward. It is possible, however, to treat claims receivable in a number of ways, which will have an impact on either the weight assigned to insurance or the weight assigned to the items insured. Spending on insurance can be weighted on either a gross basis (i.e. valued using gross premiums payable) or on a net basis (i.e. valued using gross premiums payable *less* claims receivable). Likewise, items which are insured against loss may also be weighted gross or net (in the latter case, excluding purchases explicitly financed by insurance claims receivable). Taken together, this suggests three plausible alternative treatments:

– gross premiums, net expenditures;
– net premiums, gross expenditures;
– gross premiums, gross expenditures.

10.160 *Gross premiums, net expenditures*. It may be argued that calculating expenditures net of purchases financed by insurance claims avoids double counting of that portion of gross premiums which funds the claims. There are some problems with this approach. First, it is necessary to assume that all proceeds from insurance claims are used to purchase replacement items or to repair damaged items. In some cases, claims receivable may be to compensate for damage or destruction to the property of agents beyond the scope of the index (e.g. businesses, government or even other households where the CPI reference group covers only some subset of households). Households may also choose to use the proceeds for entirely different purposes. Thus the estimation of the net expenditure weights is likely to involve some arbitrary choices. More generally, because money is fungible, attempts to restrict coverage only to those expenditures made from selected sources of funds are questionable. Finally, the potential distortion of weights for these items may reduce the usefulness of sub-indices for other purposes.

10.161 *Net premiums, gross expenditures*. Within a payments index, the "net premiums, gross expenditures" approach is based on the view that claims receivable should be regarded as negative expenditure on insurance. This may be seen as an attempt to avoid the double counting of expenditures on items financed by claims receivable and already included in gross expenditures on other items elsewhere in the index. The net premiums approach is much less problematic than the net expenditures approach (as at least the impact is restricted to the weights for insurance). It may, however, be argued that the net premiums approach is inconsistent with approaches adopted for other items in a payments index, in particular mortgage interest and consumer credit charges, where weights are based on gross payments. Any allowance for interest receipts would be likely to yield negative weights since households are generally net savers overall.

10.162 The fact that the net premiums approach effectively measures the value of the insurance service as required for indices constructed according to both the acquisitions and use approaches is incidental. The task here is to determine the appropriate treatment for a payments-based index.

10.163 *Gross premiums, gross expenditures*. The "gross premiums, gross expenditures" approach is based on the view that the claims receivable by households simply represent one of the sources of funds from which expenditures are made. This is the most appealing approach for a payments index, as it recognizes the fungible nature of money and provides a consistent means of identifying both the item coverage of the index and the relative weights by reference only to the actual outlays of households.

Use

10.164 Under the *use* approach, dwelling insurance is out of scope as an input cost of the notional landlord. The weights should relate to the value of the insurance service consumed by households. This is defined as being equal to: gross insurance premiums payable by households, *plus* premium supplements, *less* provisions for claims, *less* changes in actuarial reserves.

10.165 It is not possible to estimate the nominal value of the net insurance service from household expenditure surveys alone. For weighting purposes, the most appealing approach is to obtain data from a sample of insurance providers, permitting estimation of the ratio of net insurance services to gross premiums, and to apply this ratio to the estimated value of gross premiums obtained from household expenditure surveys. However, it has not been possible to devise a corresponding price measure that is conceptually sound. For this reason, those countries that have adopted the net concept for weighting purposes are using movements in gross insurance premiums as a proxy price measure.

Acquisitions

10.166 Under the *acquisitions* approach, all three items are in scope. Because the objective is to measure price inflation for the household sector, the expenditures required for weighting purposes should reflect the insurance companies' contribution to the inflationary process, which equates to the value of the insurance service as per the use approach.

Pricing gross insurance premiums

10.167 The gross insurance premium payable by households in any one period is determined by the conditions of the policy, the administration costs and profit objectives of the insurance provider, the risk of a claim being made and any relevant taxes. For any single policy, the principal quality-determining characteristics (generally specified in the conditions of the policy) can be summarized as being:

– the type of property being covered (dwellings, motor vehicles, etc.);
– the type of cover provided (physical damage, liability, etc.);
– the nature of the compensation (replacement cost, current market value, etc.);
– any limits on the amount claimable;
– the location of the property;
– amount of any excess payable by the insured;
– risks (or events) covered.

10.168 While it is clear that pricing to constant quality requires these conditions to be held fixed, there is also a question about whether the risk of a claim being made should be held constant. In other words, if the incidence of, say, vehicle theft increases, should this be regarded as a quality improvement or simply a price change? If, on the one hand, it is argued that as the consumers' decision to insure is based on their assessment of the likelihood of suffering a loss compared to the premium charged, the risk factors should be held constant. On the other hand, it may be argued that, once insured, the consumer simply expects to be compensated for any loss. From the perspective of the consumer, any increase in risk simply represents an increase in the insurer's cost base (which may or may not be passed on to the consumer by way of a price change). Obtaining data of sufficient reliability to make quality adjustments in response to changes in risk is problematic, so in practice most indices reflect changes in risk as a price change.

10.169 In pricing insurance policies, the approach should be to select a sample of policies representative of those policies held in the base period and to reprice these in subsequent periods. Taking dwelling insurance as an example, base period insurance policies would be taken out to insure dwellings of various values and types (e.g. timber or brick) in different locations. The price samples should therefore consist of specifications that aim to cover, in aggregate, as many combinations of these variables as is reasonable. While the conditions of the policy, the dwelling type and location should be held constant over time, the value of the dwelling should be updated each period to reflect changes in house prices (i.e. the underlying real quantity needs to be preserved). It is important to note that, as the premiums will be related in some way to the value of the insured property, the price index for insurance can change without there being any change in premium schedules.

10.170 Every effort should be made to identify any changes in the conditions applying to selected policies in order to facilitate appropriate quality adjustments. Examples would include cessation of coverage for specific conditions and changing the excess (or deductible) paid by the consumer when a claim is made. Estimates of the value of such changes may be based on the insurance company's own assessments of their likely impact on the value of total claims payable. If it is assumed that the change in the aggregate value of claims can be equated to the change in service to the consumer (compared to the service that would have been provided prior to policy renewal), then an appropriate adjustment can be made to the premium to provide a (quality-adjusted) movement in price. For example, consider the case where the excess on a policy is doubled and advice from the company is that this will result in a 3 per cent drop in the aggregate value of claims payable. This could be considered as equivalent to a 3 per cent increase in price.

Using gross premiums as a proxy for the net insurance service

10.171 The net insurance service charge captures the administration costs and profits of the insurance provider along with any taxes. The problem is that taxes on insurance are normally levied on the gross premiums. Therefore, if the gross insurance premiums are subject to a high rate of tax, then the taxes will account for an even higher proportion of the net insurance service charge. Simply using the gross insurance premium inclusive of taxes as the price measure understates the real effect of any increase in the tax rates. This is best illustrated by way of an example.

10.172 For the sake of simplicity, assume that there are no premium supplements and no actuarial reserves. Then the insurance service charge is given by gross premiums less provisions for claims. Suppose the only change between two periods is a change in the tax rate – from 5 per cent of gross premiums to 20 per cent. Then the values in Table 10.10 are likely to be observed. Under this scenario it is clear that the insurance service charge has increased from \$45 to \$60 (an increase of 33.3 per cent), yet gross premiums have only increased by 14.3 per cent.

Table 10.10 Illustration of the impact of taxes on measures of insurance services ($)

Period	Premiums before tax	Tax	Gross premiums	Claims	Insurance service
1	100	5	105	60	45
2	100	20	120	60	60

10.173 Given that changes in the tax rates on gross insurance premiums are often subject to significant variation, this is a non-trivial problem. A practical solution is to decompose insurance service into two components – insurance service before tax (or net of tax) and tax on insurance services. The price measure for the first is constructed by reference to movements in gross premiums net of tax, and the price measure for the second is given by changes in taxes on gross premiums. Further research is required to develop a workable methodology for directly measuring changes in prices of insurance services before tax.

Appendix 10.1 Calculation of a price index for a deposit product

(a) Base period sample account. Only a single month's data is used in this example. In practice, many accounts would be sampled with each account containing data for a full year.

Taxes

Date	Debit (D) or Credit (C)	Transaction	Transaction value ($)	Tax ($)	Balance ($)
					456.23
2 Jan	D	Over the counter withdrawal	107.05	0.70	348.48
12 Jan	C	Deposit	4 000.00	2.40	4 346.08
13 Jan	D	EFTPOS[1] transaction	50.62	0.30	4 295.16
13 Jan	D	Over the counter withdrawal	371.00	0.70	3 923.46
14 Jan	D	Own ATM[2] cash	300.00	0.70	3 622.76
14 Jan	D	Own ATM cash	100.00	0.70	3 522.06
16 Jan	D	Own ATM cash	100.00	0.70	3 421.36
16 Jan	D	Over the counter withdrawal	371.00	0.70	3 049.66
16 Jan	D	Cheque	90.00	0.30	2 959.36
19 Jan	D	Own ATM cash	100.00	0.70	2 858.66
19 Jan	D	Own ATM cash	100.00	0.70	2 757.96
19 Jan	C	Deposit	4 000.00	2.40	6 755.56
19 Jan	D	Cheque	740.00	1.50	6 014.06
20 Jan	D	EFTPOS transaction	76.42	0.30	5 937.34
21 Jan	D	Other ATM cash	20.00	0.30	5 917.04
21 Jan	D	Cheque	100.00	0.70	5 816.34
22 Jan	D	Cheque	43.40	0.30	5 772.64
22 Jan	D	Cheque	302.00	0.70	5 469.94
22 Jan	D	Cheque	37.00	0.30	5 432.64
23 Jan	D	Over the counter withdrawal	371.00	0.70	5 060.94
23 Jan	D	Cheque	72.00	0.30	4 988.64
27 Jan	D	Own ATM cash	150.00	0.70	4 837.94
27 Jan	D	Cheque	73.50	0.30	4 764.14
27 Jan	D	Cheque	260.00	0.70	4 503.44
27 Jan	D	EFTPOS transaction	51.45	0.30	4 451.69
28 Jan	D	Over the counter withdrawal	19.95	0.30	4 431.44
28 Jan	D	Cheque	150.00	0.70	4 280.74
29 Jan	D	Cheque	140.00	0.70	4 140.04
30 Jan	D	Over the counter withdrawal	371.00	0.70	3 768.34
30 Jan	D	Cheque	8.00	0.30	3 760.04
30 Jan	D	Cheque	60.00	0.30	3 699.74
Total taxes				**21.10**	

[1]EFTPOS (Electronic Funds Transfer Point Of Sale).
[2]ATM (Automatic Teller Machine).

Fees

Activity	Total no.	No. charged	Amount($)
Over the counter withdrawal	6	2	6.00
EFTPOS transaction	3	0	0.00
Own ATM cash	6	0	0.00
Own ATM cash	1	1	1.20
Cheque	13	3	3.00
Deposit	2	2	0.00
Total fees			**10.20**

Fees and taxes are calculated using data in tables (b) and (c), respectively.
Source: Woolford (2001)

(b) Fee schedule. This is a summary of the information typically available from financial institutions. For each period, the table includes the number of free transactions and the per transaction charge for additional transactions. A zero number free indicates that no transactions are free and a zero charge indicates that all transactions are free.

Description	Base period		Current period	
	No. free	Charge ($)	No. free	Charge ($)
Over the counter withdrawal	4	3.00	4	3.00
EFTPOS transaction	10	0.50	9	0.50
Own ATM cash	10	0.50	9	0.50
Other ATM cash	0	1.20	0	1.20
Cheque	10	1.00	9	1.00
Deposit	0	0.00	0	0.00

Source: Woolford (2001).

(c) Tax schedule. This is a table of tax rates of the type that used to be employed in Australia. The debits tax is levied on all debit transactions to eligible accounts, with the amount charged being set for ranges of transaction values (i.e. using a step function). Financial institutions duty is levied on all deposits, the amount being determined as a percentage of the value of the deposit.

Bank accounts debit tax

Transaction value ($)		Tax ($)	
Min.	Max.	Base period	Current period
0	1	0.00	0.00
1	100	0.30	0.30
100	500	0.70	0.70
500	5 000	1.50	1.50
5 000	10 000	3.00	3.00
10 000+		4.00	4.00

Financial institutions duty (%)

Base period	Current period
0.06	0.06

Source: Woolford (2001).

(d) Interest data. The table presents, in summary form, the balances and annualized interest flows derived by taking moving averages of data reported by financial institutions. Interest rates and margins are calculated from the balances and flows.

	Base period				Current period			
	Balance ($ million)	Interest ($ million)	Interest rate (%)	Margin (%)	Balance ($ million)	Interest ($ million)	Interest rate (%)	Margin (%)
Deposit products								
Personal accounts	22 000	740	3.3636	2.4937	23 600	775	3.2839	2.3971
Current accounts	6 000	68	1.1333	4.7241	6 600	75	1.1364	4.5446
Other accounts	16 000	672	4.2000	1.6574	17 000	700	4.1176	1.5634
Business accounts	25 000	920	3.6800	2.1774	28 000	1 000	3.5714	2.1096
Total deposit accounts	47 000	1 660	3.5319	2.3255	51 600	1 775	3.4399	2.2411
Loan products								
Personal accounts	42 000	3 188	7.5905	1.7331	46 000	3 400	7.3913	1.7103
Business accounts	28 000	2 540	9.0714	3.2140	31 000	2 700	8.7097	3.0287
Total loan accounts	70 000	5 728	8.1829	2.3255	77 000	6 100	7.9221	2.2411
Reference rate			**5.8574**				**5.6810**	

Source: Woolford (2001).

(e) CPI data. The table presents data required to derive the indexation factor. This example follows the Australian practice of a quarterly CPI. If a monthly CPI is produced, 12-term moving averages would be required.

	t–5	*t*–4	*t*–3	*t*–2	*t*–1
All groups	117.5	121.2	123.4	127.6	129.1
4-term moving average				122.4	125.3
Indexation factor (movement)					1.0237

Source: Woolford (2001).

(f) Projected current period sample account. The opening balance and transaction values are derived by applying the indexation factor to the base period amounts. The tax payable is determined by reference to the data in table (c). Fees payable are determined by reference to the data in table (b).

Taxes

Date	Debit (D) or Credit (C)	Transaction	Transaction value ($)	Tax ($)	Balance ($)
					467.04
2 Jan	D	Over the counter withdrawal	109.59	0.70	356.75
12 Jan	C	Deposit	4 094.75	2.46	4 449.05
13 Jan	D	EFTPOS transaction	51.82	0.30	4 396.93
13 Jan	D	Over the counter withdrawal	379.79	0.70	4 016.44
14 Jan	D	Own ATM cash	307.11	0.70	3 708.63
14 Jan	D	Own ATM cash	102.37	0.70	3 605.56
16 Jan	D	Own ATM cash	102.37	0.70	3 502.50
16 Jan	D	Over the counter withdrawal	379.79	0.70	3 122.01
16 Jan	D	Cheque	92.13	0.30	3 029.57
19 Jan	D	Own ATM cash	102.37	0.70	2 926.51
19 Jan	D	Own ATM cash	102.37	0.70	2 823.44
19 Jan	C	Deposit	4 094.75	2.46	6 915.73
19 Jan	D	Cheque	757.53	1.50	6 156.70
20 Jan	D	EFTPOS transaction	78.23	0.30	6 078.17
21 Jan	D	Other ATM cash	20.47	0.30	6 057.40
21 Jan	D	Cheque	102.37	0.70	5 954.33
22 Jan	D	Cheque	44.43	0.30	5 909.60
22 Jan	D	Cheque	309.15	0.70	5 599.75
22 Jan	D	Cheque	37.88	0.30	5 561.57
23 Jan	D	Over the counter withdrawal	379.79	0.70	5 181.08
23 Jan	D	Cheque	73.71	0.30	5 107.08
27 Jan	D	Own ATM cash	153.55	0.70	4 952.83
27 Jan	D	Cheque	75.24	0.30	4 877.28
27 Jan	D	Cheque	266.16	0.70	4 610.43
27 Jan	D	EFTPOS transaction	52.67	0.30	4 557.46
28 Jan	D	Over the counter withdrawal	20.42	0.30	4 536.73
28 Jan	D	Cheque	153.55	0.70	4 382.48
29 Jan	D	Cheque	143.32	0.70	4 238.46
30 Jan	D	Over the counter withdrawal	379.79	0.70	3 857.98
30 Jan	D	Cheque	8.19	0.30	3 849.49
30 Jan	D	Cheque	61.42	0.30	3 787.77
Total taxes				**21.21**	

Fees

Activity	Total No.	No. charged	Amount ($)
Over the counter withdrawal	6	2	6.00
EFTPOS transaction	3	0	0.00
Own ATM cash	6	0	0.00
Own ATM cash	1	1	1.20
Cheque	13	4	4.00
Deposit	2	2	0.00
Total fees			**11.20**

Source: Woolford (2001).

(g) Indices for current accounts. This table brings the results together. The current period value aggregates are derived as follows. For margins – the base period aggregate is multiplied by the product of the indexation factor (e) and the ratio of the current and base period margins for current accounts (d). For fees – the base period aggregate is multiplied by the ratio of total fees payable on the sample account in the current period (f) and the base period (a). For taxes – the same procedure is followed as for fees.

Component	Base period		Current period	
	Value aggregate ($)	Index	Value aggregate ($)	Index
Margins	28 344	100.0	27 913	98.5
Fees	11 904	100.0	13 071	109.8
Taxes	14 739	100.0	14 818	100.5
Total	54 987	100.0	55 803	101.5

Source: Woolford (2001).

15 BASIC INDEX NUMBER THEORY

Introduction

> The answer to the question what is the Mean of a given set of magnitudes cannot in general be found, unless there is given also the object for the sake of which a mean value is required. There are as many kinds of average as there are purposes; and we may almost say in the matter of prices as many purposes as writers. Hence much vain controversy between persons who are literally at cross purposes. (Edgeworth (1888, p. 347)).

15.1 The number of physically distinct goods and unique types of services that consumers can purchase is in the millions. On the business or production side of the economy, there are even more commodities that are actively traded. This is because firms not only produce commodities for final consumption, but they also produce exports and intermediate commodities that are demanded by other producers. Firms collectively also use millions of imported goods and services, thousands of different types of labour services and hundreds of thousands of specific types of capital. If we further distinguish physical commodities by their geographical location or by the season or time of day that they are produced or consumed, then there are billions of commodities that are traded within each year in any advanced economy. For many purposes, it is necessary to summarize this vast amount of price and quantity information into a much smaller set of numbers. The question that this chapter addresses is: how exactly should the microeconomic information involving possibly millions of prices and quantities be aggregated into a smaller number of price and quantity variables? This is the basic problem of index numbers.

15.2 It is possible to pose the index number problem in the context of microeconomic theory; i.e., given that we wish to implement some economic model based on producer or consumer theory, what is the "best" method for constructing a set of aggregates for the model? When constructing aggregate prices or quantities, however, other points of view (that do not rely on economics) are possible. Some of these alternative points of view are considered in this chapter and the next. Economic approaches are pursued in Chapters 17 and 18.

15.3 The index number problem can be framed as the problem of decomposing the value of a well-defined set of transactions in a period of time into an aggregate price term times an aggregate quantity term. It turns out that this approach to the index number problem does not lead to any useful solutions. So, in paragraphs 15.7 to 15.17, the problem of decomposing a value ratio pertaining to two periods of time into a component that measures the overall change in prices between the two periods (this is the price index) times a term that measures the overall change in quantities between the two periods (this is the quantity index) is considered. The simplest price index is a fixed basket type index; i.e., fixed amounts of the n quantities in the value aggregate are chosen and then the values of this fixed basket of quantities at the prices of period 0 and at the prices of period 1 are calculated. The fixed basket price index is simply the ratio of these two values where the prices vary but the quantities are held fixed. Two natural choices for the fixed basket are the quantities transacted in the base period, period 0, or the quantities transacted in the current period, period 1. These two choices lead to the Laspeyres (1871) and Paasche (1874) price indices, respectively.

15.4 Unfortunately, the Paasche and Laspeyres measures of aggregate price change can differ, sometimes substantially. Thus in paragraphs 15.18 to 15.32, taking an average of these two indices to come up with a single measure of price change is considered. In paragraphs 15.18 to 15.23, it is argued that the "best" average to take is the geometric mean, which is Irving Fisher's (1922) ideal price index. In paragraphs 15.24 to 15.32, instead of averaging the Paasche and Laspeyres measures of price change, taking an average of the two baskets is considered. This fixed basket approach to index number theory leads to a price index advocated by Correa Moylan Walsh (1901; 1921a). Other fixed basket approaches are, however, also possible. Instead of choosing the basket of period 0 or 1 (or an average of these two baskets), it is possible to choose a basket that pertains to an entirely different period, say period *b*. In fact, it is typical statistical agency practice to pick a basket that pertains to an entire year (or even two years) of transactions in a year prior to period 0, which is usually a month. Indices of this type, where the weight reference period differs from the price reference period, were originally proposed by Joseph Lowe (1823), and indices of this type are studied in paragraphs 15.24 to 15.53. Such indices are also evaluated from the axiomatic perspective in Chapter 16 and from the economic perspective in Chapter 17.[1]

15.5 In paragraphs 15.65 to 15.75, another approach to the determination of the *functional form* or the *formula* for the price index is considered. This approach is attributable to the French economist Divisia (1926) and is based on the assumption that price and quantity data are available as continuous functions of time. The theory of differentiation is used in order to decompose the rate of change of a continuous time value aggregate into two components that reflect aggregate price and quantity change. Although the approach of Divisia offers some insights,[2] it does not offer much guidance to statistical agencies in terms of leading to a definite choice of index number formula.

15.6 In paragraphs 15.76 to 15.97, the advantages and disadvantages of using a *fixed base* period in the bilateral index number comparison are considered versus always comparing the current period with the previous period, which is called the *chain system*. In the chain system, a *link* is an index number comparison of one period with the previous period. These links are multiplied together in order to make comparisons over many periods.

The decomposition of value aggregates into price and quantity components

The decomposition of value aggregates and the product test

15.7 A *price index* is a measure or function which summarizes the *change* in the prices of many commodities from one situation 0 (a time period or place) to another situation 1. More specifically, for most practical purposes, a price index can be regarded as a weighted mean of the change in the relative prices of the commodities under consideration in the two situations. To determine a price index, it is necessary to know:

- which commodities or items to include in the index;
- how to determine the item prices;
- which transactions that involve these items to include in the index;
- how to determine the weights and from which sources these weights should be drawn;

[1] Although indices of this type do not appear in Chapter 19, where most of the index number formulae exhibited in Chapters 15–18 are illustrated using an artificial data set, indices where the weight reference period differs from the price reference period are illustrated numerically in Chapter 22, in which the problem of seasonal commodities is discussed.

[2] In particular, it can be used to justify the chain system of index numbers (discussed in paragraphs 15.86 to 15.97).

- what formula or type of mean should be used to average the selected item relative prices.

All the above questions regarding the definition of a price index, except the last, can be answered by appealing to the definition of the *value aggregate* to which the price index refers. A value aggregate V for a given collection of items and transactions is computed as:

$$V = \sum_{i=1}^{n} p_i q_i \qquad (15.1)$$

where p_i represents the price of the ith item in national currency units, q_i represents the corresponding quantity transacted in the time period under consideration and the subscript i identifies the ith elementary item in the group of n items that make up the chosen value aggregate V. Included in this definition of a value aggregate is the specification of the group of included commodities (which items to include) and of the economic agents engaging in transactions involving those commodities (which transactions to include), as well as principles of the valuation and time of recording that motivate the behaviour of the economic agents undertaking the transactions (determination of prices). The included elementary items, their valuation (the p_i), the eligibility of the transactions and the item weights (the q_i) are all within the domain of definition of the value aggregate. The precise determination of the p_i and q_i is discussed in more detail elsewhere in this manual, in particular in Chapter 5.[3]

15.8 The value aggregate V defined by equation (15.1) refers to a certain set of transactions pertaining to a single (unspecified) time period. Now the same value aggregate for two places or time periods, periods 0 and 1, is considered. For the sake of convenience, period 0 is called the *base period* and period 1 is called the *current period* and it is assumed that observations on the base period price and quantity vectors, $p^0 \equiv [p_1^0,\ldots,p_n^0]$ and $q^0 \equiv [q_1^0,\ldots,q_n^0]$ respectively, have been collected.[4] The value aggregates in the base and current periods are defined in the obvious way as:

$$V^0 \equiv \sum_{i=1}^{n} p_i^0 q_i^0 ; \quad V^1 \equiv \sum_{i=1}^{n} p_i^1 q_i^1 \qquad (15.2)$$

In the previous paragraph, a price index was defined as a function or measure which summarizes the change in the prices of the n commodities in the value aggregate from situation 0 to situation 1. In this paragraph, a *price index* $P(p^0,p^1,q^0,q^1)$ along with the corresponding *quantity index* (or *volume index*) $Q(p^0,p^1,q^0,q^1)$ is defined to be two functions of the $4n$ variables p^0,p^1,q^0,q^1 (these variables describe the prices and quantities pertaining to the value aggregate for periods 0 and 1) where these two functions satisfy the following equation:[5]

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[3] Ralph Turvey has noted that some values may be difficult to decompose into unambiguous price and quantity components. Examples of difficult-to-decompose values are bank charges, gambling expenditures and life insurance payments.

[4] Note that it is assumed that there are no new or disappearing commodities in the value aggregates. Approaches to the “new goods problem” and the problem of accounting for quality change are discussed in Chapters 7, 8 and 21.

[5] The first person to suggest that the price and quantity indices should be jointly determined in order to satisfy equation (15.3) was Fisher (1911, p. 418). Frisch (1930, p. 399) called equation (15.3) the *product test*.

$$V^1/V^0 = P(p^0,p^1,q^0,q^1)\,Q(p^0,p^1,q^0,q^1) \qquad (15.3)$$

If there is only one item in the value aggregate, then the price index P should collapse down to the single price ratio, $p_1^{\,1}/p_1^{\,0}$, and the quantity index Q should collapse down to the single quantity ratio, $q_1^{\,1}/q_1^{\,0}$. In the case of many items, the price index P is to be interpreted as some sort of weighted average of the individual price ratios, $p_1^{\,1}/p_1^{\,0},\ldots, p_n^{\,1}/p_n^{\,0}$.

15.9 Thus the first approach to index number theory can be regarded as the problem of decomposing the change in a value aggregate, V^1/V^0, into the product of a part that is attributable to *price change*, $P(p^0,p^1,q^0,q^1)$, and a part that is attributable to *quantity change*, $Q(p^0,p^1,q^0,q^1)$. This approach to the determination of the price index is the approach that is taken in the national accounts, where a price index is used to deflate a value ratio in order to obtain an estimate of quantity change. Thus, in this approach to index number theory, the primary use for the price index is as a *deflator*. Note that once the functional form for the price index $P(p^0,p^1,q^0,q^1)$ is known, then the corresponding quantity or volume index $Q(p^0,p^1,q^0,q^1)$ is completely determined by P; i.e., rearranging equation (15.3):

$$Q(p^0,p^1,q^0,q^1) = \left(V^1/V^0\right)/P(p^0,p^1,q^0,q^1) \qquad (15.4)$$

Conversely, if the functional form for the quantity index $Q(p^0,p^1,q^0,q^1)$ is known, then the corresponding price index $P(p^0,p^1,q^0,q^1)$ is completely determined by Q. Thus using this deflation approach to index number theory, separate theories for the determination of the price and quantity indices are not required: if either P or Q is determined, then the other function is implicitly determined by the product test equation (15.4).

15.10 In the next section, two concrete choices for the price index $P(p^0,p^1,q^0,q^1)$ are considered and the corresponding quantity indices $Q(p^0,p^1,q^0,q^1)$ that result from using equation (15.4) are also calculated. These are the two choices used most frequently by national income accountants.

The Laspeyres and Paasche indices

15.11 One of the simplest approaches to the determination of the price index formula was described in great detail by Lowe (1823). His approach to measuring the price change between periods 0 and 1 was to specify an approximate *representative commodity basket*,[6] which is a quantity vector $q \equiv [q_1,\ldots,q_n]$ that is representative of purchases made during the two periods under consideration, and then calculate the level of prices in period 1 relative to period 0 as the ratio of the period 1 cost of the basket, $\sum_{i=1}^{n} p_i^1 q_i$, to the period 0 cost of the basket, $\sum_{i=1}^{n} p_i^0 q_i$. This *fixed basket approach* to the determination of the price index leaves open the question as to how exactly is the fixed basket vector q to be chosen.

15.12 As time passed, economists and price statisticians demanded a little more precision with respect to the specification of the basket vector q. There are two natural choices for the reference basket: the base period commodity vector q^0 or the current period commodity

[6] Lowe (1823, Appendix, p. 95) suggested that the commodity basket vector q should be updated every five years. Lowe indices are studied in more detail in paragraphs 15.24 to 15.53.

vector q^1. These two choices lead to the Laspeyres (1871) price index[7] P_L defined by equation (15.5) and the Paasche (1874) price index[8] P_P defined by equation (15.6):[9]

$$P_L(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0} \qquad (15.5)$$

$$P_P(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^1} \qquad (15.6)$$

15.13 The formulae (15.5) and (15.6) can be rewritten in an alternative manner that is more useful for statistical agencies. Define the period *t* expenditure share on commodity *i* as follows:

$$s_i^t \equiv p_i^t q_i^t \Big/ \sum_{j=1}^{n} p_j^t q_j^t \quad \text{for } i = 1,\ldots,n \quad \text{and} \quad t = 0,1 \qquad (15.7)$$

Then the Laspeyres index (15.5) can be rewritten as follows:[10]

$$\begin{aligned} P_L(p^0,p^1,q^0,q^1) &= \sum_{i=1}^{n} p_i^1 q_i^0 \Big/ \sum_{i=1}^{n} p_j^0 q_j^0 \\ &= \sum_{i=1}^{n} (p_i^1/p_i^0) p_i^0 q_i^0 \Big/ \sum_{j=1}^{n} p_j^0 q_j^0 \qquad (15.8) \\ &= \sum_{i=1}^{n} (p_i^1/p_i^0) s_i^0 \end{aligned}$$

using definitions (15.7). The Laspeyres price index P_L can thus be written as an arithmetic average of the *n* price ratios, p_i^1/p_i^0, weighted by base period expenditure shares. The Laspeyres formula (until very recently) has been widely used as the intellectual base for consumer price indices (CPIs) around the world. To implement it, a statistical agency needs only to collect information on expenditure shares s_n^0 for the index domain of definition for

[7] This index was actually introduced and justified by Drobisch (1871a, p. 147) slightly earlier than Laspeyres. Laspeyres (1871, p. 305) in fact explicitly acknowledged that Drobisch showed him the way forward. However, the contributions of Drobisch have been forgotten for the most part by later writers because Drobisch aggressively pushed for the ratio of two unit values as being the "best" index number formula. While this formula has some excellent properties where all the *n* commodities being compared have the same unit of measurement, it is useless when, say, both goods and services are in the index basket.

[8] Drobisch (1871b, p. 424) also appears to have been the first to define explicitly and justify the Paasche price index formula, but he rejected this formula in favour of his preferred formula, the ratio of unit values, and so again he did not gain any credit for his early suggestion of the Paasche formula.

[9] Note that $P_L(p^0,p^1,q^0,q^1)$ does not actually depend on q^1 and $P_P(p^0,p^1,q^0,q^1)$ does not actually depend on q^0. It does no harm to include these vectors, however, and the notation indicates that the reader is in the realm of bilateral index number theory; i.e., the prices and quantities for a value aggregate pertaining to two periods are being compared.

[10] This method of rewriting the Laspeyres index (or any fixed basket index) as a share weighted arithmetic average of price ratios is attributable to Fisher (1897, p. 517) (1911, p. 397) (1922, p. 51) and Walsh (1901, p. 506; 1921a, p. 92).

the base period 0, and then collect information on item *prices* alone on an ongoing basis. *Thus the Laspeyres CPI can be produced on a timely basis without having quantity information for the current period.*

15.14 The Paasche index can also be written in expenditure share and price ratio form as follows:[11]

$$
\begin{aligned}
P_P(p^0,p^1,q^0,q^1) &= \frac{1}{\left\{ \sum_{i=1}^{n} p_i^0 q_i^1 \Big/ \sum_{j=1}^{n} p_j^1 q_j^1 \right\}} \\
&= \frac{1}{\left\{ \sum_{i=1}^{n} \left(p_i^0/p_i^1\right) p_i^1 q_i^1 \Big/ \sum_{j=1}^{n} p_j^1 q_j^1 \right\}} \\
&= \frac{1}{\left\{ \sum_{i=1}^{n} \left(p_i^1/p_i^0\right)^{-1} s_i^1 \right\}} \\
&= \left\{ \sum_{i=1}^{n} \left(p_i^1/p_i^0\right)^{-1} s_i^1 \right\}^{-1}
\end{aligned}
\qquad (15.9)
$$

using definitions (15.7). The Paasche price index P_P can thus be written as a *harmonic* average of the *n* item price ratios, p_i^1/p_i^0, weighted by period 1 (current period) expenditure shares.[12] The lack of information on current period quantities prevents statistical agencies from producing Paasche indices on a timely basis.

15.15 The quantity index that corresponds to the Laspeyres price index using the product test in equation (15.3) is the Paasche quantity index; i.e., if P in equation (15.4) is replaced by P_L defined by equation (15.5), then the following quantity index is obtained:

$$
Q_P(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^1 q_i^0} \qquad (15.10)
$$

Note that Q_P is the value of the period 1 quantity vector valued at the period 1 prices, $\sum_{i=1}^{n} p_i^1 q_i^1$, divided by the (hypothetical) value of the period 0 quantity vector valued at the period 1 prices, $\sum_{i=1}^{n} p_i^1 q_i^0$. Thus the period 0 and 1 quantity vectors are valued at the same set of prices, the current period prices, p^1.

[11] This method of rewriting the Paasche index (or any fixed basket index) as a share weighted harmonic average of the price ratios is attributable to Walsh (1901, p. 511; 1921a, p. 93) and Fisher (1911, p. 397-398).

[12] Note that the derivation in the formula (15.9) shows how harmonic averages arise in index number theory in a very natural way.

15.16 The quantity index that corresponds to the Paasche price index using the product test (15.3) is the Laspeyres quantity index; i.e., if P in equation (15.4) is replaced by P_P defined by equation (15.6), then the following quantity index is obtained:

$$Q_L(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} p_i^0 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0} \tag{15.11}$$

Note that Q_L is the (hypothetical) value of the period 1 quantity vector valued at the period 0 prices, $\sum_{i=1}^{n} p_i^0 q_i^1$, divided by the value of the period 0 quantity vector valued at the period 0 prices, $\sum_{i=1}^{n} p_i^0 q_i^0$. Thus the period 0 and 1 quantity vectors are valued at the same set of prices, the base period prices, p^0.

15.17 The problem with the Laspeyres and Paasche index number formulae is that, although they are equally plausible, in general they will give different answers. For most purposes, it is not satisfactory for the statistical agency to provide two answers to the question[13]: What is the "best" overall summary measure of price change for the value aggregate over the two periods in question? In the following section, we consider how "best" averages of these two estimates of price change can be constructed. Before doing so, we ask: What is the "normal" relationship between the Paasche and Laspeyres indices? Under "normal" economic conditions when the price ratios pertaining to the two situations under consideration are negatively correlated with the corresponding quantity ratios, it can be shown that the Laspeyres price index will be larger than the corresponding Paasche index.[14] A precise statement of this result is presented in Appendix 15.1.[15] The divergence between P_L and P_P suggests that if a *single estimate* for the price change between the two periods is required, then some sort of evenly weighted average of the Laspeyres and Paasche indices should be taken as the final estimate of price change between periods 0 and 1. As mentioned above, this

[13] In principle, instead of averaging the Paasche and Laspeyres indices, the statistical agency could think of providing both (the Paasche index on a delayed basis). This suggestion would lead to a matrix of price comparisons between every pair of periods instead of a time series of comparisons. Walsh (1901, p. 425) noted this possibility: "In fact, if we use such direct comparisons at all, we ought to use all possible ones."

[14] Peter Hill (1993, p. 383) summarized this inequality as follows:

> It can be shown that relationship (13) [i.e., that P_L is greater than P_P] holds whenever the price and quantity relatives (weighted by values) are negatively correlated. Such negative correlation is to be expected for price takers who react to changes in relative prices by substituting goods and services that have become relatively less expensive for those that have become relatively more expensive. In the vast majority of situations covered by index numbers, the price and quantity relatives turn out to be negatively correlated so that Laspeyres indices tend systematically to record greater increases than Paasche with the gap between them tending to widen with time.

[15] There is another way to see why P_P will often be less than P_L. If the period 0 expenditure shares s_i^0 are exactly equal to the corresponding period 1 expenditure shares s_i^1, then by Schlömilch's (1858) Inequality (see Hardy, Littlewood and Polyá (1934, p. 26)), it can be shown that a weighted harmonic mean of n numbers is equal to or less than the corresponding arithmetic mean of the n numbers and the inequality is strict if the n numbers are not all equal. If expenditure shares are approximately constant across periods, then it follows that P_P will usually be less than P_L under these conditions (see paragraphs 15.70 to 15.84).

strategy will be pursued in the following section. It should, however, be kept in mind that statistical agencies will not usually have information on current expenditure weights, hence averages of Paasche and Laspeyres indices can be produced only on a delayed basis (perhaps using national accounts information) or not at all.

Symmetric averages of fixed basket price indices

The Fisher index as an average of the Paasche and Laspeyres indices

15.18 As mentioned above, since the Paasche and Laspeyres price indices are equally plausible but often give different estimates of the amount of aggregate price change between periods 0 and 1, it is useful to consider taking an evenly weighted average of these fixed basket price indices as a single estimator of price change between the two periods. Examples of such *symmetric averages*[16] are the arithmetic mean, which leads to the Drobisch (1871b, p. 425), Sidgwick (1883, p. 68) and Bowley (1901, p. 227)[17] index, $P_D \equiv (1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the Fisher (1922)[18] ideal index, P_F, defined as

$$P_F(p^0,p^1,q^0,q^1) \equiv \{P_L(p^0,p^1,q^0,q^1)\,P_P(p^0,p^1,q^0,q^1)\}^{1/2} \qquad (15.12)$$

At this point, the fixed basket approach to index number theory is transformed into the *test approach* to index number theory; i.e., in order to determine which of these fixed basket indices or which averages of them might be "best", desirable *criteria* or *tests* or *properties* are needed for the price index. This topic will be pursued in more detail in the next chapter, but an introduction to the test approach is provided in the present section because a test is used to determine which average of the Paasche and Laspeyres indices might be "best".

15.19 What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical cost of living index? It is very desirable for a price index formula that depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test.*[19] An index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

$$P(p^1,p^0,q^1,q^0) = 1/\,P(p^0,p^1,q^0,q^1) \qquad (15.13)$$

i.e., if the period 0 and period 1 price and quantity data are interchanged, and then the index number formula is evaluated, then this new index $P(p^1,p^0,q^1,q^0)$ is equal to the reciprocal of

[16] For a discussion of the properties of symmetric averages, see Diewert (1993c). Formally, an average m(a,b) of two numbers a and b is symmetric if m(a,b) = m(b,a). In other words, the numbers a and b are treated in the same manner in the average. An example of a nonsymmetric average of a and b is (1/4)a + (3/4)b. In general, Walsh (1901, p. 105) argued for a symmetric treatment if the two periods (or countries) under consideration were to be given equal importance.

[17] Walsh (1901, p. 99) also suggested the arithmetic mean index P_D (see Diewert (1993a, p. 36) for additional references to the early history of index number theory).

[18] Bowley (1899, p.641) appears to have been the first to suggest the use of the geometric mean index P_F. Walsh (1901, p. 428-429) also suggested this index while commenting on the big differences between the Laspeyres and Paasche indices in one of his numerical examples: "The figures in columns (2) [Laspeyres] and (3) [Paasche] are, singly, extravagant and absurd. But there is order in their extravagance; for the nearness of their means to the more truthful results shows that they straddle the true course, the one varying on the one side about as the other does on the other."

[19] See Diewert (1992a, p. 218) for early references to this test. If we want the price index to have the same property as a single price ratio, then it is important to satisfy the time reversal test. However, other points of view are possible. For example, we may want to use our price index for compensation purposes, in which case satisfaction of the time reversal test may not be so important.

the original index $P(p^0,p^1,q^0,q^1)$. This is a property that is satisfied by a single price ratio, and it seems desirable that the measure of aggregate price change should also satisfy this property so that it does not matter which period is chosen as the base period. Put another way, the index number comparison between any two points of time should not depend on the choice of which period we regard as the base period: if the other period is chosen as the base period, then the new index number should simply equal the reciprocal of the original index. It should be noted that the Laspeyres and Paasche price indices do not satisfy this time reversal property.

15.20 Having defined what it means for a price index P to satisfy the time reversal test, then it is possible to establish the following result.[20] The Fisher ideal price index defined by equation (15.12) is the *only* index that is a homogeneous[21] symmetric average of the Laspeyres and Paasche price indices, P_L and P_P, and satisfies the time reversal test (15.13). The Fisher ideal price index thus emerges as perhaps the "best" evenly weighted average of the Paasche and Laspeyres price indices.

15.21 It is interesting to note that this *symmetric basket approach* to index number theory dates back to one of the early pioneers of index number theory, Arthur L. Bowley, as the following quotations indicate:

> If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean … as a first approximation (Bowley (1901, p. 227)).
>
> When estimating the factor necessary for the correction of a change found in money wages to obtain the change in real wages, statisticians have not been content to follow Method II only [to calculate a Laspeyres price index], but have worked the problem backwards [to calculate a Paasche price index] as well as forwards. … They have then taken the arithmetic, geometric or harmonic mean of the two numbers so found (Bowley (1919, p. 348)).[22]

15.22 The quantity index that corresponds to the Fisher price index using the product test (15.3) is the Fisher quantity index; i.e., if P in equation (15.4) is replaced by P_F defined by equation (15.12), the following quantity index is obtained:

$$Q_F(p^0,p^1,q^0,q^1) \equiv \{Q_L(p^0,p^1,q^0,q^1)Q_P(p^0,p^1,q^0,q^1)\}^{1/2} \qquad (15.14)$$

Thus the Fisher quantity index is equal to the square root of the product of the Laspeyres and Paasche quantity indices. It should also be noted that $Q_F(p^0,p^1,q^0,q^1) = P_F(q^0,q^1,p^0,p^1)$; i.e., if the role of prices and quantities is interchanged in the Fisher price index formula, then the Fisher quantity index is obtained. [23]

[20] See Diewert (1997, p. 138))

[21] An average or mean of two numbers *a* and *b*, *m(a,b),* is *homogeneous* if when both numbers *a* and *b* are multiplied by a positive number λ, then the mean is also multiplied by λ; i.e., m satisfies the following property: $m(\lambda a,\lambda b) = \lambda m(a,b)$.

[22] Fisher (1911, p. 417-418; 1922) also considered the arithmetic, geometric and harmonic averages of the Paasche and Laspeyres indices.

[23] Fisher (1922, p. 72) said that P and Q satisfied the *factor reversal test* if $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ and P and Q satisfied the product test (15.3) as well.

15.23 Rather than take a symmetric average of the two basic fixed basket price indices pertaining to two situations, P_L and P_P, it is also possible to return to Lowe's basic formulation and choose the basket vector q to be a symmetric average of the base and current period basket vectors, q^0 and q^1. This approach to index number theory is pursued in the following section.

The Walsh index and the theory of the "pure" price index

15.24 Price statisticians tend to be very comfortable with a concept of the price index that is based on pricing out a constant "representative" basket of commodities, $q \equiv (q_1,q_2,\ldots,q_n)$, at the prices of periods 0 and 1, $p^0 \equiv (p_1{}^0,p_2{}^0,\ldots,p_n{}^0)$ and $p^1 \equiv (p_1{}^1,p_2{}^1,\ldots,p_n{}^1)$ respectively. Price statisticians refer to this type of index as a *fixed basket index* or a *pure price index*[24] and it corresponds to Sir George H. Knibbs's (1924, p. 43) *unequivocal price index*.[25] Since Lowe (1823) was the first person to describe systematically this type of index, it is referred to as a Lowe index. Thus the general functional form for the *Lowe price index* is

$$P_{Lo}(p^0,p^1,q) \equiv \sum_{i=1}^{n} p_i^1 q_i \Big/ \sum_{i=1}^{n} p_i^0 q_i = \sum_{i=1}^{n} s_i (p_i^1/p_i^0) \qquad (15.15)$$

where the (hypothetical) *hybrid expenditure shares* s_i[26] corresponding to the quantity weights vector q are defined by:

$$s_i \equiv p_i^0 q_i \Big/ \sum_{j=1}^{n} p_j^0 q_j \quad \text{for } i = 1,2,\ldots,n \qquad (15.16)$$

15.25 The main reason why price statisticians might prefer a member of the family of Lowe or fixed basket price indices defined by equation (15.15) is that the fixed basket concept is easy to explain to the public. Note that the Laspeyres and Paasche indices are special cases of the pure price concept if we choose $q = q^0$ (which leads to the Laspeyres index) or if we choose $q = q^1$ (which leads to the Paasche index).[27] The practical problem of picking q remains to be resolved, and that is the problem that will be addressed in this section.

---

[24] See section 7 in Diewert (2001).

[25] "Suppose however that, for each commodity, Q′ = Q, then the fraction, ∑(P′Q) / ∑(PQ), viz., the ratio of aggregate value for the second unit-period to the aggregate value for the first unit-period is no longer merely a ratio of totals, it also shows unequivocally the effect of the change in price. Thus it is an unequivocal price index for the quantitatively unchanged complex of commodities, A, B, C, etc.

It is obvious that if the quantities were different on the two occasions, and if at the same time the prices had been unchanged, the preceding formula would become ∑(PQ′) / ∑(PQ). It would still be the ratio of the aggregate value for the second unit-period to the aggregate value for the first unit period. But it would be also more than this. It would show in a generalized way the ratio of the quantities on the two occasions. Thus it is an unequivocal quantity index for the complex of commodities, unchanged as to price and differing only as to quantity.

Let it be noted that the mere algebraic form of these expressions shows at once the logic of the problem of finding these two indices is identical" (Knibbs (1924, p. 43–44)).

[26] Note that Fisher (1922, p. 53) used the terminology "weighted by a hybrid value", while Walsh (1932, p. 657) used the term "hybrid weights".

[27] Note that the *i*th share defined by equation (15.16) in this case is the hybrid share $s_i \equiv p_i^0 q_i^1 / \sum_{i=1}^{n} p_i^0 q_i^1$, which uses the prices of period 0 and the quantities of period 1.

15.26 It should be noted that Walsh (1901, p. 105; 1921a) also saw the price index number problem in the above framework:

> Commodities are to be weighted according to their importance, or their full values. But the problem of axiometry always involves at least two periods. There is a first period, and there is a second period which is compared with it. Price variations have taken place between the two, and these are to be averaged to get the amount of their variation as a whole. But the weights of the commodities at the second period are apt to be different from their weights at the first period. Which weights, then, are the right ones—those of the first period? Or those of the second? Or should there be a combination of the two sets? There is no reason for preferring either the first or the second. Then the combination of both would seem to be the proper answer. And this combination itself involves an averaging of the weights of the two periods (Walsh (1921a, p. 90)).

Walsh's suggestion will be followed and thus the *i*th quantity weight, q_i, is restricted to be an average or *mean* of the base period quantity q_i^0 and the current period quantity for commodity i q_i^1, say $m(q_i^0,q_i^1)$, for $i = 1,2,\ldots, n$.[28] Under this assumption, the Lowe price index (15.15) becomes:

$$P_{Lo}(p^0,p^1,q^0,q^1) = \frac{\sum_{i=1}^{n} p_i^1 m(q_i^0,q_i^1)}{\sum_{j=1}^{n} p_j^0 m(q_j^0,q_j^1)}. \qquad (15.17)$$

15.27 In order to determine the functional form for the mean function *m*, it is necessary to impose some *tests* or *axioms* on the pure price index defined by equation (15.17). As above, we ask that P_{Lo} satisfy the *time reversal test* (15.13). Under this hypothesis, it is immediately obvious that the mean function m must be a *symmetric mean*[29]; i.e., *m* must satisfy the following property: $m(a,b) = m(b,a)$ for all $a > 0$ and $b > 0$. This assumption still does not pin down the functional form for the pure price index defined by equation (15.17). For example, the function $m(a,b)$ could be the *arithmetic mean*, $(1/2)a + (1/2)b$, in which case equation (15.17) reduces to the *Marshall* (1887) and *Edgeworth* (1925) *price index* P_{ME}, which was the pure price index preferred by Knibbs (1924, p. 56):

$$P_{ME}(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 \left\{\left(q_i^0+q_i^1\right)/2\right\}}{\sum_{j=1}^{n} p_j^0 \left\{\left(q_j^0+q_j^1\right)/2\right\}} \qquad (15.18)$$

15.28 On the other hand, the function $m(a,b)$ could be the *geometric mean*, $(ab)^{1/2}$, in which case equation (15.17) reduces to the *Walsh* (1901, p. 398; 1921a, p. 97) *price index*, P_W:[30]

[28] Note that we have chosen the mean function $m(q_i^0,q_i^1)$ to be the same for each item i. We assume that $m(a,b)$ has the following two properties: $m(a,b)$ is a positive and continuous function, defined for all positive numbers a and b and $m(a,a) = a$ for all $a > 0$.

[29] For more on symmetric means, see Diewert (1993c, p. 361).

[30] Walsh (1921a, p. 103) endorsed P_W as being the best index number formula: "We have seen reason to believe formula 6 better than formula 7. Perhaps formula 9 is the best of the rest, but between it and Nos. 6 and 8 it would be difficult to decide with assurance". His formula 6 is P_W defined by equation (15.19) and his 9 is the Fisher ideal defined by equation (15.12). The *Walsh quantity index*, $Q_W(p^0,p^1,q^0,q^1)$ is defined as $P_W(q^0,q^1,p^0,p^1)$; i.e., the role of prices and quantities in definition (15.19) is interchanged. If the Walsh quantity index is used to deflate the value ratio, an implicit price index is obtained, which is Walsh's formula 8.

$$P_W(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 \sqrt{q_i^0 q_i^1}}{\sum_{j=1}^{n} p_j^0 \sqrt{q_j^0 q_j^1}} \qquad (15.19)$$

15.29 There are many other possibilities for the mean function m, including the mean of order r, $[(1/2)a^r + (1/2)b^r]^{1/r}$ for $r \neq 0$. Obviously, in order to completely determine the functional form for the pure price index P_{Lo}, it is necessary to impose at least one additional test or axiom on $P_{Lo}(p^0,p^1,q^0,q^1)$.

15.30 There is a potential problem with the use of the Edgeworth-Marshall price index (15.18) that has been noticed in the context of using the formula to make international comparisons of prices. If the price levels of a very large country are compared to the price levels of a small country using formula (15.18), then the quantity vector of the large country may totally overwhelm the influence of the quantity vector corresponding to the small country.[31] In technical terms, the Edgeworth-Marshall formula is not homogeneous of degree 0 in the components of both q^0 and q^1. To prevent this problem from occurring in the use of the pure price index $P_K(p^0,p^1,q^0,q^1)$ defined by equation (15.17), it is asked that P_{Lo} satisfy the following *invariance to proportional changes in current quantities test:*[32]

$$P_{Lo}(p^0,p^1,q^0,\lambda q^1) = P_{Lo}(p^0,p^1,q^0,q^1) \text{ for all } p^0,p^1,q^0,q^1 \text{ and all } \lambda > 0 \qquad (15.20)$$

The two tests, the time reversal test (15.13) and the invariance test (15.20), make it possible to determine the precise functional form for the pure price index P_{Lo} defined by formula (15.17): the pure price index P_K must be the Walsh index P_W defined by formula (15.19).[33]

15.31 In order to be of practical use by statistical agencies, an index number formula must be able to be expressed as a function of the base period expenditure shares, s_i^0, the current period expenditure shares, s_i^1, and the n price ratios, p_i^1/p_i^0. The Walsh price index defined by the formula (15.19) can be rewritten in the following format:

[31] This is not likely to be a severe problem in the time series context, however, where the change in quantity vectors going from one period to the next is small.

[32] This is the terminology used by Diewert (1992a, p. 216); Vogt (1980) was the first to propose this test.

[33] See section 7 in Diewert (2001).

$$P_W(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1\sqrt{q_i^0 q_i^1}}{\sum_{j=1}^{n} p_j^0\sqrt{q_j^0 q_j^1}}$$

$$= \frac{\sum_{i=1}^{n}\left(p_i^1 \Big/ \sqrt{p_i^0 p_i^1}\right)\sqrt{s_i^0 s_i^1}}{\sum_{j=1}^{n}\left(p_j^0 \Big/ \sqrt{p_j^0 p_j^1}\right)\sqrt{s_j^0 s_j^1}} \tag{15.21}$$

$$= \frac{\sum_{i=1}^{n}\sqrt{s_i^0 s_i^1}\sqrt{p_i^1/p_i^0}}{\sum_{j=1}^{n}\sqrt{s_j^0 s_j^1}\sqrt{p_j^0/p_j^1}}$$

15.32 The approach taken to index number theory in this section was to consider averages of various fixed basket type price indices. The first approach was to take an even-handed average of the two primary fixed basket indices: the Laspeyres and Paasche price indices. These two primary indices are based on pricing out the baskets that pertain to the two periods (or locations) under consideration. Taking an average of them led to the Fisher ideal price index P_F defined by equation (15.12). The second approach was to average the basket quantity weights and then price out this average basket at the prices pertaining to the two situations under consideration. This approach led to the Walsh price index, $P_{W,}$ defined by equation (15.19). Both of these indices can be written as a function of the base period expenditure shares, s_i^0, the current period expenditure shares, s_i^1, and the *n* price ratios, p_i^1/p_i^0. Assuming that the statistical agency has information on these three sets of variables, which index should be used? Experience with normal time series data has shown that these two indices will not differ substantially and thus it is a matter of indifference which of these indices is used in practice.[34] Both of these indices are examples of *superlative indices*, which are defined in Chapter 17. Note, however, that both of these indices treat the data pertaining to the two situations in a *symmetric* manner. Hill[35] commented on superlative price indices and the importance of a symmetric treatment of the data as follows:

> Thus economic theory suggests that, in general, a symmetric index that assigns equal weight to the two situations being compared is to be preferred to either the Laspeyres or Paasche indices on their own. The precise choice of superlative index—whether Fisher, Törnqvist or other superlative index—may be of only secondary importance as all the symmetric indices are likely to approximate each other, and the underlying theoretic index fairly closely, at least when the index number spread between the Laspeyres and Paasche is not very great (Hill (1993, p. 384)).

[34] Diewert (1978, pp. 887-889) showed that these two indices will approximate each other to the second order around an equal price and quantity point. Thus for normal time series data where prices and quantities do not change much going from the base period to the current period, the indices will approximate each other quite closely.

[35] See also Hill (1988).

Annual weights and monthly price indices

The Lowe index with monthly prices and annual base year quantities

15.33 It is now necessary to discuss a major practical problem with the above theory of basket type indices. Up to now, it has been assumed that the quantity vector $q \equiv (q_1,q_2,\ldots,q_n)$ that appeared in the definition of the Lowe index, $P_{Lo}(p^0,p^1,q)$ defined by equation (15.15), is either the base period quantity vector q^0 or the current period quantity vector q^1 or an average of these two quantity vectors. In fact, in terms of actual statistical agency practice, the quantity vector q is usually taken to be an annual quantity vector that refers to a *base year*, say b, that is prior to the base period for the prices, period 0. Typically, a statistical agency will produce a consumer price index at a monthly or quarterly frequency, but for the sake of argument a monthly frequency will be assumed in what follows. Thus a typical price index will have the form $P_{Lo}(p^0,p^t,q^b)$, where p^0 is the price vector pertaining to the base period month for prices, month 0, p^t is the price vector pertaining to the current period month for prices, say month t, and q^b is a reference basket quantity vector that refers to the base year b, which is equal to or prior to month 0.[36] Note that this Lowe index $P_{Lo}(p^0,p^t,q^b)$ is *not* a true Laspeyres index (because the annual quantity vector q^b is not equal to the monthly quantity vector q^0 in general).[37]

15.34 The question is: why do statistical agencies *not* pick the reference quantity vector q in the Lowe formula to be the monthly quantity vector q^0 that pertains to transactions in month 0 (so that the index would reduce to an ordinary Laspeyres price index)? There are two main reasons why this is not done:

- Most economies are subject to seasonal fluctuations, and so picking the quantity vector of month 0 as the reference quantity vector for all months of the year would not be representative of transactions made throughout the year.
- Monthly household quantity or expenditure weights are usually collected by the statistical agency using a household expenditure survey with a relatively small sample. Hence the resulting weights are usually subject to very large sampling errors and so standard practice is to average these monthly expenditure or quantity weights over an entire year (or in some cases, over several years), in an attempt to reduce these sampling errors.

The index number problems that are caused by seasonal monthly weights are studied in more detail in Chapter 22. For now, it can be argued that the use of annual weights in a monthly index number formula is simply a method for dealing with the seasonality problem.[38]

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[36] Month 0 is called the price reference period and year b is called the weight reference period.

[37] Triplett (1981, p. 12) defined the Lowe index, calling it a Laspeyres index, and calling the index that has the weight reference period equal to the price reference period, a pure Laspeyres index. Balk (1980c, p. 69), however, asserted that although the Lowe index is of the fixed base type; it is not a Laspeyres price index. Triplett also noted the hybrid share representation for the Lowe index defined by equations (15.15) and (15.16). Triplett noted that the ratio of two Lowe indices using the same quantity weights was also a Lowe index. Baldwin (1990, p. 255) called the Lowe index an *annual basket index*.

[38] In fact, the use of the Lowe index $P_{Lo}(p^0,p^t,q^b)$ in the context of seasonal commodities corresponds to Bean and Stine's (1924, p. 31) Type A index number formula. Bean and Stine made three additional suggestions for price indices in the context of seasonal commodities. Their contributions are evaluated in Chapter 22.

15.35 One problem with using annual weights corresponding to a perhaps distant year in the context of a monthly consumer price index must be noted at this point: if there are systematic (but divergent) trends in commodity prices and households increase their purchases of commodities that decline (relatively) in price and reduce their purchases of commodities that increase (relatively) in price, then the use of distant quantity weights will tend to lead to an upward bias in this Lowe index compared to one that used more current weights, as will be shown below. This observation suggests that statistical agencies should strive to get up-to-date weights on an ongoing basis.

15.36 It is useful to explain how the annual quantity vector q^b could be obtained from monthly expenditures on each commodity during the chosen base year b. Let the month m expenditure of the reference population in the base year b for commodity i be $v_i^{b,m}$ and let the corresponding price and quantity be $p_i^{b,m}$ and $q_i^{b,m}$ respectively. Of course, value, price and quantity for each commodity are related by the following equations:

$$v_i^{b,m} = p_i^{b,m} q_i^{b,m} \qquad \text{where} \quad i = 1,\ldots,n \quad \text{and} \quad m = 1,\ldots,12 \qquad (15.22)$$

For each commodity i, the annual total, q_i^b can be obtained by price deflating monthly values and summing over months in the base year b as follows:

$$q_i^b = \sum_{m=1}^{12} \frac{v_i^{b,m}}{p_i^{b,m}} = \sum_{m=1}^{12} q_i^{b,m}; \qquad i = 1,\ldots,n \qquad (15.23)$$

where equation (15.22) was used to derive the second equation in (15.23). In practice, the above equations will be evaluated using aggregate expenditures over closely related commodities and the price $p_i^{b,m}$ will be the month m price index for this elementary commodity group i in year b relative to the first month of year b.

15.37 For some purposes, it is also useful to have annual prices by commodity to match up with the annual quantities defined by equation (15.23). Following national income accounting conventions, a reasonable[39] price p_i^b to match up with the annual quantity $q_i^{\mathbf{b}}$ is the value of total consumption of commodity i in year b divided by q_i^b.
Thus we have:

$$\begin{aligned} p_i^b &\equiv \sum_{m=1}^{12} v_i^{b,m} / q_i^b && i = 1,\ldots,n \\ &= \frac{\sum_{m=1}^{12} v_i^{b,m}}{\sum_{m=1}^{12} v_i^{b,m} / p_i^{b,m}} && \text{using (15.23)} \qquad (15.24) \\ &= \left[\sum_{m=1}^{12} s_i^{b,m} (p_i^{b,m})^{-1} \right]^{-1} \end{aligned}$$

where the share of annual expenditure on commodity i in month m of the base year is

[39] These annual commodity prices are essentially unit value prices. Under conditions of high inflation, the annual prices defined by equation (15.24) may no longer be "reasonable" or representative of prices during the entire base year because the expenditures in the final months of the high inflation year will be somewhat artificially blown up by general inflation. Under these conditions, the annual prices and annual commodity expenditure shares should be interpreted with caution. For more on dealing with situations where there is high inflation within a year, see Hill (1996).

$$s_i^{b,m} \equiv \frac{v_i^{b,m}}{\sum_{k=1}^{12} v_i^{b,k}}; \qquad i = 1,...,n \qquad (15.25)$$

Thus the annual base year price for commodity i, p_i^b, turns out to be a monthly expenditure weighted *harmonic mean* of the monthly prices for commodity i in the base year, $p_i^{b,1}$, $p_i^{b,2}$,…, $p_i^{b,12}$.

Using the annual commodity prices for the base year defined by equation (15.24), a vector of these prices can be defined as $p^b \equiv [p_1^b,\ldots,p_n^b]$. Using this definition, the Lowe index $P_{Lo}(p^0,p^t,q^b)$ can be expressed as a ratio of two Laspeyres indices, where the price vector p^b plays the role of base period prices in each of the two Laspeyres indices:

$$\begin{aligned} P_{Lo}(p^0,p^t,q^b) &= \frac{\sum_{i=1}^n p_i^t q_i^b}{\sum_{i=1}^n p_i^0 q_i^b} = \frac{\sum_{i=1}^n p_i^t q_i^b / \sum_{i=1}^n p_i^b q_i^b}{\sum_{i=1}^n p_i^0 q_i^b / \sum_{i=1}^n p_i^b q_i^b} \\ &= \frac{\sum_{i=1}^n s_i^b (p_i^t / p_i^b)}{\sum_{i=1}^n s_i^b (p_i^0 / p_i^b)} \qquad (15.26) \\ &= P_L(p^b,p^t,q^b)/P_L(p^b,p^0,q^b) \end{aligned}$$

where the Laspeyres formula P_L was defined by equation (15.5). Thus the above equation shows that the Lowe monthly price index comparing the prices of month 0 to those of month t using the quantities of base year b as weights, $P_{Lo}(p^0,p^t,q^b)$, is equal to the Laspeyres index that compares the prices of month t to those of year b, $P_L(p^b,p^t,q^b)$, divided by the Laspeyres index that compares the prices of month 0 to those of year b, $P_L(p^b,p^0,q^b)$. Note that the Laspeyres index in the numerator can be calculated if the base year commodity expenditure shares, s_i^b, are known along with the price ratios that compare the prices of commodity i in month t, p_i^t, with the corresponding annual average prices in the base year b, p_i^b. The Laspeyres index in the denominator can be calculated if the base year commodity expenditure shares, s_i^b, are known along with the price ratios that compare the prices of commodity i in month 0, p_i^0, with the corresponding annual average prices in the base year b, p_i^b.

15.39 There is another convenient formula for evaluating the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, and that is to use the hybrid weights formula (15.15). In the present context, the formula becomes:

$$P_{Lo}(p^0,p^t,q^b) \equiv \frac{\sum_{i=1}^n p_i^t q_i^b}{\sum_{i=1}^n p_i^0 q_i^b} = \frac{\sum_{i=1}^n \left(p_i^t / p_i^0\right) p_i^0 q_i^b}{\sum_{i=1}^n p_i^0 q_i^b} = \sum_{i=1}^n \left(\frac{p_i^t}{p_i^0}\right) s_i^{0b} \qquad (15.27)$$

where the hybrid weights s_i^{0b} using the prices of month 0 and the quantities of year b are defined by

$$s_i^{0b} \equiv \frac{p_i^0 q_i^b}{\sum_{j=1}^{n} p_j^0 q_j^b}; \qquad i = 1,\ldots,n$$
$$= \frac{p_i^b q_i^b (p_i^0 / p_i^b)}{\sum_{j=1}^{n} \left[p_j^b q_j^b (p_j^0 / p_j^b)\right]}. \tag{15.28}$$

The second equation in (15.28) shows how the base year expenditures, $p_i^b q_i^b$, can be multiplied by the commodity price indices, p_i^0/p_i^b, in order to calculate the hybrid shares.

15.40 There is one additional formula for the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, that will be exhibited. Note that the Laspeyres decomposition of the Lowe index defined by the third term in equation (15.26) involves the long-term price relatives, p_i^t/p_i^b, which compare the prices in month t, p_i^t, with the possibly distant base year prices, p_i^b, and that the hybrid share decomposition of the Lowe index defined by the third term in equation (15.27) involves the long-term monthly price relatives, p_i^t/p_i^0, which compare the prices in month t, p_i^t, with the base month prices, p_i^0. Both of these formulae are unsatisfactory in practice because of sample attrition: each month, a substantial fraction of commodities disappears from the marketplace. Thus it is useful to have a formula for updating the previous month's price index using just month-over-month price relatives. In other words, long-term price relatives disappear at too fast a rate to make it viable, in practice, to base an index number formula on their use. The Lowe index for month t+1, $P_{Lo}(p^0,p^{t+1},q^b)$, can be written in terms of the Lowe index for month t, $P_{Lo}(p^0,p^t,q^b)$, and an updating factor as follows:

$$P_{Lo}(p^0,p^{t+1},q^b) \equiv \frac{\sum_{i=1}^{n} p_i^{t+1} q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} = \left[\frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b}\right]\left[\frac{\sum_{i=1}^{n} p_i^{t+1} q_i^b}{\sum_{i=1}^{n} p_i^t q_i^b}\right]$$
$$= P_{Lo}(p^0,p^t,q^b)\left[\frac{\sum_{i=1}^{n} p_i^{t+1} q_i^b}{\sum_{i=1}^{n} p_i^t q_i^b}\right]$$
$$= P_{Lo}(p^0,p^t,q^b)\left[\frac{\sum_{i=1}^{n} \left(\frac{p_i^{t+1}}{p_i^t}\right) p_i^t q_i^b}{\sum_{i=1}^{n} p_i^t q_i^b}\right] \tag{15.29}$$
$$= P_{Lo}(p^0,p^t,q^b)\left[\sum_{i=1}^{n} \left(\frac{p_i^{t+1}}{p_i^t}\right) s_i^{tb}\right]$$

where the hybrid weights s_i^{tb} are defined by:

$$s_i^{tb} \equiv \frac{p_i^t q_i^b}{\sum_{j=1}^{n} p_j^t q_j^b}; \qquad i = 1,\ldots,n \tag{15.30}$$

Thus the required updating factor, going from month t to month t+1, is the chain link index $\sum_{i=1}^{n} s_i^{tb}(p_i^{t+1}/p_i^t)$, which uses the hybrid share weights s_i^{tb} corresponding to month t and base year b.

15.41 The Lowe index $P_{Lo}(p^0, p^t, q^b)$ can be regarded as an approximation to the ordinary Laspeyres index, $P_L(p^0, p^t, q^0)$, that compares the prices of the base month 0, p^0, to those of month t, p^t, using the quantity vectors of month 0, q^0, as weights. It turns out that there is a relatively simple formula that relates these two indices. In order to explain this formula, it is first necessary to make a few definitions. Define the ith price relative between month 0 and month as

$$r_i \equiv p_i^t / p_i^0; \qquad i = 1,...,n \tag{15.31}$$

The ordinary Laspeyres price index, going from month 0 to t, can be defined in terms of these price relatives as follows:

$$\begin{aligned} P_L(p^0, p^t, q^0) &\equiv \frac{\sum_{i=1}^{n} p_i^t q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0} = \frac{\sum_{i=1}^{n} \left(\frac{p_i^t}{p_i^0}\right) p_i^0 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0} \\ &= \sum_{i=1}^{n} \left(\frac{p_i^t}{p_i^0}\right) s_i^0 = \sum_{i=1}^{n} s_i^0 r_i \equiv r^* \end{aligned} \tag{15.32}$$

where the month 0 expenditure shares s_i^0 are defined as follows:

$$s_i^0 \equiv \frac{p_i^0 q_i^0}{\sum_{j=1}^{n} p_j^0 q_j^0}; \qquad i = 1,...,n \tag{15.33}$$

15.42 Define the ith quantity relative t_i as the ratio of the quantity of commodity i used in the base year b, q_i^b, to the quantity used in month 0, q_i^0, as follows:

$$t_i \equiv q_i^b / q_i^0; \qquad i = 1,...,n \tag{15.34}$$

The Laspeyres quantity index, $Q_L(q^0, q^b, p^0)$, that compares quantities in year b, q^b, to the corresponding quantities in month 0, q^0, using the prices of month 0, p^0, as weights can be defined as a weighted average of the quantity ratios t_i as follows:

$$\begin{aligned} Q_L(q^0, q^b, p^0) &\equiv \frac{\sum_{i=1}^{n} p_i^0 q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^0} = \frac{\sum_{i=1}^{n} \left(\frac{q_i^b}{q_i^0}\right) p_i^0 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0} = \sum_{i=1}^{n} \left(\frac{q_i^b}{q_i^0}\right) s_i^0 \\ &= \sum\nolimits_{i=1}^{n} s_i^0 t_i \qquad \text{using definition (15.34)} \\ &\equiv t^* \end{aligned} \tag{15.35}$$

15.43 Using formula (A15.2.4) in Appendix 15.2 to this chapter, the relationship between the Lowe index $P_{Lo}(p^0,p^t,q^b)$ that uses the quantities of year b as weights to compare the prices of month t to month 0, and the corresponding ordinary Laspeyres index $P_L(p^0,p^t,q^0)$ that uses the quantities of month 0 as weights is the following one:

$$P_{Lo}(p^0,p^t,q^b) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} \qquad (15.36)$$

$$= P_L(p^0,p^t,q^0) + \frac{\sum_{i=1}^{n}(r_i - r^*)(t_i - t^*)s_i^0}{Q_L(q^0,q^b,p^0)}$$

Thus the Lowe price index using the quantities of year b as weights, $P_{Lo}(p^0,p^t,q^b)$, is equal to the usual Laspeyres index using the quantities of month 0 as weights, $P_L(p^0,p^t,q^0)$, plus a covariance term $\sum_{i=1}^{n}(r_i - r^*)(t_i - t^*)s_i^0$ between the price relatives $r_i \equiv p_i^t/p_i^0$ and the quantity relatives $t_i \equiv q_i^b/q_i^0$, divided by the Laspeyres quantity index $Q_L(q^0, q^b, p^0)$ between month 0 and base year b.

15.44 Formula (15.36) shows that the Lowe price index will coincide with the Laspeyres price index if the covariance or correlation between the month 0 to t price relatives $r_i \equiv p_i^t/p_i^0$ and the month 0 to year b quantity relatives $t_i \equiv q_i^b/q_i^0$ is zero. Note that this covariance will be zero under three different sets of conditions:

- if the month t prices are proportional to the month 0 prices so that all $r_i = r^*$;
- if the base year b quantities are proportional to the month 0 quantities so that all $t_i = t^*$;
- if the distribution of the relative prices r_i is independent of the distribution of the relative quantities t_i.

The first two conditions are unlikely to hold empirically, but the third is possible, at least approximately, if consumers do not systematically change their purchasing habits in response to changes in relative prices.

15.45 If this covariance in formula (15.36) is negative, then the Lowe index will be less than the Laspeyres index. Finally, if the covariance is positive, then the Lowe index will be greater than the Laspeyres index. Although the sign and magnitude of the covariance term, $\sum_{i=1}^{n}(r_i - r^*)(t_i - t^*)s_i^0$, is ultimately an empirical matter, it is possible to make some reasonable conjectures about its likely sign. If the base year b precedes the price reference month 0 and there are long-term trends in prices, then it is likely that this covariance is positive and hence that the Lowe index will exceed the corresponding Laspeyres price index;[40] i.e.,

$$P_{Lo}(p^0,p^t,q^b) > P_L(p^0,p^t,q^0) \qquad (15.37)$$

To see why the covariance is likely to be positive, suppose that there is a long-term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With normal

[40] For this relationship to hold, it is also necessary to assume that households have normal substitution effects in response to these long-term trends in prices; i.e., if a commodity increases (relatively) in price, its consumption will decline (relatively) and if a commodity decreases relatively in price, its consumption will increase relatively.

consumer substitution responses[41], q_i^t/q_i^0 less an average quantity change of this type is likely to be negative, or, upon taking reciprocals, q_i^0/q_i^t less an average quantity change of this (reciprocal) type is likely to be positive. But if the long-term upward trend in prices has persisted back to the base year b, then $t_i - t^* \equiv (q_i^b/q_i^0) - t^*$ is also likely to be positive. Hence, the covariance will be positive under these circumstances. Moreover, the more distant is the base year b from the base month 0, the bigger the residuals $t_i - t^*$ are likely to be and the bigger will be the positive covariance. Similarly, the more distant is the current period month t from the base period month 0, the bigger the residuals $r_i - r^*$ are likely to be and the bigger will be the positive covariance. Thus, under the assumptions that there are long-term trends in prices and normal consumer substitution responses, the Lowe index will normally be greater than the corresponding Laspeyres index.

15.46 Define the Paasche index between months 0 and t as follows:

$$P_P(p^0, p^t, q^t) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^t}{\sum_{i=1}^{n} p_i^0 q_i^t} \qquad (15.38)$$

As discussed in paragraphs 15.18 to 15.23, a reasonable target index to measure the price change going from month 0 to t is some sort of symmetric average of the Paasche index $P_P(p^0,p^t,q^t)$, defined by formula (15.38), and the corresponding Laspeyres index, $P_L(p^0,p^t,q^0)$, defined by formula (15.32). Adapting equation (A15.1.5) in Appendix 15.1, the relationship between the Paasche and Laspeyres indices can be written as follows:

$$P_P(p^0, p^t, q^t) = P_L(p^0, p^t, q^0) + \frac{\sum_{i=1}^{n}(r_i - r^*)(u_i - u^*)s_i^0}{Q_L(q^0, q^t, p^0)} \qquad (15.39)$$

where the price relatives $r_i \equiv p_i^t/p_i^0$ are defined by equation (15.31) and their share-weighted average r^* by equation (15.32) and the u_i, u^* and Q_L are defined as follows:

$$u_i \equiv q_i^t / q_i^0; \qquad i = 1,\ldots,n \qquad (15.40)$$

$$u^* \equiv \sum_{i=1}^{n} s_i^0 u_i = Q_L(q^0, q^t, p^0) \qquad (15.41)$$

and the month 0 expenditure shares s_i^0 are defined by the identity (15.33). Thus u^* is equal to the Laspeyres quantity index between months 0 and t. This means that the Paasche price index that uses the quantities of month t as weights, $P_P(p^0,p^t,q^t)$, is equal to the usual Laspeyres index using the quantities of month 0 as weights, $P_L(p^0,p^t,q^0)$, plus a covariance term $\sum_{i=1}^{n}(r_i - r^*)(u_i - u^*)s_i^0$ between the price relatives $r_i \equiv p_i^t/p_i^0$ and the quantity relatives $u_i \equiv q_i^t/q_i^0$, divided by the Laspeyres quantity index $Q_L(q^0,q^t,p^0)$ between month 0 and month t.

[41] Walsh (1901, pp. 281-282) was well aware of consumer substitution effects, as can be seen in the following comment which noted the basic problem with a fixed basket index that uses the quantity weights of a single period: “The argument made by the arithmetic averagist supposes that we buy the same quantities of every class at both periods in spite of the variation in their prices, which we rarely, if ever, do. As a rough proposition, we – a community – generally spend more on articles that have risen in price and get less of them, and spend less on articles that have fallen in price and get more of them.”

15.47 Although the sign and magnitude of the covariance term, $\sum_{i=1}^{n}(r_i - r^*)(u_i - u^*)s_i^0$, is again an empirical matter, it is possible to make a reasonable conjecture about its likely sign. If there are long-term trends in prices and consumers respond normally to price changes in their purchases, then it is likely that this covariance is negative and hence the Paasche index will be less than the corresponding Laspeyres price index; i.e.,

$$P_P(p^0, p^t, q^t) < P_L(p^0, p^t, q^0) \qquad (15.42)$$

To see why this covariance is likely to be negative, suppose that there is a long-term upward trend in the price of commodity i[42] so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With normal consumer substitution responses, q_i^t/q_i^0 less an average quantity change of this type is likely to be negative. Hence $u_i - u^* \equiv (q_i^t/q_i^0) - u^*$ is likely to be negative. Thus, the covariance will be negative under these circumstances. Moreover, the more distant is the base month 0 from the current month t, the bigger in magnitude the residuals $u_i - u^*$ are likely to be and the bigger in magnitude will be the negative covariance.[43] Similarly, the more distant is the current period month t from the base period month 0, the bigger the residuals $r_i - r^*$ will probably be and the bigger in magnitude will be the covariance. Thus under the assumptions that there are long-term trends in prices and normal consumer substitution responses, the Laspeyres index will be greater than the corresponding Paasche index, with the divergence likely to grow as month t becomes more distant from month 0.

15.48 Putting the arguments in the three previous paragraphs together, it can be seen that under the assumptions that there are long-term trends in prices and normal consumer substitution responses, the Lowe price index between months 0 and t will exceed the corresponding Laspeyres price index, which in turn will exceed the corresponding Paasche price index; i.e., under these hypotheses,

$$P_{Lo}(p^0, p^t, q^b) > P_L(p^0, p^t, q^0) > P_P(p^0, p^t, q^t) \qquad (15.43)$$

Thus, if the long-run target price index is an average of the Laspeyres and Paasche indices, it can be seen that the Laspeyres index will have an upward bias relative to this target index and the Paasche index will have a downward bias. In addition, if the base year b is prior to the price reference month, month 0, then the Lowe index will also have an upward bias relative to the Laspeyres index and hence also to the target index.

The Lowe index and mid-year indices

15.49 The discussion in the previous paragraph assumed that the base year b for quantities preceded the base month for prices, month 0. If the current period month t is quite distant from the base month 0, however, then it is possible to think of the base year b as referring to a year that lies between months 0 and t. If the year **b** does fall between months 0 and t, then the Lowe index becomes a *mid-year index*.[44] It turns out that the Lowe mid-year index no longer

[42] The reader can carry through the argument if there is a long-term relative decline in the price of the ith commodity. The argument required to obtain a negative covariance requires that there be some differences in the long-term trends in prices; i.e., if all prices grow (or fall) at the same rate, there will be price proportionality and the covariance will be zero.

[43] However, $Q_L = u^*$ may also be growing in magnitude, so the net effect on the divergence between P_L and P_P is ambiguous.

[44] The concept of the mid-year index can be traced to Hill (1998, p. 46):

(continued)

has the upward biases indicated by the inequalities in the inequality (15.43) under the assumption of long-term trends in prices and normal substitution responses by quantities.

15.50 It is now assumed that the base year quantity vector q^b corresponds to a year that lies between months 0 and t. Under the assumption of long-term trends in prices and normal substitution effects so that there are also long-term trends in quantities (in the opposite direction to the trends in prices so that if the ith commodity price is trending up, then the corresponding ith quantity is trending down), it is likely that the intermediate year quantity vector will lie between the monthly quantity vectors q^0 and q^t. The mid-year Lowe index, $P_{Lo}(p^0,p^t,q^b)$, and the Laspeyres index going from month 0 to t, $P_L(p^0,p^t,q^0)$, will still satisfy the exact relationship given by equation (15.36). Thus $P_{Lo}(p^0,p^t,q^b)$ will equal $P_L(p^0,p^t,q^0)$ plus the covariance term $[\sum_{i=1}^n (r_i - r^*)(t_i - t^*)s_i^0]/Q_L(q^0,q^b,p^0)$, where $Q_L(q^0,q^b,p^0)$ is the Laspeyres quantity index going from month 0 to t. This covariance term is likely to be negative so that

$$P_L(p^0, p^t, q^0) > P_{Lo}(p^0, p^t, q^b). \qquad (15.44)$$

To see why this covariance is likely to be negative, suppose that there is a long-term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With normal consumer substitution responses, q_i will tend to decrease relatively over time and since q_i^b is assumed to be between q_i^0 and q_i^t, q_i^b/q_i^0 less an average quantity change of this type is likely to be negative. Hence $t_i - t^* \equiv (q_i^b/q_i^0) - t^*$ is likely to be negative. Thus, the covariance is likely to be negative under these circumstances. Therefore, under the assumptions that the quantity base year falls between months 0 and t and that there are long-term trends in prices and normal consumer substitution responses, the Laspeyres index will normally be larger than the corresponding Lowe mid-year index, with the divergence probably growing as month t becomes more distant from month 0.

15.51 It can also be seen that under the above assumptions, the mid-year Lowe index is likely to be greater than the Paasche index between months 0 and t; i.e.,

$$P_{Lo}(p^0, p^t, q^b) > P_P(p^0, p^t, q^t) \qquad (15.45)$$

> When inflation has to be measured over a specified sequence of years, such as a decade, a pragmatic solution to the problems raised above would be to take the middle year as the base year. This can be justified on the grounds that the basket of goods and services purchased in the middle year is likely to be much more representative of the pattern of consumption over the decade as a whole than baskets purchased in either the first or the last years. Moreover, choosing a more representative basket will also tend to reduce, or even eliminate, any bias in the rate of inflation over the decade as a whole as compared with the increase in the CoL index.

Thus, in addition to introducing the concept of a mid-year index, Hill also introduced the terminology *representativity bias*. Baldwin (1990, pp. 255-256) also introduced the term *representativeness*: "Here representativeness [in an index number formula] requires that the weights used in any comparison of price levels are related to the volume of purchases in the periods of comparison."

However, this basic idea dates back to Walsh (1901, p.104;1921a, p. 90). Baldwin (1990, p. 255) also noted that his concept of representativeness was the same as Drechsler's (1973, p. 19) concept of *characteristicity*. For additional material on mid-year indices, see Schultz (1999) and Okamoto (2001). Note that the mid-year index concept could be viewed as a close competitor to Walsh's (1901, p. 431) multi-year fixed basket index where the quantity vector was chosen to be an arithmetic or geometric average of the quantity vectors in the span of periods under consideration.

To see why the above inequality is likely to hold, think of q^b starting at the month 0 quantity vector q^0 and then trending smoothly to the month t quantity vector q^t. When $q^b = q^0$, the Lowe index $P_{Lo}(p^0,p^t,q^b)$ becomes the Laspeyres index $P_L(p^0,p^t,q^0)$. When $q^b = q^t$, the Lowe index $P_{Lo}(p^0,p^t,q^b)$ becomes the Paasche index $P_P(p^0,p^t,q^t)$. Under the assumption of trending prices and normal substitution responses to these trending prices, it was shown earlier that the Paasche index will be less than the corresponding Laspeyres price index; i.e., that $P_P(p^0,p^t,q^t)$ was less than $P_L(p^0,p^t,q^0)$, recalling the inequality (15.42). Thus, under the assumption of smoothly trending prices and quantities between months 0 and t, and assuming that q^b is between q^0 and q^t, we will have

$$P_P(p^0,p^t,q^t) < P_{Lo}(p^0,p^t,q^b) < P_L(p^0,p^t,q^0) \qquad (15.46)$$

Thus if the base year for the Lowe index is chosen to be in between the base month for the prices, month 0, and the current month for prices, month t, and there are trends in prices with corresponding trends in quantities that correspond to normal consumer substitution effects, then the resulting Lowe index is likely to lie between the Paasche and Laspeyres indices going from months 0 to t. If the trends in prices and quantities are smooth, then choosing the base year half-way between periods 0 and t should give a Lowe index that is approximately half-way between the Paasche and Laspeyres indices; hence it will be very close to an ideal target index between months 0 and t. This basic idea has been implemented by Okamoto (2001), using Japanese consumer data, and he found that the resulting mid-year indices approximated very closely to the corresponding Fisher ideal indices.

15.52 It should be noted that these mid-year indices can only be computed on a retrospective basis; i.e., they cannot be calculated in a timely fashion, as can Lowe indices that use a base year that is prior to month 0. Thus mid-year indices cannot be used to replace the more timely Lowe indices. The above material indicates, however, that these timely Lowe indices are likely to have an upward bias that is even bigger than the usual Laspeyres upward bias compared to an ideal target index, which was taken to be an average of the Paasche and Laspeyres indices.

15.53 All the inequalities derived in this section rest on the assumption of long-term trends in prices (and corresponding economic responses in quantities). If there are no systematic long-run trends in prices, but only random fluctuations around a common trend in all prices, then the above inequalities are not valid and the Lowe index using a prior base year will probably provide a perfectly adequate approximation to both the Paasche and Laspeyres indices. There are, however, reasons for believing that there are some long-run trends in prices. In particular:

- The computer chip revolution of the past 40 years has led to strong downward trends in the prices of products that use these chips intensively. As new uses for chips have been developed over the years, the share of products that are chip intensive has grown and this implies that what used to be a relatively minor problem has become a more major problem.
- Other major scientific advances have had similar effects. For example, the invention of fibre optic cable (and lasers) has led to a downward trend in telecommunications prices as obsolete technologies based on copper wire are gradually replaced.
- Since the end of the Second World War, a series of international trade agreements has dramatically reduced tariffs around the world. These reductions, combined with improvements in transport technologies, have led to a very rapid growth of international trade and remarkable improvements in international specialization. Manufacturing activities in the more developed economies have gradually been

outsourced to lower-wage countries, leading to deflation in goods prices in most countries around the world. In contrast, many services cannot be readily outsourced[45] and so, on average, the price of services trends upwards while the price of goods trends downwards.

- At the microeconomic level, there are tremendous differences in growth rates of firms. Successful firms expand their scale, lower their costs, and cause less successful competitors to wither away with their higher prices and lower volumes. This leads to a systematic negative correlation between changes in item prices and the corresponding changes in item volumes that can be very large indeed.

Thus there is some a priori basis for assuming long-run divergent trends in prices. Hence there is some basis for concern that a Lowe index that uses a base year for quantity weights that is prior to the base month for prices may be upwardly biased, compared to a more ideal target index.

The Young index

15.54 Recall the definitions for the base year quantities, q_i^b, and the base year prices, p_i^b, given by equations (15.23) and (15.24) above. The base year expenditure shares can be defined in the usual way as follows:

$$s_i^b \equiv \frac{p_i^b q_i^b}{\sum_{k=1}^{n} p_k^b q_k^b}; \qquad i = 1,\ldots,n \qquad (15.47)$$

Define the vector of base year expenditure shares in the usual way as $s^b \equiv [s_1^b,\ldots,s_n^b]$. These base year expenditure shares were used to provide an alternative formula for the base year b Lowe price index going from month 0 to t, defined in equation (15.26) as $P_{Lo}(p^0,p^t,q^b) = \left[\sum_{i=1}^{n} s_i^b \left(p_i^t/p_i^b\right)\right] / \left[\sum_{i=1}^{n} s_i^b \left(p_i^0/p_i^b\right)\right]$. Rather than using this index as their short-run target index, many statistical agencies use the following closely related index:

$$P_Y(p^0,p^t,s^b) \equiv \sum_{i=1}^{n} s_i^b \left(p_i^t/p_i^0\right) \qquad (15.48)$$

This type of index was first defined by the English economist, Arthur Young (1812).[46] Note that there is a change in focus when the Young index is used compared to the other indices proposed earlier in this chapter. Up to this point, the indices proposed have been of the fixed basket type (or averages of such indices) where a *commodity basket* that is somehow representative for the two periods being compared is chosen and then "purchased" at the prices of the two periods and the index is taken to be the ratio of these two costs. In contrast, for the Young index, *representative expenditure shares* are chosen that pertain to the two periods under consideration, and then these shares are used to calculate the overall index as a share-weighted average of the individual price ratios, p_i^t/p_i^0. Note that this view of index number theory, based on the share-weighted average of price ratios, is a little different from the view taken at the beginning of this chapter, which saw the index number problem as that of decomposing a value ratio into the product of two terms, one of which expresses the

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[45] Some services, however, can be internationally outsourced; e.g., call centres, computer programming and airline maintenance.

[46] This formula is attributed to Young by Walsh (1901, p. 536; 1932, p. 657).

amount of price change between the two periods and the other which expresses the amount of quantity change.[47]

15.55 Statistical agencies sometimes regard the Young index, defined above, as an approximation to the Laspeyres price index $P_L(p^0,p^t,q^0)$. Hence, it is of interest to see how the two indices compare. Defining the long-term monthly price relatives going from month 0 to t as $r_i \equiv p_i^t/p_i^0$ and using definitions (15.32) and (15.48):

$$\begin{aligned} P_Y(p^0,p^t,s^b) - P_L(p^0,p^t,q^0) &\equiv \sum_{i=1}^{n} s_i^b\left(\frac{p_i^t}{p_i^0}\right) - \sum_{i=1}^{n} s_i^0\left(\frac{p_i^t}{p_i^0}\right) \\ &= \sum_{i=1}^{n}\left[s_i^b - s_i^0\right]\left(\frac{p_i^t}{p_i^0}\right) \\ &= \sum_{i=1}^{n}\left[s_i^b - s_i^0\right] r_i \qquad (15.49) \\ &= \sum_{i=1}^{n}\left[s_i^b - s_i^0\right]\left[r_i - r^*\right] + r^*\sum_{i=1}^{n}\left[s_i^b - s_i^0\right] \\ &= \sum_{i=1}^{n}\left[s_i^b - s_i^0\right]\left[r_i - r^*\right] \end{aligned}$$

since $\sum_{i=1}^{n} s_i^b = \sum_{i=1}^{n} s_i^0 = 1$ and using (15.32) which defined $r^* \equiv \sum_{i=1}^{n} s_i^0 r_i = P_L(p^0,p^t,q^0)$. Thus the Young index $P_Y(p^0,p^t,s^b)$ is equal to the Laspeyres index $P_L(p^0,p^t,q^0)$, plus the *covariance* between the difference in the annual shares pertaining to year b and the month 0 shares, $s_i^b - s_i^0$, and the deviations of the relative prices from their mean, $r_i - r^*$.

15.56 It is no longer possible to guess at what the likely sign of the covariance term is. The question is no longer whether the *quantity* demanded goes down as the price of commodity i goes up (the answer to this question is usually "yes") but the new question is: does the *share* of expenditure go down as the price of commodity i goes up? The answer to this question depends on the elasticity of demand for the product. Let us provisionally assume, however, that there are long-run trends in commodity prices and if the trend in prices for commodity i is above the mean, then the expenditure share for the commodity trends *down* (and vice versa). Thus we are assuming high elasticities or very strong substitution effects. Assuming also that the base year b is prior to month 0, then under these conditions, suppose that there is

[47] Fisher's 1922 book is famous for developing the value ratio decomposition approach to index number theory, but his introductory chapters took the share weighted average point of view: "An index number of prices, then shows the *average percentage change* of prices from one point of time to another" (Fisher (1922, p. 3)). Fisher went on to note the importance of economic weighting: "The preceding calculation treats all the commodities as equally important; consequently, the average was called 'simple'. If one commodity is more important than another, we may treat the more important as though it were two or three commodities, thus giving it two or three times as much 'weight' as the other commodity" (Fisher (1922, p. 6)). Walsh (1901, pp. 430-431) considered both approaches: "We can either (1) draw some average of the total money values of the classes during an epoch of years, and with weighting so determined employ the geometric average of the price variations [ratios]; or (2) draw some average of the mass quantities of the classes during the epoch, and apply to them Scrope's method." Scrope's method is the same as using the Lowe index. Walsh (1901, pp. 88-90) consistently stressed the importance of weighting price ratios by their economic importance (rather than using equally weighted averages of price relatives). Both the value ratio decomposition approach and the share-weighted average approach to index number theory are studied from the axiomatic perspective in Chapter 16.

a long-term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With the assumed very elastic consumer substitution responses, s_i will tend to decrease relatively over time and since s_i^b is assumed to be prior to s_i^0, s_i^0 is expected to be less than s_i^b or $s_i^b - s_i^0$ will probably be positive. Thus, the covariance is likely to be positive under these circumstances. *Hence with long-run trends in prices and very elastic responses of consumers to price changes, the Young index is likely to be greater than the corresponding Laspeyres index.*

15.57 Assume that there are long-run trends in commodity prices. If the trend in prices for commodity i is above the mean, then suppose that the expenditure share for the commodity trends *up* (and vice versa). Thus we are assuming low elasticities or very weak substitution effects. Assume also that the base year b is prior to month 0 and suppose that there is a long-term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With the assumed very inelastic consumer substitution responses, s_i will tend to increase relatively over time and since s_i^b is assumed to be prior to s_i^0, it will be the case that s_i^0 is greater than s_i^b or $s_i^b - s_i^0$ is negative. Thus, the covariance is likely to be negative under these circumstances. *Hence with long-run trends in prices and very inelastic responses of consumers to price changes, the Young index is likely to be less than the corresponding Laspeyres index.*

15.58 The previous two paragraphs indicate that, a priori, it is not known what the likely difference between the Young index and the corresponding Laspeyres index will be. If elasticities of substitution are close to one, then the two sets of expenditure shares, s_i^b and s_i^0, will be close to each other and the difference between the two indices will be close to zero. If monthly expenditure shares have strong seasonal components, however, then the annual shares s_i^b could differ substantially from the monthly shares s_i^0.

15.59 It is useful to have a formula for updating the previous month's Young price index using just month-over-month price relatives. The Young index for month t+1, $P_Y(p^0,p^{t+1},s^b)$, can be written in terms of the Young index for month t, $P_Y(p^0,p^t,s^b)$, and an updating factor as follows:

$$
\begin{aligned}
P_Y(p^0, p^{t+1}, s^b) &\equiv \sum_{i=1}^{n} s_i^b \left(\frac{p_i^{t+1}}{p_i^0} \right) \\
&= P_Y(p^0, p^t, s^b) \frac{\sum_{i=1}^{n} s_i^b \left(p_i^{t+1} / p_i^0 \right)}{\sum_{i=1}^{n} s_i^b \left(p_i^t / p_i^0 \right)} \\
&= P_Y(p^0, p^t, s^b) \frac{\sum_{i=1}^{n} p_i^b q_i^b \left(p_i^{t+1} / p_i^0 \right)}{\sum_{i=1}^{n} p_i^b q_i^b \left(p_i^t / p_i^0 \right)}
\end{aligned}
$$

using definition (15.47)

$$
\begin{aligned}
&= P_Y(p^0, p^t, s^b) \frac{\sum_{i=1}^{n} p_i^b q_i^b \left(\frac{p_i^t}{p_i^0} \right) \left(\frac{p_i^{t+1}}{p_i^t} \right)}{\sum_{i=1}^{n} p_i^b q_i^b (p_i^t / p_i^0)} \\
&= P_Y(p^0, p^t, s^b) \left[\sum_{i=1}^{n} s_i^{b0t} (p_i^{t+1} / p_i^t) \right] \qquad (15.50)
\end{aligned}
$$

where the hybrid weights s_i^{b0t} are defined by

$$
s_i^{b0t} \equiv \frac{p_i^b q_i^b \left(p_i^t / p_i^0 \right)}{\sum_{k=1}^{n} p_k^b q_k^b \left(p_k^t / p_k^0 \right)} = \frac{s_i^b \left(p_i^t / p_i^0 \right)}{\sum_{k=1}^{n} s_k^b \left(p_k^t / p_k^0 \right)} \qquad i = 1,...,n \qquad (15.51)
$$

Thus the hybrid weights s_i^{b0t} can be obtained from the base year weights s_i^b by updating them; i.e., by multiplying them by the price relatives (or *indices* at higher levels of aggregation), p_i^t/p_i^0. Thus the required updating factor, going from month t to month t+1, is the chain link index, $\sum_{i=1}^{n} s_i^{b0t} (p_i^{t+1} / p_i^t)$, which uses the hybrid share weights s_i^{b0t} defined by equation (15.51).

15.60 Even if the Young index provides a close approximation to the corresponding Laspeyres index, it is difficult to recommend the use of the Young index as a final estimate of the change in prices going from period 0 to t, just as it was difficult to recommend the use of the Laspeyres index as the *final* estimate of inflation going from period 0 to t. Recall that the problem with the Laspeyres index was its lack of symmetry in the treatment of the two periods under consideration; i.e., using the justification for the Laspeyres index as a good fixed basket index, there was an identical justification for the use of the Paasche index as an equally good fixed basket index to compare periods 0 and t. The Young index suffers from a similar lack of symmetry with respect to the treatment of the base period. The problem can be explained as follows. The Young index, $P_Y(p^0,p^t,s^b)$ defined by equation (15.48) calculates the price change between months 0 and t treating month 0 as the base. But there is no particular reason to necessarily treat month 0 as the base month other than convention. Hence, if we treat month t as the base and use the same formula to measure the price change from month t back to month 0, the index $P_Y(p^t, p^0, s^b) = \sum_{i=1}^{n} s_i^b \left(p_i^0 / p_i^t \right)$ would be appropriate. This estimate of price change can then be made comparable to the original

Young index by taking its reciprocal, leading to the following *rebased Young index*[48], $P_Y{}^*(p^0,p^t,s^b)$, defined as

$$P_Y^*(p^0,p^t,s^b) \equiv 1 \Big/ \sum_{i=1}^{n} s_i^b (p_i^0 / p_i^t) = \left[\sum_{i=1}^{n} s_i^b (p_i^t / p_i^0)^{-1} \right]^{-1} \qquad (15.52)$$

The rebased Young index, $P_Y{}^*(p^0,p^t,s^b)$, which uses the current month as the initial base period, is a *share-weighted harmonic mean* of the price relatives going from month 0 to month *t*, whereas the original Young index, $P_Y(p^0,p^t,s^b)$, is a *share-weighted arithmetic mean* of the same price relatives.

15.61 Fisher argued as follows that an index number formula should give the same answer no matter which period was chosen as the base:

> Either one of the two times may be taken as the "base". Will it make a difference which is chosen? Certainly, it *ought* not and our Test 1 demands that it shall not. More fully expressed, the test is that the formula for calculating an index number should be such that it will give the same ratio between one point of comparison and the other point, *no matter which of the two is taken as the base* (Fisher (1922, p. 64)).

15.62 The problem with the Young index is that not only does it not coincide with its rebased counterpart, but there is a definite inequality between the two indices, namely:

$$P_Y^*(p^0,p^t,s^b) \le P_Y(p^0,p^t,s^b) \qquad (15.53)$$

with a strict inequality provided that the period *t* price vector p^t is not proportional to the period 0 price vector p^0.[49] A statistical agency that uses the direct Young index $P_Y(p^0,p^t,s^b)$ will generally show a higher inflation rate than a statistical agency that uses the same raw data but uses the rebased Young index, $P_Y{}^*(p^0,p^t,s^b)$.

15.63 The inequality (15.53) does not tell us by how much the Young index will exceed its rebased time antithesis. In Appendix 15.3, however, it is shown that to the accuracy of a certain second-order Taylor series approximation, the following relationship holds between the direct Young index and its time antithesis:

$$P_Y(p^0,p^t,s^b) \approx P_Y^*(p^0,p^t,s^b) + P_Y(p^0,p^t,s^b)\,\mathrm{Var}\, e \qquad (15.54)$$

where Var *e* is defined as

[48] Using Fisher's (1922, p. 118) terminology, $P_Y{}^*(p^0,p^t,s^b) \equiv 1/[P_Y(p^t,p^0,s^b)]$ is the *time antithesis* of the original Young index, $P_Y(p^0,p^t,s^b)$.

[49] These inequalities follow from the fact that a harmonic mean of M positive numbers is always equal to or less than the corresponding arithmetic mean; see Walsh (1901, p.517) or Fisher (1922, pp. 383-384). This inequality is a special case of Schlömilch's (1858) inequality; see Hardy, Littlewood and Polya (1934, p. 26). Walsh (1901, pp. 330-332) explicitly noted the inequality (15.53) and also noted that the corresponding geometric average would fall between the harmonic and arithmetic averages. Walsh (1901, p. 432) computed some numerical examples of the Young index and found big differences between it and his "best" indices, even using weights that were representative for the periods being compared. Recall that the Lowe index becomes the Walsh index when geometric mean quantity weights are chosen and so the Lowe index can perform well when representative weights are used. This is not necessarily the case for the Young index, even using representative weights. Walsh (1901, p. 433) summed up his numerical experiments with the Young index as follows: "In fact, Young's method, in every form, has been found to be bad."

$$\text{Var } e \equiv \sum_{i=1}^{n} s_i^b \left[e_i - e^* \right]^2 \tag{15.55}$$

The deviations e_i are defined by $1+e_i = r_i/r^*$ for $i = 1,\ldots, n$ where the r_i and their weighted mean r^* are defined by

$$r_i \equiv p_i^t / p_i^0; \qquad i = 1,\ldots,n; \tag{15.56}$$

$$r^* \equiv \sum_{i=1}^{n} s_i^b r_i \tag{15.57}$$

which turns out to equal the direct Young index, $P_Y(p^0,p^t,s^b)$. The weighted mean of the e_i is defined as

$$e^* \equiv \sum_{i=1}^{n} s_i^b e_i \tag{15.58}$$

which turns out to equal 0. *Hence the more dispersion there is in the price relatives p_i^t/p_i^0, to the accuracy of a second-order approximation, the more the direct Young index will exceed its counterpart that uses month t as the initial base period rather than month 0.*

15.64 Given two a priori equally plausible index number formulae that give different answers, such as the Young index and its time antithesis, Fisher (1922, p. 136) generally suggested taking the geometric average of the two indices.[50] A benefit of this averaging is that the resulting formula will satisfy the time reversal test. Thus rather than using *either* the base period 0 Young index, $P_Y(p^0,p^t,s^b)$, *or* the current period t Young index, $P_Y^*(p^0,p^t,s^b)$, which is always below the base period 0 Young index if there is any dispersion in relative prices, it seems preferable to use the following index, which is the *geometric average* of the two alternatively based Young indices.[51]

$$P_Y^{**}(p^0,p^t,s^b) \equiv \left[P_Y(p^0,p^t,s^b) P_Y^*(p^0,p^t,s^b) \right]^{1/2} \tag{15.59}$$

If the base year shares s_i^b happen to coincide with both the month 0 and month t shares, s_i^0 and s_i^t respectively, it can be seen that the time-rectified Young index $P_Y^{**}(p^0,p^t,s^b)$ defined by equation (15.59) will coincide with the Fisher ideal price index between months 0 and t, $P_F(p^0,p^t,q^0,q^t)$ (which will also equal the Laspeyres and Paasche indices under these conditions). Note also that the index P_Y^{**} defined by equation (15.59) can be produced on a timely basis by a statistical agency.

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[50] "We now come to a third use of these tests, namely, to 'rectify' formulae, i.e., to derive from any given formula which does not satisfy a test another formula which does satisfy it; …. This is easily done by 'crossing', that is, by averaging antitheses. If a given formula fails to satisfy Test 1 [the time reversal test], its time antithesis will also fail to satisfy it; but the two will fail, as it were, in opposite ways, so that a cross between them (obtained by *geometrical* averaging) will give the golden mean which does satisfy" (Fisher (1922, p. 136)).

Actually the basic idea behind Fisher's rectification procedure was suggested by Walsh, who was a discussant for Fisher (1921), where Fisher gave a preview of his 1922 book: "We merely have to take any index number, find its antithesis in the way prescribed by Professor Fisher, and then draw the geometric mean between the two" (Walsh (1921b, p. 542)).

[51] This index is a base year weighted counterpart to an equally weighted index proposed by Carruthers, Sellwood and Ward (1980, p. 25) and Dalén (1992, p. 140) in the context of elementary index number formulae. See Chapter 20 for further discussion of this unweighted index.

The Divisia index and discrete approximations to it
The Divisia price and quantity indices

15.65 The second broad approach to index number theory relies on the assumption that price and quantity data change in a more or less continuous way.

15.66 Suppose that the price and quantity data on the *n* commodities in the chosen domain of definition can be regarded as continuous functions of (continuous) time, say $p_i(t)$ and $q_i(t)$ for i = 1,…,*n*. The value of consumer expenditure at time *t* is $V(t)$ defined in the obvious way as:

$$V(t) \equiv \sum_{i=1}^{n} p_i(t)q_i(t) \qquad (15.60)$$

15.67 Now suppose that the functions $p_i(t)$ and $q_i(t)$ are differentiable. Then both sides of the definition (15.60) can be differentiated with respect to time to obtain:

$$V'(t) = \sum_{i=1}^{n} p_i'(t)q_i(t) + \sum_{i=1}^{n} p_i(t)q_i'(t) \qquad (15.61)$$

Divide both sides of equation (15.61) through by $V(t)$ and using definition (15.60), the following equation is obtained:

$$\frac{V'(t)}{V(t)} = \frac{\sum_{i=1}^{n} p_i'(t)q_i(t) + \sum_{i=1}^{n} p_i(t)q_i'(t)}{\sum_{j=1}^{n} p_j(t)q_j(t)} = \sum_{i=1}^{n} \frac{p_i'(t)}{p_i(t)} s_i(t) + \sum_{i=1}^{n} \frac{q_i'(t)}{q_i(t)} s_i(t) \qquad (15.62)$$

where the time *t* expenditure share on commodity *i*, $s_i(t)$, is defined as:

$$s_i(t) \equiv \frac{p_i(t)q_i(t)}{\sum_{m=1}^{n} p_m(t)q_m(t)} \quad \text{for } i = 1,2,\ldots,n \qquad (15.63)$$

15.68 Divisia (1926, p. 39) argued as follows: *suppose* the aggregate value at time *t*, $V(t)$, can be written as the product of a time *t* price level function, $P(t)$ say, times a time *t* quantity level function, $Q(t)$ say; i.e., we have:

$$V(t) = P(t)Q(t) \qquad (15.64)$$

Suppose further that the functions $P(t)$ and $Q(t)$ are differentiable. Then differentiating the equation (15.64) yields:

$$V'(t) = P'(t)Q(t) + P(t)Q'(t) \qquad (15.65)$$

Dividing both sides of equation (15.65) by $V(t)$ and using equation (15.64) leads to the following equation:

$$\frac{V'(t)}{V(t)} = \frac{P'(t)}{P(t)} + \frac{Q'(t)}{Q(t)} \qquad (15.66)$$

15.69 Divisia compared the two expressions for the logarithmic value derivative, $V'(t)/V(t)$, given by equations (15.62) and (15.66), and he simply defined the logarithmic rate of change of the *aggregate price level*, $P'(t)/P(t)$, as the first set of terms on the right-hand side of (15.62). He also simply defined the logarithmic rate of change of the *aggregate quantity* *level*, $Q'(t)/Q(t)$, as the second set of terms on the right-hand side of equation (15.62). That is, he made the following definitions:

$$\frac{P'(t)}{P(t)} \equiv \sum_{i=1}^{n} s_i(t)\frac{p_i'(t)}{p_i(t)} \tag{15.67}$$

$$\frac{Q'(t)}{Q(t)} \equiv \sum_{i=1}^{n} s_i(t)\frac{q_i'(t)}{q_i(t)} \tag{15.68}$$

15.70 Definitions (15.67) and (15.68) are reasonable definitions for the proportional changes in the aggregate price and quantity (or quantity) levels, $P(t)$ and $Q(t)$.[52] The problem with these definitions is that economic data are not collected in *continuous* time; they are collected in *discrete* time. In other words, even though transactions can be thought of as occurring in continuous time, no consumer records his or her purchases as they occur in continuous time; rather, purchases over a finite time period are cumulated and then recorded. A similar situation occurs for producers or sellers of commodities; firms cumulate their sales over discrete periods of time for accounting or analytical purposes. If it is attempted to approximate continuous time by shorter and shorter discrete time intervals, empirical price and quantity data can be expected to become increasingly erratic since consumers only make purchases at discrete points of time (and producers or sellers of commodities only make sales at discrete points of time). It is, however, still of some interest to approximate the continuous time price and quantity levels, $P(t)$ and $Q(t)$ defined implicitly by equations (15.67) and (15.68), by discrete time approximations. This can be done in two ways. Either methods of numerical approximation can be used or assumptions can be made about the path taken through time by the functions $p_i(t)$ and $q_i(t)$ ($i = 1,\ldots, n$). The first strategy is used in the following section. For discussions of the second strategy, see Vogt (1977; 1978), Van Ijzeren (1987, pp. 8-12), Vogt and Barta (1997) and Balk (2000a).

15.71 There is a connection between the Divisia price and quantity levels, $P(t)$ and $Q(t)$, and the economic approach to index number theory. This connection is, however, best made after studying the economic approach to index number theory. Since this material is rather technical, it has been relegated to Appendix 15.4.

Discrete approximations to the continuous time Divisia index

15.72 In order to make operational the continuous time Divisia price and quantity levels, $P(t)$ and $Q(t)$ defined by the differential equations (15.67) and (15.68), it is necessary to convert to discrete time. Divisia (1926, p. 40) suggested a straightforward method for doing this conversion, which we now outline.

15.73 Define the following price and quantity (forward) differences:

$$\Delta P \equiv P(1) - P(0) \tag{15.69}$$

$$\Delta p_i \equiv p_i(1) - p_i(0); \quad i = 1,\ldots,n \tag{15.70}$$

Using the above definitions:

[52] If these definitions are applied (approximately) to the Young index studied in the previous section, then it can be seen that in order for the Young price index to be consistent with the Divisia price index, the base year shares should be chosen to be average shares that apply to the entire time period between months 0 and *t*.

$$\frac{P(1)}{P(0)} = \frac{P(0)+\Delta P}{P(0)} = 1 + \frac{\Delta P}{P(0)} \approx 1 + \frac{\sum_{i=1}^{n} \Delta p_i q_i(0)}{\sum_{m=1}^{n} p_m(0) q_m(0)}$$

using (15.67) when $t = 0$ and approximating $p_i'(0)$ by the difference Δp_i (15.71)

$$= \frac{\sum_{i=1}^{n} \{p_i(0) + \Delta p_i\} q_i(0)}{\sum_{m=1}^{n} p_m(0) q_m(0)} = \frac{\sum_{i=1}^{n} p_i(1) q_i(0)}{\sum_{m=1}^{n} p_m(0) q_m(0)} = P_L(p^0, p^1, q^0, q^1)$$

where $p^t \equiv [p_1(t),\ldots,p_n(t)]$ and $q^t \equiv [q_1(t),\ldots,q_n(t)]$ for $t = 0,1$. Thus, it can be seen that Divisia's discrete approximation to his continuous time price index is just the Laspeyres price index, P_L, defined above by equation (15.5).

15.74 But now a problem noted by Frisch (1936, p. 8) occurs: instead of approximating the derivatives by the discrete (forward) differences defined by equations (15.69) and (15.70), other approximations could be used and a wide variety of discrete time approximations could be obtained. For example, instead of using forward differences and evaluating the index at time $t = 0$, it would be possible to use backward differences and evaluate the index at time $t = 1$. These backward differences are defined as:

$$\Delta_b p_i \equiv p_i(0) - p_i(1); \quad i = 1, \ldots, n \qquad (15.72)$$

This use of backward differences leads to the following approximation for $P(0)/P(1)$:

$$\frac{P(0)}{P(1)} = \frac{P(1)+\Delta_b P}{P(1)} = 1 + \frac{\Delta_b P}{P(1)} \approx 1 + \frac{\sum_{i=1}^{n} \Delta_b p_i q_i(1)}{\sum_{m=1}^{n} p_m(1) q_m(1)}$$

using (15.67) when $t = 1$ and approximating $p_i'(1)$ by the difference $\Delta_b p_i$ (15.73)

$$= \frac{\sum_{i=1}^{n} \{p_i(1) + \Delta_b p_i\} q_i(1)}{\sum_{m=1}^{n} p_m(1) q_m(1)} = \frac{\sum_{i=1}^{n} p_i(0) q_i(1)}{\sum_{m=1}^{n} p_m(1) q_m(1)} = \frac{1}{P_P(p^0, p^1, q^0, q^1)}$$

where P_P is the Paasche index defined above by equation (15.6). Taking reciprocals of both sides of equation (15.73) leads to the following discrete approximation to $P(1)/P(0)$:

$$\frac{P(1)}{P(0)} \approx P_P \qquad (15.74)$$

15.75 Thus, as Frisch[53] noted, both the Paasche and Laspeyres indices can be regarded as (equally valid) approximations to the continuous time Divisia price index.[54] Since the

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[53] "As the elementary formula of the chaining, we may get Laspeyres' or Paasche's or Edgeworth's or nearly any other formula, according as we choose the approximation principle for the steps of the numerical integration" (Frisch (1936, p. 8)).

[54] Diewert (1980, p. 444) also obtained the Paasche and Laspeyres approximations to the Divisia index, using a somewhat different approximation argument. He also showed how several other popular discrete time index number formulae could be regarded as approximations to the continuous time Divisia index.

Paasche and Laspeyres indices can differ considerably in some empirical applications, it can be seen that Divisia's idea is not all that helpful in determining a *unique* discrete time index number formula.[55] What is useful about the Divisia indices is the idea that as the discrete unit of time gets smaller, discrete approximations to the Divisia indices can approach meaningful economic indices under certain conditions. Moreover, if the Divisia concept is accepted as the "correct" one for index number theory, then the corresponding "correct" discrete time counterpart might be taken as a weighted average of the chain price relatives pertaining to the adjacent periods under consideration, where the weights are somehow representative of the two periods under consideration.

Fixed base versus chain indices

15.76 In this section[56], we discuss the merits of using the chain system for constructing price indices in the time series context versus using the fixed base system.[57]

15.77 The chain system[58] measures the change in prices going from one period to a subsequent period using a bilateral index number formula involving the prices and quantities pertaining to the two adjacent periods. These one-period rates of change (the links in the chain) are then cumulated to yield the relative levels of prices over the entire period under consideration. Thus if the bilateral price index is P, the chain system generates the following pattern of price levels for the first three periods:

$$1,\ P(p^0,p^1,q^0,q^1),\ \ P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2) \qquad (15.75)$$

15.78 In contrast, the fixed base system of price levels, using the same bilateral index number formula P, simply computes the level of prices in period t relative to the base period 0 as $P(p^0,p^t,q^0,q^t)$. Thus the fixed base pattern of price levels for periods 0,1 and 2 is:

$$1, P(p^0,p^1,q^0,q^1),\ \ P(p^0,p^2,q^0,q^2) \qquad (15.76)$$

15.79 Note that in both the chain system and the fixed base system of price levels defined by the formulae (15.75) and (15.76), the base period price level is set equal to 1. The usual practice in statistical agencies is to set the base period price level equal to 100. If this is done,

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[55] Trivedi (1981) systematically examined the problems involved in finding a "best" discrete time approximation to the Divisia indices using the techniques of numerical analysis. These numerical analysis techniques depend on the assumption that the "true" continuous time micro-price functions, $p_i(t)$, can be adequately represented by a polynomial approximation. Thus we are led to the conclusion that the "best" discrete time approximation to the Divisia index depends on assumptions that are difficult to verify.

[56] This section is largely based on the work of Hill (1988; 1993, p.385-390).

[57] The results in Appendix 15.4 provide some theoretical support for the use of chain indices in that it is shown that under certain conditions, the Divisia index will equal an economic index. Hence any discrete approximation to the Divisia index will approach the economic index as the time period gets shorter. Thus under certain conditions, chain indices will approach an underlying economic index.

[58] The chain principle was introduced independently into the economics literature by Lehr (1885. pp. 45-46) and Marshall (1887, p. 373). Both authors observed that the chain system would mitigate the difficulties arising from the introduction of new commodities into the economy, a point also mentioned by Hill (1993, p. 388). Fisher (1911, p. 203) introduced the term "chain system".

then it is necessary to multiply each of the numbers in the formulae (15.75) and (15.76) by 100.

15.80 Because of the difficulties involved in obtaining current period information on quantities (or equivalently, on expenditures), many statistical agencies loosely base their consumer price index on the use of the Laspeyres formula (15.5) and the fixed base system. Therefore, it is of interest to look at some of the possible problems associated with the use of fixed base Laspeyres indices.

15.81 The main problem with the use of fixed base Laspeyres indices is that the period 0 fixed basket of commodities that is being priced out in period t can often be quite different from the period t basket. Thus if there are systematic trends in at least some of the prices and quantities[59] in the index basket, the fixed base Laspeyres price index $P_L(p^0,p^t,q^0,q^t)$ can be quite different from the corresponding fixed base Paasche price index, $P_P(p^0,p^t,q^0,q^t)$.[60] This means that both indices are likely to be an inadequate representation of the movement in average prices over the time period under consideration.

15.82 The fixed base Laspeyres quantity index cannot be used for ever: eventually, the base period quantities q^0 are so far removed from the current period quantities q^t that the base must be changed. Chaining is merely the limiting case where the base is changed each period.[61]

15.83 The main advantage of the chain system is that under normal conditions, chaining will reduce the spread between the Paasche and Laspeyres indices.[62] These two indices each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration and it could be expected that a single point estimate of the aggregate price change should lie between these two estimates. Thus the use of either a chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth".[63]

15.84 Hill (1993, p. 388), drawing on the earlier research of Szulc (1983) and Hill (1988, pp. 136-137), noted that it is not appropriate to use the chain system when prices oscillate or bounce. This phenomenon can occur in the context of regular seasonal fluctuations or in the context of price wars. However, in the context of roughly monotonically changing prices and quantities, Hill (1993, p. 389) recommended the use of chained symmetrically weighted

[59] Examples of rapidly downward trending prices and upward trending quantities are computers, electronic equipment of all types, Internet access and telecommunication charges.

[60] Note that $P_L(p^0,p^t,q^0,q^t)$ will equal $P_P(p^0,p^t,q^0,q^t)$ if *either* the two quantity vectors q^0 and q^t are proportional *or* the two price vectors p^0 and p^t are proportional. Thus in order to obtain a difference between the Paasche and Laspeyres indices, nonproportionality in *both* prices and quantities is required.

[61] Regular seasonal fluctuations can cause monthly or quarterly data to "bounce" – using the term coined by Szulc (1983, p. 548) – and chaining bouncing data can lead to a considerable amount of index "drift"; i.e., if after 12 months, prices and quantities return to their levels of a year earlier, then a chained monthly index will usually not return to unity. Hence, the use of chained indices for "noisy" monthly or quarterly data is not recommended without careful consideration.

[62] See Diewert (1978, p. 895) and Hill (1988; 1993, pp. 387-388).

[63] This observation will be illustrated with an artificial data set in Chapter 19.

indices (see paragraphs 15.18 to 15.32). The Fisher and Walsh indices are examples of symmetrically weighted indices.

15.85 It is possible to be a little more precise about the conditions under which to chain or not to chain. Basically, chaining is advisable if the prices and quantities pertaining to adjacent periods are *more similar* than the prices and quantities of more distant periods, since this strategy will lead to a narrowing of the spread between the Paasche and Laspeyres indices at each link.[64] Of course, one needs a measure of how similar are the prices and quantities pertaining to two periods. The similarity measures could be *relative* ones or *absolute* ones. In the case of absolute comparisons, two vectors of the same dimension are similar if they are identical and dissimilar otherwise. In the case of relative comparisons, two vectors are similar if they are proportional and dissimilar if they are non-proportional.[65] Once a similarity measure has been defined, the prices and quantities of each period can be compared to each other using this measure, and a "tree" or path that links all of the observations can be constructed where the most similar observations are compared with each other using a bilateral index number formula.[66] Hill (1995) defined the price structures between two countries to be more dissimilar the bigger the spread between P_L and P_P; i.e., the bigger is $\{P_L/P_P, P_P/P_L\}$. The problem with this measure of dissimilarity in the price structures of the two countries is that it could be the case that $P_L = P_P$ (so that the Hill measure would register a maximal degree of similarity), but p^0 could be very different from p^t. Thus there is a need for a more systematic study of similarity (or dissimilarity) measures in order to pick the "best" one that could be used as an input into Hill's (1999a; 1999b; 2001) spanning tree algorithm for linking observations.

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[64] Walsh, in discussing whether fixed base or chained index numbers should be constructed, took for granted that the precision of all reasonable bilateral index number formulae would improve, provided that the two periods or situations being compared were more similar, and hence favoured the use of chained indices: "The question is really, in which of the two courses [fixed base or chained index numbers] are we likely to gain greater exactness in the comparisons actually made? Here the probability seems to incline in favor of the second course; for the conditions are likely to be less diverse between two contiguous periods than between two periods say fifty years apart" (Walsh (1901, p. 206)).

Walsh (1921a, pp. 84-85) later reiterated his preference for chained index numbers. Fisher also made use of the idea that the chain system would usually make bilateral comparisons between price and quantity data that were more similar, and hence the resulting comparisons would be more accurate:

> The index numbers for 1909 and 1910 (each calculated in terms of 1867-1877) are compared with each other. But direct comparison between 1909 and 1910 would give a different and more valuable result. To use a common base is like comparing the relative heights of two men by measuring the height of each above the floor, instead of putting them back to back and directly measuring the difference of level between the tops of their heads (Fisher (1911, p. 204)).
>
> It seems, therefore, advisable to compare each year with the next, or, in other words, to make each year the base year for the next. Such a procedure has been recommended by Marshall, Edgeworth and Flux. It largely meets the difficulty of non-uniform changes in the Q's, for any inequalities for successive years are relatively small (Fisher (1911, pp. 423-424)).

[65] Diewert (2002b) takes an axiomatic approach to defining various *indices* of absolute and relative dissimilarity.

[66] Fisher (1922, pp.271-276) hinted at the possibility of using spatial linking; i.e., of linking countries that are similar in structure. The modern literature has, however, grown as a result of the pioneering efforts of Robert Hill (1995; 1999a; 1999b; 2001). Hill (1995) used the spread between the Paasche and Laspeyres price indices as an indicator of similarity, and showed that this criterion gives the same results as a criterion that looks at the spread between the Paasche and Laspeyres quantity indices.

15.86 The method of linking observations explained in the previous paragraph, based on the similarity of the price and quantity structures of any two observations, may not be practical in a statistical agency context since the addition of a new period may lead to a reordering of the previous links. The above "scientific" method for linking observations may be useful, however, in deciding whether chaining is preferable or whether fixed base indices should be used while making month-to-month comparisons within a year.

15.87 Some index number theorists have objected to the chain principle on the grounds that it has no counterpart in the spatial context:

> They [chain indices] only apply to intertemporal comparisons, and in contrast to direct indices they are not applicable to cases in which no natural order or sequence exists. Thus the idea of a chain index for example has no counterpart in interregional or international price comparisons, because countries cannot be sequenced in a "logical" or "natural" way (there is no k+1 nor k−1country to be compared with country k) (von der Lippe (2001, p. 12)).[67]

This is of course correct, but the approach of Hill does lead to a "natural" set of spatial links. Applying the same approach to the time series context will lead to a set of links between periods which may not be month-to-month but it will in many cases justify year-over-year linking of the data pertaining to the same month. This problem is reconsidered in Chapter 22.

15.88 It is of some interest to determine if there are index number formulae that give the same answer when either the fixed base or chain system is used. Comparing the sequence of chain indices defined by the expression (15.75) to the corresponding fixed base indices, it can be seen that we will obtain the same answer in all three periods if the index number formula P satisfies the following functional equation for all price and quantity vectors:

$$P(p^0,p^2,q^0,q^2) = P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2) \qquad (15.77)$$

If an index number formula P satisfies the equation (15.77), then P satisfies the *circularity test*.[68]

15.89 If it is assumed that the index number formula P satisfies certain properties or tests in addition to the circularity test above,[69] then Funke, Hacker and Voeller (1979) showed that P

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[67] It should be noted that von der Lippe (2001, pp. 56-58) is a vigorous critic of all index number tests based on symmetry in the time series context, although he is willing to accept symmetry in the context of making international comparisons. "But there are good reasons *not* to insist on such criteria in the *intertemporal* case. When no symmetry exists between 0 and t, there is no point in interchanging 0 and t" (von der Lippe (2001, p. 58)).

[68] The test name is attributable to Fisher (1922, p. 413) and the concept originated from Westergaard (1890, pp. 218-219).

[69] The additional tests referred to above are: (i) positivity and continuity of $P(p^0,p^1,q^0,q^1)$ for all strictly positive price and quantity vectors p^0,p^1,q^0,q^1; (ii) the identity test; (iii) the commensurability test; (iv) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree one in the components of p^1, and (v) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree zero in the components of q^1.

must have the following functional form, originally established by Konüs and Byushgens[70] (1926, pp. 163-166):[71]

$$P_{KB}(p^0, p^1, q^0, q^1) \equiv \prod_{i=1}^{n}\left(\frac{p_i^1}{p_i^0}\right)^{\alpha_i} \qquad (15.78)$$

where the n constants α_i satisfy the following restrictions:

$$\sum_{i=1}^{n}\alpha_i = 1 \quad \text{and} \quad \alpha_i > 0 \quad \text{for} \quad i = 1,...,n \qquad (15.79)$$

Thus under very weak regularity conditions, the only price index satisfying the circularity test is a weighted geometric average of all the individual price ratios, the weights being constant through time.

15.90 An interesting special case of the family of indices defined by equation (15.78) occurs when the weights α_i are all equal. In this case, P_{KB} reduces to the Jevons (1865) index:

$$P_J(p^0, p^1, q^0, q^1) \equiv \prod_{i=1}^{n}\left(\frac{p_i^1}{p_i^0}\right)^{\frac{1}{n}} \qquad (15.80)$$

15.91 The problem with the indices defined by Konüs and Byushgens, and Jevons is that the individual price ratios, p_i^1/p_i^0, have weights (either α_i or $1/n$) that are *independent* of the economic importance of commodity i in the two periods under consideration. Put another way, these price weights are independent of the quantities of commodity i consumed or the expenditures on commodity i during the two periods. Hence, these indices are not really suitable for use by statistical agencies at higher levels of aggregation when expenditure share information is available.[72]

15.92 The above results indicate that it is not useful to ask that the price index P satisfy the circularity test *exactly*. It is nevertheless of some interest to find index number formulae that satisfy the circularity test to some degree of approximation, since the use of such an index number formula will lead to measures of aggregate price change that are more or less the same no matter whether we use the chain or fixed base systems. Fisher (1922, p. 284) found that deviations from circularity using his data set and the Fisher ideal price index P_F defined

[70] Konüs and Byushgens show that the index defined by equation (15.78) is exact for Cobb-Douglas (1928) preferences; see also Pollak (1983, pp. 119-120). The concept of an exact index number formula is explained in Chapter 17.

[71] The result in equation (15.78) can be derived using results in Eichhorn (1978, pp. 167-168) and Vogt and Barta (1997, p. 47). A simple proof can be found in Balk (1995). This result vindicates Irving Fisher's (1922, p. 274) intuition that "the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*…". Fisher (1922, p. 275) went on to assert: "But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915, we need, theoretically at least, another set of weights. … Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another."

[72] When there are only two periods being compared and expenditure share information is available for both periods, then the economic approach will suggest in Chapter 17 that good choices for the weights α_i are the arithmetic averages of the period 0 and 1 expenditure shares, s_i^0 and s_i^1.

by equation (15.12) above were quite small. This relatively high degree of correspondence between fixed base and chain indices has been found to hold for other symmetrically weighted formulae, such as the Walsh index P_W defined by equation (15.19).[73] In most time series applications of index number theory where the base year in fixed base indices is changed every five years or so, it will not matter very much whether the statistical agency uses a fixed base price index or a chain index, provided that a symmetrically weighted formula is used.[74] The choice between a fixed base price index or chain index will depend, of course, on the length of the time series considered and the degree of variation in the prices and quantities as we go from period to period. The more prices and quantities are subject to large fluctuations (rather than smooth trends), the less the correspondence.[75]

15.93 It is possible to give a theoretical explanation for the approximate satisfaction of the circularity test for symmetrically weighted index number formulae. Another symmetrically weighted formula is the Törnqvist index P_T.[76] The natural logarithm of this index is defined as follows:

$$\ln P_T(p^0, p^1, q^0, q^1) \equiv \sum_{i=1}^{n} \frac{1}{2}\left(s_i^0 + s_i^1\right)\ln\left(\frac{p_i^1}{p_i^0}\right) \qquad (15.81)$$

where the period t expenditure shares s_i^t are defined by equation (15.7). Alterman, Diewert and Feenstra (1999, p. 61) show that if the logarithmic price ratios $\ln(p_i^t/p_i^{t-1})$ trend linearly with time t and the expenditure shares s_i^t also trend linearly with time, then the Törnqvist index P_T will satisfy the circularity test exactly.[77] Since many economic time series on prices and quantities satisfy these assumptions approximately, the Törnqvist index P_T will satisfy the circularity test approximately. As is seen in Chapter 19, the Törnqvist index generally closely approximates the symmetrically weighted Fisher and Walsh indices, so that for many economic time series (with smooth trends), all three of these symmetrically weighted indices will satisfy the circularity test to a high enough degree of approximation so that it will not matter whether we use the fixed base or chain principle.

15.94 Walsh (1901, p. 401; 1921a, p. 98; 1921b, p. 540) introduced the following useful variant of the circularity test:

$$1 = P(p^0, p^1, q^0, q^1)P(p^1, p^2, q^1, q^2)...P(p^T, p^0, q^T, q^0) \qquad (15.82)$$

[73] See, for example, Diewert (1978, p. 894)). Walsh (1901, pp. 424 and 429) found that his three preferred formulae all approximated each other very well, as did the Fisher ideal for his artificial data set.

[74] More specifically, most superlative indices (which are symmetrically weighted) will satisfy the circularity test to a high degree of approximation in the time series context. See Chapter 17 for the definition of a superlative index. It is worth stressing that fixed base Paasche and Laspeyres indices are very likely to diverge considerably over a five-year period if computers (or any other commodity which has price and quantity trends that are quite different from the trends in the other commodities) are included in the value aggregate under consideration (see Chapter 19 for some "empirical" evidence on this topic).

[75] Again, see Szulc (1983) and Hill (1988).

[76] This formula was implicitly introduced in Törnqvist (1936) and explicitly defined in Törnqvist and Törnqvist (1937).

[77] This exactness result can be extended to cover the case when there are monthly proportional variations in prices, and the expenditure shares have constant seasonal effects in addition to linear trends; see Alterman, Diewert and Feenstra (1999, p. 65).

The motivation for this test is the following. Use the bilateral index formula $P(p^0,p^1,q^0,q^1)$ to calculate the change in prices going from period 0 to 1, use the same formula evaluated at the data corresponding to periods 1 and 2, $P(p^1,p^2,q^1,q^2)$, to calculate the change in prices going from period 1 to 2, … , use $P(p^{T-1},p^T,q^{T-1},q^T)$ to calculate the change in prices going from period $T-1$ to T, introduce an artificial period $T+1$ that has exactly the price and quantity of the initial period 0 and use $P(p^T,p^0,q^T,q^0)$ to calculate the change in prices going from period T to $T+1$. Finally, multiply all of these indices together. Since we end up where we started, the product of all of these indices should ideally be one. Diewert (1993a, p. 40) called this test a *multiperiod identity test*.[78] Note that if $T = 2$ (so that the number of periods is three in total), then Walsh's test reduces to Fisher's (1921, p. 534; 1922, p. 64) time reversal test.[79]

15.95 Walsh (1901, pp. 423-433) showed how his circularity test could be used in order to evaluate how "good" any bilateral index number formula was. What he did was invent artificial price and quantity data for five periods, and he added a sixth period that had the data of the first period. He then evaluated the right-hand side of equation (15.82) for various formulae, $P(p^0,p^1,q^0,q^1)$, and determined how far from unity the results were. His "best" formulae had products that were close to one.[80]

15.96 This same framework is often used to evaluate the efficacy of chained *indices* versus their direct counterparts. Thus if the right-hand side of equation (15.82) turns out to be different from unity, the chained indices are said to suffer from "chain drift". If a formula does suffer from chain drift, it is sometimes recommended that fixed base indices be used in place of chained ones. However, this advice, if accepted, would *always* lead to the adoption of fixed base indices, provided that the bilateral index formula satisfies the identity test, $P(p^0,p^0,q^0,q^0) = 1$. Thus it is not recommended that Walsh's circularity test be used to decide whether fixed base or chained indices should be calculated. It is fair to use Walsh's circularity test, as he originally used it as an approximate method for deciding how "good" a particular index number formula is. To decide whether to chain or use fixed base indices, look at how similar the observations being compared are and choose the method which will best link up the most similar observations.

15.97 Various properties, axioms or tests that an index number formula could satisfy have been introduced in this chapter. In the following chapter, the test approach to index number theory is studied in a more systematic manner.

Appendix 15.1 The relationship between the Paasche and Laspeyres indices

1. Recall the notation used in paragraphs 15.11 to 15.17, above. Define the ith relative price or price relative r_i and the ith quantity relative t_i as follows:

$$r_i \equiv \frac{p_i^1}{p_i^0};\quad t_i \equiv \frac{q_i^1}{q_i^0};\qquad i = 1,\ldots,n \qquad \text{(A15.1.1)}$$

[78] Walsh (1921a, p. 98) called his test the *circular test*,but since Fisher also used this term to describe his transitivity test defined earlier by equation (15.77), it seems best to stick to Fisher's terminology since it is well established in the literature.

[79] Walsh (1921b, pp. 540-541) noted that the time reversal test was a special case of his circularity test.

[80] This is essentially a variant of the methodology that Fisher (1922, p- 284) used to check how well various formulae corresponded to his version of the circularity test.

Using formula (15.8) for the Laspeyres price index P_L and definitions (A15.1.1), we have:

$$P_L = \sum_{i=1}^{n} r_i s_i^0 \equiv r^* \qquad \text{(A15.1.2)}$$

i.e., we define the "average" price relative r^* as the base period expenditure share-weighted average of the individual price relatives, r_i .

2. Using formula (15.6) for the Paasche price index P_P, we have:

$$\begin{aligned} P_P &\equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{m=1}^{n} p_m^0 q_m^1} = \frac{\sum_{i=1}^{n} r_i t_i p_i^0 q_i^0}{\sum_{m=1}^{n} t_m p_m^0 q_m^0} \qquad \text{using definitions (A15.1.1)} \\ &= \frac{\sum_{i=1}^{n} r_i t_i s_i^0}{\sum_{m=1}^{n} t_m s_m^0} = \left\{ \frac{1}{\sum_{m=1}^{n} t_m s_m^0} \sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0 \right\} + r^* \end{aligned} \qquad \text{(A15.1.3)}$$

using (A15.1.2) and $\sum_{i=1}^{n} s_i^0 = 1$ and where the "average" quantity relative t^* is defined as

$$t^* \equiv \sum_{i=1}^{n} t_i s_i^0 = Q_L \qquad \text{(A15.1.4)}$$

where the last equality follows using equation (15.11), the definition of the Laspeyres quantity index Q_L.

3. Taking the difference between P_P and P_L and using equations (A15.1.2)–(A15.1.4) yields:

$$P_P - P_L = \frac{1}{Q_L} \sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0 \qquad \text{(A15.1.5)}$$

Now let r and t be discrete random variables that take on the n values r_i and t_i respectively. Let s_i^0 be the joint probability that $r = r_i$ and $t = t_i$ for $i = 1,\ldots,n$ and let the joint probability be 0 if $r = r_i$ and $t = t_j$ where $i \neq j$. It can be verified that the summation $\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0$ on the right-hand side of equation (A15.1.5) is the covariance between the price relatives r_i and the corresponding quantity relatives t_i. This covariance can be converted into a correlation coefficient.[81] If this covariance is negative, which is the usual case in the consumer context, then P_P will be less than P_L.

Appendix 15.2 The relationship between the Lowe and Laspeyres indices

1. Recall the notation used in paragraphs 15.33 to 15.48, above. Define the ith relative price relating the price of commodity i of month t to month 0, r_i, and the ith quantity relative, t_i, relating quantity of commodity i in base year b to month 0 t_i as follows:

$$r_i \equiv \frac{p_i^t}{p_i^0}\; t_i \equiv \frac{q_i^b}{q_i^0};\;\; i = 1,\ldots, n \qquad \text{(A15.2.1)}$$

As in Appendix A15.1, the Laspeyres price index $P_L(p^0,p^t,q^0)$ can be defined as r^*, the month 0 expenditure share-weighted average of the individual price relatives r_i defined in (A15.2.1)

[81] See Bortkiewicz (1923, pp. 374-375) for the first application of this correlation coefficient decomposition technique.

except that the month t price, p_i^t, now replaces period 1 price, p_i^1, in the definition of the ith price relative r_i:

$$r^* \equiv \sum_{i=1}^{n} r_i s_i^0 = P_L \tag{A15.2.2}$$

2. The "average" quantity relative t^* relating the quantities of base year b to those of month 0 is defined as the month 0 expenditure share-weighted average of the individual quantity relatives t_i, defined in (A15.2.1):

$$t^* \equiv \sum_{i=1}^{n} t_i s_i^0 = Q_L \tag{A15.2.3}$$

where $Q_L = Q_L(q^0,q^b,p^0)$ is the Laspeyres quantity index relating the quantities of month 0, q^0, to those of the year b, q^b, using the prices of month 0, p^0, as weights.

3. Using definition (15.26), the Lowe index comparing the prices in month t to those of month 0, using the quantity weights of the base year b, is equal to:

$$P_{Lo}(p^0,p^t,q^b) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} = \frac{\sum_{i=1}^{n} p_i^t t_i q_i^0}{\sum_{i=1}^{n} p_i^0 t_i q_i^0} \qquad \text{using (A15.2.1)}$$

$$= \left\{\frac{\sum_{i=1}^{n} p_i^t t_i q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0}\right\}\left\{\frac{\sum_{i=1}^{n} p_i^0 t_i q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0}\right\}^{-1}$$

$$= \left\{\frac{\sum_{i=1}^{n}\left(\frac{p_i^t}{p_i^0}\right) t_i p_i^0 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0}\right\} \Big/ t^* \quad \text{using (A15.2.3)}$$

$$= \left\{\frac{\sum_{i=1}^{n} r_i t_i p_i^0 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0}\right\} \Big/ t^* \quad \text{using (A15.2.1)}$$

$$= \frac{\sum_{i=1}^{n} r_i t_i s_i^0}{t^*} = \frac{\sum_{i=1}^{n} (r_i - r^*) t_i s_i^0}{t^*} + \frac{\sum_{i=1}^{n} r^* t_i s_i^0}{t^*}$$

$$= \frac{\sum_{i=1}^{n} (r_i - r^*) t_i s_i^0}{t^*} + \frac{r^*\left[\sum_{i=1}^{n} t_i s_i^0\right]}{t^*}$$

$$= \frac{\sum_{i=1}^{n} (r_i - r^*) t_i s_i^0}{t^*} + \frac{r^*\left[t^*\right]}{t^*} \qquad \text{using (A15.2.3)}$$

$$= \frac{\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0}{t^*} + \frac{\sum_{i=1}^{n} (r_i - r^*) t^* s_i^0}{t^*} + r^*$$

$$= \frac{\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0}{t^*} + \frac{t^*\left[\sum_{i=1}^{n} r_i s_i^0 - r^*\right]}{t^*} + r^*$$

$$= \frac{\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0}{t^*} + r^* \quad \text{since} \sum_{\mathrm{i}=1}^{\mathrm{n}} r_i s_i^0 = r^*$$

$$= P_L(p^0,p^t,q^0) + \frac{\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0}{Q_L(q^0,q^b,p^0)} \qquad \text{(A15.2.4)}$$

since using (A15.2.2), r^* equals the Laspeyres price index, $P_L(p^0,p^t,q^0)$, and using (A15.2.3), t^* equals the Laspeyres quantity index, $Q_L(q^0,q^b,p^0)$. Thus equation (A15.2.4) tells us that the Lowe price index using the quantities of year b as weights, $P_{Lo}(p^0,p^t,q^b)$, is equal to the usual

Laspeyres index using the quantities of month 0 as weights, $P_L(p^0,p^t,q^0)$, plus a covariance term $\sum_{i=1}^{n}(r_i - r^*)(t_i - t^*)s_i^0$ between the price relatives $r_i \equiv p_i^t/p_i^0$ and the quantity relatives $t_i \equiv q_i^b/q_i^0$, divided by the Laspeyres quantity index $Q_L(q^0,q^b,p^0)$ between month 0 and base year b.

Appendix 15.3 The relationship between the Young index and its time antithesis

1. Recall that the direct Young index, $P_Y(p^0,p^t,s^b)$, was defined by equation (15.48) and its time antithesis, $P_Y^*(p^0,p^t,s^b)$, was defined by equation (15.52). Define the ith relative price between months 0 and t as

$$r_i \equiv p_i^t / p_i^0; \qquad i = 1,...,n \qquad \text{(A15.3.1)}$$

and define the weighted average (using the base year weights s_i^b) of the r_i as

$$r^* \equiv \sum_{i=}^{n} s_i^b r_i \qquad \text{(A15.3.2)}$$

which turns out to equal the direct Young index, $P_Y(p^0,p^t,s^b)$. Define the deviation e_i of r_i from their weighted average r^* using the following equations:

$$r_i = r^*(1+e_i); \qquad i = 1,...,n \qquad \text{(A15.3.3)}$$

If equation (A15.3.3) is substituted into equation (A15.3.2), the following equation is obtained:

$$\begin{aligned} r^* &\equiv \sum_{i=1}^{n} s_i^b r^*(1+e_i) \\ &= r^* + r^* \sum_{i=1}^{n} s_i^b e_i \qquad \text{since } \sum_{i=1}^{n} s_i^b = 1 \end{aligned} \qquad \text{(A15.3.4)}$$

$$e^* \equiv \sum_{i=1}^{n} s_i^b e_i = 0 \qquad \text{(A15.3.5)}$$

Thus the weighted mean e^* of the deviations e_i equals 0.

2. The direct Young index, $P_Y(p^0,p^t,s^b)$, and its time antithesis, $P_Y^*(p^0,p^t,s^b)$, can be written as functions of r^*, the weights s_i^b and the deviations of the price relatives e_i as follows:

$$P_Y(p^0, p^t, s^b) = r^* \qquad \text{(A15.3.6)}$$

$$\begin{aligned} P_Y^*(p^0, p^t, s^b) &= \left[\sum_{i=1}^{n} s_i^b \left\{ r^*(1+e_i) \right\}^{-1} \right]^{-1} \\ &= r^* \left[\sum_{i=1}^{n} s_i^b (1+e_i)^{-1} \right]^{-1} \end{aligned} \qquad \text{(A15.3.7)}$$

3. Now regard $P_Y^*(p^0,p^t,s^b)$ as a function of the vector of deviations, $e \equiv [e_1,\ldots,e_n]$, say $P_Y^*(e)$. The second-order Taylor series approximation to $P_Y^*(e)$ around the point $e = 0_n$ is given by the following expression:[82]

[82] This type of second order approximation is attributable to Dalén (1992; 143) for the case r^* =1 and to Diewert (1995a, p. 29) for the case of a general r^*.

$$P_Y^*(e) \approx r^* + r^*\sum_{i=1}^{n} s_i^b e_i + r^*\sum_{i=1}^{n}\sum_{j=1}^{n} s_i^b s_j^b e_i e_j - r^*\sum_{i=1}^{n} s_i^b [e_i]^2$$

$$= r^* + r^* 0 + r^*\sum_{i=1}^{n} s_i^b \left[\sum_{j=1}^{n} s_j^b e_j\right] e_i - r^*\sum_{i=1}^{n} s_i^b \left[e_i - e^*\right]^2 \quad \text{using (A15.3.5)}$$

$$= r^* + r^*\sum_{i=1}^{n} s_i^b [0] e_i - r^*\sum_{i=1}^{n} s_i^b \left[e_i - e^*\right]^2 \quad \text{using (A15.3.5)}$$

$$= P_Y(p^0, p^t, s^b) - P_Y(p^0, p^t, s^b)\sum_{i=1}^{n} s_i^b \left[e_i - e^*\right]^2 \quad \text{using (A15.3.6)}$$

$$= P_Y(p^0, p^t, s^b) - P_Y(p^0, p^t, s^b)\operatorname{Var} e \qquad \text{(A15.3.8)}$$

where the weighted sample variance of the vector e of price deviations is defined as

$$\operatorname{Var} e \equiv \sum_{i=1}^{n} s_i^b \left[e_i - e^*\right]^2 \qquad \text{(A15.3.9)}$$

4. Rearranging equation (A15.3.8) gives the following approximate relationship between the direct Young index $P_Y(p^0,p^t,s^b)$ and its time antithesis $P_Y*(p^0,p^t,s^b)$, to the accuracy of a second-order Taylor series approximation about a price point where the month t price vector is proportional to the month 0 price vector:

$$P_Y(p^0, p^t, s^b) \approx P_Y^*(p^0, p^t, s^b) + P_Y(p^0, p^t, s^b)\ \operatorname{Var} e \qquad \text{(A15.3.10)}$$

Thus, to the accuracy of a second-order approximation, the direct Young index will exceed its time antithesis by a term equal to the direct Young index times the weighted variance of the deviations of the price relatives from their weighted mean. Thus the bigger is the dispersion in relative prices, the more the direct Young index will exceed its time antithesis.

Appendix 15.4 The relationship between the Divisia and economic approaches

1. Divisia's approach to index number theory relied on the theory of differentiation. Thus it does not appear to have any connection with economic theory. However, starting with Ville (1946), a number of economists[83] have established that the Divisia price and quantity indices *do* have a connection with the economic approach to index number theory. This connection is outlined in this appendix.

2. The economic approach to the determination of the price level and the quantity level is first outlined. The particular economic approach that is used here is attributable to Shephard (1953; 1970), Samuelson (1953) and Samuelson and Swamy (1974).

3. It is assumed that "the" consumer has well-defined *preferences* over different combinations of the n consumer commodities or items. Each combination of items can be represented by a positive vector $q \equiv [q_1,\ldots,q_n]$. The consumer's preferences over alternative possible consumption vectors q are assumed to be representable by a continuous, non-decreasing and concave utility function f. It is further assumed that the consumer minimizes the cost of achieving the period t utility level $u^t \equiv f(q^t)$ for periods $t = 0,1,\ldots,T$. Thus it is

[83] See for example Malmquist (1953, p. 227), Wold (1953, pp. 134-147), Solow (1957), Jorgenson and Griliches (1967) and Hulten (1973), and see Balk (2000a) for a recent survey of work on Divisia price and quantity indices.

assumed that the observed period t consumption vector q^t solves the following period t cost minimization problem:

$$C(u^t, p^t) \equiv \min_q \left\{ \sum_{i=1}^{n} p_i^t q_i : f(q) = u^t = f(q^t) \right\}$$
$$= \sum_{i=1}^{n} p_i^t q_i^t ; \quad t = 0,1,...,T \qquad \text{(A15.4.1)}$$

The period t price vector for the n commodities under consideration that the consumer faces is p^t. Note that the solution to the period t cost or expenditure minimization problem defines the *consumer's cost function*, $C(u^t,p^t)$.

4. An additional regularity condition is placed on the consumer's utility function f. It is assumed that f is (positively) linearly homogeneous for strictly positive quantity vectors. Under this assumption, the consumer's expenditure or cost function, $C(u,p)$, decomposes into $uc(p)$ where $c(p)$ is the consumer's unit cost function.[84] The following equation is obtained:

$$\sum_{i=1}^{n} p_i^t q_i^t = c(p^t) f(q^t) \quad \text{for } t = 0,1,\ldots,T \qquad \text{(A15.4.2)}$$

Thus the period t total expenditure on the n commodities in the aggregate, $\sum_{i=1}^{n} p_i^t q_i^t$, decomposes into the product of two terms, $c(p^t)f(q^t)$. The period t unit cost, $c(p^t)$, can be identified as the period t price level P^t and the period t level of utility, $f(q^t)$, can be identified as the period t quantity level Q^t.

5. The economic price level for period t, $P^t \equiv c(p^t)$, defined in the previous paragraph, is now related to the Divisia price level for time t, $P(t)$, that was implicitly defined by the differential equation (15.67). As in paragraphs 15.65 to 15.71, think of the prices as being continuous, differentiable functions of time, $p_i(t)$ say, for $i = 1,\ldots, n$. Thus the unit cost function can be regarded as a function of time t as well; i.e., define the unit cost function as a function of t as

$$c^*(t) \equiv c\left[p_1(t), p_2(t),..., p_n(t) \right] \qquad \text{(A15.4.3)}$$

6. Assuming that the first-order partial derivatives of the unit cost function $c(p)$ exist, calculate the logarithmic derivative of $c^*(t)$ as follows:

$$\frac{d \ln c^*(t)}{dt} \equiv \frac{1}{c^*(t)} \frac{dc^*(t)}{dt}$$
$$= \frac{\sum_{i=1}^{n} c_i \left[p_1(t), p_2(t),..., p_n(t) \right] p_i'(t)}{c\left[p_1(t), p_2(t),..., p_n(t) \right]} \qquad \text{(A15.4.4)}$$

where $c_i[p_1(t),p_2(t),\ldots,p_n(t)] \equiv \partial c[p_1(t),p_2(t),\ldots,p_n(t)]/\partial p_i$ is the partial derivative of the unit cost function with respect to the ith price, p_i, and $p_i'(t) \equiv dp_i(t)/dt$ is the time derivative of the ith price function, $p_i(t)$. Using Shephard's (1953, p. 11) Lemma, the consumer's cost-minimizing demand for commodity i at time t is:

$$q_i(t) = u(t) c_i \left[p_1(t), p_2(t),..., p_n(t) \right] \text{ for } i = 1,...,n \qquad \text{(A15.4.5)}$$

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[84] See Diewert (1993b, pp.120-121) for material on unit cost functions. This material will also be covered in Chapter 17.

where the utility level at time t is $u(t) = f[q_1(t),q_2(t),\ldots,q_n(t)]$. The continuous time counterpart to equations (A15.4.2) above is that total expenditure at time t is equal to total cost at time t which in turn is equal to the utility level, $u(t)$, times the period t unit cost, $c^*(t)$:

$$\sum_{i=1}^{n} p_i(t)q_i(t) = u(t)c^*(t) = u(t)c[p_1(t), p_2(t),\ldots, p_n(t)]. \qquad \text{(A15.4.6)}$$

7. The logarithmic derivative of the Divisia price level $P(t)$ can be written as (recall equation (15.67) above):

$$\begin{aligned}
\frac{P'(t)}{P(t)} &= \frac{\sum_{i=1}^{n} p_i'(t)q_i(t)}{\sum_{i=1}^{n} p_i(t)q_i(t)} = \frac{\sum_{i=1}^{n} p_i'(t)q_i(t)}{u(t)c^*(t)} \text{ using (A15.4.6)} \\
&= \frac{\sum_{i=1}^{n} p_i'(t)\{u(t)c[p_1(t), p_2(t),\ldots, p_n(t)]\}}{u(t)c^*(t)} \text{ using (A15.4.5)} \qquad \text{(A15.4.7)} \\
&= \frac{\sum_{i=1}^{n} c_i[p_1(t), p_2(t),\ldots, p_n(t)]p_i'(t)}{c^*(t)} = \frac{1}{c^*(t)}\frac{dc^*(t)}{dt} \text{ using (A15.4.4)} \\
&\equiv \frac{c^{*\prime}(t)}{c^*(t)}.
\end{aligned}$$

Thus under the above continuous time cost-minimizing assumptions, the Divisia price level, $P(t)$, is essentially equal to the unit cost function evaluated at the time t prices, $c^*(t) \equiv c[p_1(t),p_2(t),\ldots,p_n(t)]$.

8. If the Divisia price level $P(t)$ is set equal to the unit cost function $c^*(t) \equiv c[p_1(t),p_2(t),\ldots,p_n(t)]$, then from equation (A15.4.2), it follows that the Divisia quantity level $Q(t)$ defined by equation (15.68) will equal the consumer's utility function regarded as a function of time, $f^*(t) \equiv f[q_1(t),\ldots,q_n(t)]$. Thus, under the assumption that the consumer is continuously minimizing the cost of achieving a given utility level where the utility or preference function is linearly homogeneous, it has been shown that the Divisia price and quantity levels $P(t)$ and $Q(t)$, defined implicitly by the differential equations (15.67) and (15.68), are essentially equal to the consumer's unit cost function $c^*(t)$ and utility function $f^*(t)$ respectively.[85] These are rather remarkable equalities since in principle, given the functions of time, $p_i(t)$ and $q_i(t)$, the differential equations that define the Divisia price and quantity indices can be solved numerically and hence $P(t)$ and $Q(t)$ are in principle observable (up to some normalizing constants).

9. For more on the Divisia approach to index number theory, see Vogt (1977; 1978) and Balk (2000a). An alternative approach to Divisia indices using line integrals may be found in the forthcoming companion volume *Producer price index manual* (IMF et al., 2004).

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[85] Obviously, the scale of the utility and cost functions are not uniquely determined by the differential equations (15.62) and (15.63).

16 THE AXIOMATIC AND STOCHASTIC APPROACHES TO INDEX NUMBER THEORY

Introduction

16.1 As was seen in Chapter 15, it is useful to be able to evaluate various index number formulae that have been proposed in terms of their properties. If a formula turns out to have rather undesirable properties, this casts doubts on its suitability as an index that could be used by a statistical agency as a target index. Looking at the mathematical properties of index number formulae leads to the *test* or *axiomatic approach to index number theory*. In this approach, desirable properties for an index number formula are proposed, and it is then attempted to determine whether any formula is consistent with these properties or tests. An ideal outcome is the situation where the proposed tests are both desirable and completely determine the functional form for the formula.

16.2 The axiomatic approach to index number theory is not completely straightforward, since choices have to be made in two dimensions:

- The index number framework must be determined.
- Once the framework has been decided upon, it must be decided what tests or properties should be imposed on the index number.

The second point is straightforward: different price statisticians may have different ideas about which tests are important, and alternative sets of axioms can lead to alternative "best" index number functional forms. This point must be kept in mind while reading this chapter, since there is no universal agreement on what the "best" set of "reasonable" axioms is. Hence the axiomatic approach can lead to more than one best index number formula.

16.3 The first point about choices listed above requires further discussion. In the previous chapter, for the most part, the focus was on *bilateral index number theory*; i.e., it was assumed that prices and quantities for the same *n* commodities were given for two periods and the object of the index number formula was to compare the overall level of prices in one period with the other period. In this framework, both sets of price and quantity vectors were regarded as variables which could be independently varied so that, for example, variations in the prices of one period did not affect the prices of the other period or the quantities in either period. The emphasis was on comparing the overall cost of a fixed basket of quantities in the two periods or taking averages of such fixed basket indices. This is an example of an index number framework.

16.4 However, other index number frameworks are possible. For example, instead of decomposing a value ratio into a term that represents price change between the two periods times another term that represents quantity change, an attempt could be made to decompose a value aggregate for one period into a single number that represents the price level in the period times another number that represents the quantity level in the period. In the first variant of this approach, the price index number is supposed to be a function of the *n* commodity prices pertaining to that aggregate in the period under consideration, while the quantity index number is supposed to be a function of the *n* commodity quantities pertaining to the aggregate in the period. The resulting price index function was called an *absolute index number* by Frisch (1930, p. 397), a *price level* by Eichhorn (1978, p. 141) and a *unilateral price index* by Anderson, Jones and Nesmith (1997, p. 75). In a second variant of this approach, the price and quantity functions are allowed to depend on both the price and quantity vectors pertaining to the period under consideration.[1] These two variants of unilateral index number theory will be considered in paragraphs 16.11 to 16.29.[2]

16.5 The remaining approaches in this chapter are largely bilateral approaches; i.e., the prices and quantities in an aggregate are compared for two periods. In paragraphs 16.30 to 16.73 and 16.94 to 16.129, the value ratio decomposition approach is taken.[3] In paragraphs 16.30 to 16.73, the bilateral price and quantity indices, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$, are regarded as functions of the price vectors pertaining to the two periods, p^0 and p^1, and the two quantity vectors, q^0 and q^1. Not only do the axioms or tests that are placed on the price index $P(p^0,p^1,q^0,q^1)$ reflect "reasonable" price index properties, but some tests have their origin as "reasonable" tests on the quantity index $Q(p^0,p^1,q^0,q^1)$. The approach in paragraphs 16.30 to 16.73 simultaneously determines the "best" price and quantity indices.

16.6 In paragraphs 16.74 to 16.93, attention is shifted to the *price ratios* for the n commodities between periods 0 and 1, $r_i \equiv p_i^1/p_i^0$ for $i = 1,\ldots,n$. In the *unweighted stochastic approach to index number theory*, the price index is regarded as an evenly weighted average of the n price relatives or ratios, r_i. Carli (1764) and Jevons (1863; 1865) were the earlier pioneers in this approach to index number theory, with Carli using the arithmetic average of the price relatives and Jevons endorsing the geometric average (but also considering the harmonic average). This approach to index number theory will be covered in paragraphs 16.74 to 16.79. This approach is consistent with a statistical approach that regards each price ratio r_i as a random variable with mean equal to the underlying price index.

16.7 A major problem with the unweighted average of price relatives approach to index number theory is that this approach does not take into account the economic importance of the individual commodities in the aggregate. Young (1812) did advocate some form of rough weighting of the price relatives according to their relative value over the period being considered, but the precise form of the required value weighting was not indicated.[4] It was Walsh (1901, pp. 83-121; 1921a, pp. 81-90), however, who stressed the importance of weighting the individual price ratios, where the weights are functions of the associated values for the commodities in each period and each period is to be treated symmetrically in the resulting formula:

> What we are seeking is to average the variations in the exchange value of one given total sum of money in relation to the several classes of goods, to which several variations [price ratios] must be assigned weights proportional to the relative sizes of the classes. Hence the relative sizes of the classes at both the periods must be considered (Walsh (1901, p. 104)).

[1] Eichhorn (1978, p. 144) and Diewert (1993d, p. 9) considered this approach.

[2] In these unilateral index number approaches, the price and quantity vectors are allowed to vary independently. In yet another index number framework, prices are allowed to vary freely but quantities are regarded as functions of the prices. This leads to the *economic approach to index number theory*, which is considered briefly in Appendix 15.4 of Chapter 15, and in more depth in Chapters 17 and 18.

[3] Recall paragraphs 15.7 to 15.17 of Chapter 15 for an explanation of this approach.

[4] Walsh (1901, p. 84) refers to Young's contributions as follows:

> Still, although few of the practical investigators have actually employed anything but even weighting, they have almost always recognized the theoretical need of allowing for the relative importance of the different classes ever since this need was first pointed out, near the commencement of the century just ended, by Arthur Young. … Arthur Young advised simply that the classes should be weighted according to their importance.

> Commodities are to be weighted according to their importance, or their full values. But the problem of axiometry always involves at least two periods. There is a first period and there is a second period which is compared with it. Price variations[5] have taken place between the two, and these are to be averaged to get the amount of their variation as a whole. But the weights of the commodities at the second period are apt to be different from their weights at the first period. Which weights, then, are the right ones – those of the first period or those of the second? Or should there be a combination of the two sets? There is no reason for preferring either the first or the second. Then the combination of both would seem to be the proper answer. And this combination itself involves an averaging of the weights of the two periods (Walsh (1921a, p. 90)).

16.8 Thus Walsh was the first to examine in some detail the rather intricate problems[6] involved in deciding how to weight the price relatives pertaining to an aggregate, taking into account the economic importance of the commodities in the two periods being considered. Note that the type of index number formula that Walsh was considering was of the form $P(r,v^0,v^1)$, where r is the vector of price relatives which has ith component $r_i = p_i^1/p_i^0$ and v^t is the period t value vector which has ith component $v_i^t = p_i^t q_i^t$ for $t = 0,1$. His suggested solution to this weighting problem was not completely satisfactory but he did at least suggest a very useful framework for a price index, as a value-weighted average of the n price relatives. The first satisfactory solution to the weighting problem was obtained by Theil (1967, pp. 136-137) and his solution is explained in paragraphs 16.79 to 16.93.

16.9 It can be seen that one of Walsh's approaches to index number theory[7] was an attempt to determine the "best" weighted average of the price relatives, r_i. This is equivalent to using an axiomatic approach to try to determine the "best" index of the form $P(r,v^0,v^1)$. This approach is considered in paragraphs 16.94 to 16.129.[8]

16.10 The Young and Lowe indices, discussed in Chapter 15, do not fit precisely into the bilateral framework since the value or quantity weights used in these indices do not

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[5] A price variation is a price ratio or price relative in Walsh's terminology.

[6] Walsh (1901, pp. 104-105) realized that it would not do to simply take the arithmetic average of the values in the two periods, $[v_i^0 + v_i^1]/2$, as the "correct" weight for the ith price relative r_i since, in a period of rapid inflation, this would give too much importance to the period that had the highest prices and he wanted to treat each period symmetrically:

> But such an operation is manifestly wrong. In the first place, the sizes of the classes at each period are reckoned in the money of the period, and if it happens that the exchange value of money has fallen, or prices in general have risen, greater influence upon the result would be given to the weighting of the second period; or if prices in general have fallen, greater influence would be given to the weighting of the second period. Or in a comparison between two countries greater influence would be given to the weighting of the country with the higher level of prices. But it is plain that *the one period, or the one country, is as important, in our comparison between them, as the other, and the weighting in the averaging of their weights should really be even*.

However, Walsh was unable to come up with Theil's (1967) solution to the weighting problem, which was to use the average expenditure share $[s_i^0 + s_i^1]/2$, as the "correct" weight for the ith price relative in the context of using a weighted geometric mean of the price relatives.

[7] Walsh also considered basket-type approaches to index number theory, as was seen in Chapter 15.

[8] In paragraphs 16.94 to 16.129, rather than starting with indices of the form $P(r,v^0,v^1)$, indices of the form $P(p^0,p^1,v^0,v^1)$ are considered. However, if the test of invariance to changes in the units of measurement is imposed on this index, it is equivalent to studying indices of the form $P(r,v^0,v^1)$. Vartia (1976) also used a variation of this approach to index number theory.

necessarily correspond to the values or quantities that pertain to either of the periods that correspond to the price vectors p^0 and p^1. The axiomatic properties of these two indices with respect to their price variables are studied in paragraphs 16.130 to 16.134.

The levels approach to index number theory
An axiomatic approach to unilateral price indices

16.11 Denote the price and quantity of commodity *n* in period *t* by p_i^t and q_i^t respectively for $i = 1,2,\ldots,n$ and $t = 0,1,\ldots,T$. The variable q_i^t is interpreted as the total amount of commodity *i* transacted within period *t*. In order to conserve the value of transactions, it is necessary that p_i^t be defined as a unit value; i.e., p_i^t must be equal to the value of transactions in commodity *i* for period *t* divided by the total quantity transacted, q_i^t. In principle, the period of time should be chosen so that variations in commodity prices within a period are very small compared to their variations between periods.[9] For $t = 0,1,\ldots,T$, and $i = 1,\ldots,n$, define the value of transactions in commodity *i* as $v_i^t \equiv p_i^t q_i^t$ and define the *total value of transactions in period t* as:

$$V^t \equiv \sum_{i=1}^{n} v_i^t = \sum_{i=1}^{n} p_i^t q_i^t \qquad t = 0,1,\ldots,T. \qquad (16.1)$$

16.12 Using the above notation, the following *levels version of the index number problem* is defined as follows: for $t = 0,1,\ldots,T$, find scalar numbers P^t and Q^t such that

$$V^t = P^t Q^t \qquad t = 0,1,\ldots,T. \qquad (16.2)$$

The number P^t is interpreted as an aggregate period *t* price level, while the number Q^t is interpreted as an aggregate period *t* quantity level. The aggregate price level P^t is allowed to be a function of the period *t* price vector, p^t, while the aggregate period *t* quantity level Q^t is allowed to be a function of the period *t* quantity vector, q^t; hence:

$$P^t = c(p^t) \text{ and } Q^t = f(q^t) \qquad t = 0,1,\ldots,T. \qquad (16.3)$$

16.13 The functions *c* and *f* are to be determined somehow. Note that equation (16.3) requires that the functional forms for the price aggregation function *c* and for the quantity

[9] This treatment of prices as unit values over time follows Walsh (1901, p. 96; 1921a, p. 88) and Fisher (1922, p. 318). Fisher and Hicks both had the idea that the length of the period should be short enough so that variations in price within the period could be ignored, as the following quotations indicate:

> Throughout this book "the price" of any commodity or "the quantity" of it for any one year was assumed given. But what is such a price or quantity? Sometimes it is a single quotation for January 1 or July 1, but usually it is an average of several quotations scattered throughout the year. The question arises: On what principle should this average be constructed? The *practical* answer is *any* kind of average since, ordinarily, the variation during a year, so far, at least, as prices are concerned, are too little to make any perceptible difference in the result, whatever kind of average is used. Otherwise, there would be ground for subdividing the year into quarters or months until we reach a small enough period to be considered practically a point. The quantities sold will, of course, vary widely. What is needed is their sum for the year (which, of course, is the same thing as the simple arithmetic average of the per annum rates for the separate months or other subdivisions). In short, the simple arithmetic average, both of prices and of quantities, may be used. Or, if it is worth while to put any finer point on it, we may take the weighted arithmetic average for the prices, the weights being the quantities sold (Fisher (1922, p. 318)).
>
> I shall define a week as that period of time during which variations in prices can be neglected. For theoretical purposes this means that prices will be supposed to change, not continuously, but at short intervals. The calendar length of the week is of course quite arbitrary; by taking it to be very short, our theoretical scheme can be fitted as closely as we like to that ceaseless oscillation which is a characteristic of prices in certain markets (Hicks (1946, p. 122)).

aggregation function f be independent of time. This is a reasonable requirement since there is no reason to change the method of aggregation as time changes.

16.14 Substituting equations (16.3) and (16.2) into equation (16.1) and dropping the superscripts t means that c and f must satisfy the following functional equation for all strictly positive price and quantity vectors:

$$c(p)f(q)=\sum_{i=1}^{n}p_i q_i \qquad \text{for all } p_i>0 \text{ and for all } q_i>0. \qquad (16.4)$$

16.15 It is natural to assume that the functions $c(p)$ and $f(q)$ are positive if all prices and quantities are positive:

$$c(p_1,\ldots,p_n)>0;\ f(q_1,\ldots,q_n)>0 \quad \text{if all } p_i>0 \text{ and all } q_i>0. \qquad (16.5)$$

16.16 Let 1_n denote an n-dimensional vector of ones. Then (16.5) implies that when $p = 1_n$, $c(1_n)$ is a positive number, a for example, and when $q = 1_n$, then $f(1_n)$ is also a positive number, b for example; i.e., (16.5) implies that c and f satisfy:

$$c(1_n)=a>0;\ f(1_n)=b>0. \qquad (16.6)$$

16.17 Let $p = 1_n$ and substitute the first equation in (16.6) into equation (16.4) in order to obtain the following equation:

$$f(q)=\sum_{i=1}^{n}\frac{q_i}{a} \qquad \text{for all } q_i>0. \qquad (16.7)$$

16.18 Now let $q = 1_n$ and substitute the second equation in (16.6) into equation (16.4) in order to obtain the following equation:

$$c(p)=\sum_{i=1}^{n}\frac{p_i}{b} \qquad \text{for all } p_i>0. \qquad (16.8)$$

16.19 Finally substitute equations (16.7) and (16.8) into the left-hand side of equation (16.4) to obtain the following equation:

$$\left(\sum_{i=1}^{n}\frac{p_i}{b}\right)\left(\sum_{i=1}^{n}\frac{q_i}{a}\right)=\sum_{i=1}^{n}p_i q_i \qquad \text{for all } p_i>0 \text{ and for all } q_i>0. \qquad (16.9)$$

If n is greater than one, it is obvious that equation (16.9) cannot be satisfied for all strictly positive p and q vectors. Thus if the number of commodities n exceeds one, then there do not exist any functions c and f that satisfy equations (16.4) and (16.5).[10]

16.20 Thus this levels test approach to index number theory comes to an abrupt halt; it is fruitless to look for price and quantity level functions, $P^t = c(p^t)$ and $Q^t = f(q^t)$, that satisfy equations (16.2) or (16.4) and also satisfy the very reasonable positivity requirements (16.5).

16.21 Note that the levels price index function, $c(p^t)$, did not depend on the corresponding quantity vector q^t and the levels quantity index function, $f(q^t)$, did not depend on the price vector p^t. Perhaps this is the reason for the rather negative result obtained above. Hence, in the next section, the price and quantity functions are allowed to be functions of both p^t and q^t.

[10] Eichhorn (1978, p. 144) established this result.

A second axiomatic approach to unilateral price indices

16.22 In this section, the goal is to find functions of 2*n* variables, $c(p,q)$ and $f(p,q)$ such that the following counterpart to equation (16.4) holds:

$$c(p,q)\,f(p,q)=\sum_{i=1}^{n} p_i q_i \qquad \text{for all } p_i>0 \text{ and for all } q_i>0. \qquad (16.10)$$

16.23 Again, it is natural to assume that the functions $c(p,q)$ and $f(p,q)$ are positive if all prices and quantities are positive:

$$c(p_1,\ldots,p_n;q_1,\ldots,q_n)>0;\ f(p_1,\ldots,p_n;q_1,\ldots,q_n)>0 \ \text{ if all } p_i>0 \text{ and all } q_i>0. \quad (16.11)$$

16.24 The present framework does not distinguish between the functions *c* and *f,* so it is necessary to require that these functions satisfy some "reasonable" properties. The first property imposed on *c* is that this function be homogeneous of degree one in its price components:

$$c(\lambda p,q)=\lambda\, c(p,q) \qquad \text{for all } \lambda>0. \qquad (16.12)$$

Thus, if all prices are multiplied by the positive number λ, then the resulting price index is λ times the initial price index. A similar linear homogeneity property is imposed on the quantity index *f*; i.e., *f* is to be homogeneous of degree one in its quantity components:

$$f(p,\lambda q)=\lambda\, f(p,q) \qquad \text{for all } \lambda>0. \qquad (16.13)$$

16.25 Note that properties (16.10), (16.11) and (16.13) imply that the price index $c(p,q)$ has the following homogeneity property with respect to the components of *q*:

$$\begin{aligned} c(p,\lambda q) &= \sum_{i=1}^{n}\frac{p_i\lambda q_i}{f(p,\lambda q)} && \text{where } \lambda>0 \\ &= \sum_{i=1}^{n}\frac{p_i\lambda q_i}{\lambda f(p,q)} && \text{using (16.13)} \\ &= \sum_{i=1}^{n}\frac{p_i q_i}{f(p,q)} \\ &= c(p,q) && \text{using (16.10) and (16.11).} \end{aligned} \qquad (16.14)$$

Thus $c(p,q)$ is homogeneous of degree 0 in its *q* components.

16.26 A final property that is imposed on the levels price index $c(p,q)$ is the following one. Let the positive numbers d_i be given. Then it is asked that the price index be invariant to changes in the units of measurement for the *n* commodities so that the function $c(p,q)$ has the following property:

$$c(d_1p_1,\ldots,d_np_n;q_1/d_1,\ldots,q_n/d_n)=c(p_1,\ldots,p_n;q_1,\ldots,q_n). \qquad (16.15)$$

16.27 It is now possible to show that properties (16.10), (16.11), (16.12), (16.14) and (16.15) on the price levels function $c(p,q)$ are inconsistent; i. e., there does not exist a function of 2*n* variables $c(p,q)$ that satisfies these very reasonable properties.[11]

[11] This proposition is due to Diewert (1993d, p. 9), but his proof is an adaptation of a closely related result due to Eichhorn(1978, pp. 144-145).

16.28 To see why this is so, apply the equation (16.15), setting $d_i = q_i$ for each i, to obtain the following equation:

$$c(p_1,..., p_n; q_1,..., q_n) = c(p_1q_1,..., p_nq_n;1,...,1). \qquad (16.16)$$

If $c(p,q)$ satisfies the linear homogeneity property (16.12) so that $c(\lambda p,q) = \lambda c(p,q)$, then equation (16.16) implies that $c(p,q)$ is also linearly homogeneous in q so that $c(p,\lambda q) = \lambda c(p,q)$. But this last equation contradicts equation (16.14), which establishes the impossibility result.

16.29 The rather negative results obtained in paragraphs 16.13 to 16.21 indicate that it is fruitless to pursue the axiomatic approach to the determination of price and quantity levels, where both the price and quantity vector are regarded as independent variables.[12] Hence, in the following sections of this chapter, the axiomatic approach to the determination of a *bilateral price index* of the form $P(p^0,p^1,q^0,q^1)$ will be pursued.

The first axiomatic approach to bilateral price indices

Bilateral indices and some early tests

16.30 In this section, the strategy will be to assume that the bilateral price index formula, $P(p^0,p^1,q^0,q^1)$, satisfies a sufficient number of "reasonable" tests or properties so that the functional form for P is determined.[13] The word "bilateral"[14] refers to the assumption that the function P depends only on the data pertaining to the two situations or periods being compared; i.e., P is regarded as a function of the two sets of price and quantity vectors, p^0,p^1,q^0,q^1, that are to be aggregated into a single number that summarizes the overall change in the n price ratios, $p_1^1/p_1^0,\ldots, p_n^1/p_n^0$.

16.31 In this section, the value ratio decomposition approach to index number theory will be taken; i.e., along with the price index $P(p^0,p^1,q^0,q^1)$, there is a companion quantity index $Q(p^0,p^1,q^0,q^1)$ such that the product of these two indices equals the value ratio between the two periods.[15] Thus, throughout this section, it is assumed that P and Q satisfy the following *product test*:

$$V^1/V^0 = P(p^0,p^1,q^0,q^1)\, Q(p^0,p^1,q^0,q^1). \qquad (16.17)$$

The period t values, V^t, for $t = 0,1$ are defined by equation (16.1). As soon as the functional form for the price index P is determined, then equation (16.17) can be used to determine the functional form for the quantity index Q. A further advantage of assuming that the product test holds is that, if a reasonable test is imposed on the quantity index Q, then equation

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[12] Recall that in the economic approach, the price vector p is allowed to vary independently, but the corresponding quantity vector q is regarded as being determined by p.

[13] Much of the material in this section is drawn from sections 2 and 3 of Diewert (1992a). For more recent surveys of the axiomatic approach see Balk (1995) and Auer (2001).

[14] Multilateral index number theory refers to the case where there are more than two situations whose prices and quantities need to be aggregated.

[15] See paragraphs 15.7 to 15.25 of Chapter 15 for more on this approach, which was initially due to Fisher (1911, p. 403; 1922).

(16.17) can be used to translate this test on the quantity index into a corresponding test on the price index P.[16]

16.32 If $n = 1$, so that there is only one price and quantity to be aggregated, then a natural candidate for P is p_1^1/p_1^0 , the single price ratio, and a natural candidate for Q is q_1^1/q_1^0 , the single quantity ratio. When the number of commodities or items to be aggregated is greater than 1, then what index number theorists have done over the years is propose properties or tests that the price index P should satisfy. These properties are generally multi-dimensional analogues to the one good price index formula, p_1^1/p_1^0. Below, some 20 tests are listed that turn out to characterize the Fisher ideal price index.

16.33 It will be assumed that every component of each price and quantity vector is positive; i.e., $p^t >> 0_n$ and $q^t >> 0_n$ [17] for $t = 0,1$. If it is desired to set $q^0 = q^1$, the common quantity vector is denoted by q; if it is desired to set $p^0 = p^1$, the common price vector is denoted by p.

16.34 The first two tests, denoted T1 and T2, are not very controversial, so they will not be discussed in detail.

T1: *Positivity*:[18] $P(p^0,p^1,q^0,q^1) > 0$.

T2: *Continuity*:[19] $P(p^0,p^1,q^0,q^1)$ is a continuous function of its arguments.

16.35 The next two tests, T3 and T4, are somewhat more controversial.

T3: *Identity or constant prices test*:[20] $P(p,p,q^0,q^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. The controversial aspect of this test is that the two quantity vectors are allowed to be different in the test.[21]

T4: *Fixed basket or constant quantities test*:[22] $P(p^0, p^1, q, q) = \dfrac{\sum_{i=1}^{n} p_i^1 q_i}{\sum_{i=1}^{n} p_i^0 q_i}$.

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[16] This observation was first made by Fisher (1911, pp. 400-406), and the idea was pursued by Vogt (1980) and Diewert (1992a).

[17] The notation $q >> 0_n$ means that each component of the vector q is positive; $q \geq 0_n$ means each component of q is non-negative and $q > 0_n$ means $q \geq 0_n$ and $q \neq 0_n$.

[18] Eichhorn and Voeller (1976, p. 23) suggested this test.

[19] Fisher (1922, pp. 207-215) informally suggested the essence of this test.

[20] Laspeyres (1871, p. 308), Walsh (1901, p. 308) and Eichhorn and Voeller (1976, p. 24) have all suggested this test. Laspeyres came up with this test or property to discredit the ratio of unit values index of Drobisch (1871a), which does not satisfy this test. This test is also a special case of Fisher's (1911, pp. 409-410) price proportionality test.

[21] Usually, economists assume that, given a price vector p, the corresponding quantity vector q is uniquely determined. Here, the same price vector is used but the corresponding quantity vectors are allowed to be different.

[22] The origins of this test go back at least 200 years to the Massachusetts legislature, which used a constant basket of goods to index the pay of Massachusetts soldiers fighting in the American Revolution; see Willard Fisher (1913). Other researchers who have suggested the test over the years include: Lowe (1823, Appendix, p.

(continued)

That is, if quantities are constant during the two periods so that $q^0 = q^1 \equiv q$, then the price index should equal the expenditure on the constant basket in period 1, $\sum_{i=1}^{n} p_i^1 q_i$, divided by the expenditure on the basket in period 0, $\sum_{i=1}^{n} p_i^0 q_i$.

16.36 If the price index P satisfies Test T4 and P and Q jointly satisfy the product test (16.17) above, then it is easy to show[23] that Q must satisfy the identity test $Q(p^0,p^1,q,q) = 1$ for all strictly positive vectors p^0,p^1,q. This *constant quantities test* for Q is also somewhat controversial since p^0 and p^1 are allowed to be different.

Homogeneity tests

16.37 The following four tests, T5–T8, restrict the behaviour of the price index P as the scale of any one of the four vectors p^0,p^1,q^0,q^1 changes.

T5: *Proportionality in current prices*:[24]

$$P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1) \text{ for } \lambda > 0.$$

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. Most index number theorists regard this property as a very fundamental one that the index number formula should satisfy.

16.38 Walsh (1901) and Fisher (1911, p. 418; 1922, p. 420) proposed the related proportionality test $P(p,\lambda p,q^0,q^1) = \lambda$. This last test is a combination of T3 and T5; in fact Walsh (1901, p. 385) noted that this last test implies the identity test, T3.

16.39 In the next test, instead of multiplying all period 1 prices by the same number, all period 0 prices are multiplied by the number λ.

T6: *Inverse proportionality in base period prices*:[25]

$$P(\lambda p^0,p^1,q^0,q^1) = \lambda^{-1} P(p^0,p^1,q^0,q^1) \quad \text{for } \lambda > 0.$$

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0.

16.40 The following two homogeneity tests can also be regarded as invariance tests.

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95), Scrope (1833, p. 406), Jevons (1865), Sidgwick (1883, pp. 67-68), Edgeworth (1925, p. 215) originally published in 1887, Marshall (1887, p. 363), Pierson (1895, p. 332), Walsh (1901, p. 540; 1921b, pp. 543-544), and Bowley (1901, p. 227). Vogt and Barta (1997, p. 49) correctly observe that this test is a special case of Fisher's (1911, p. 411) proportionality test for quantity indexes which Fisher (1911, p. 405) translated into a test for the price index using the product test (15.3).

[23] See Vogt (1980, p. 70).

[24] This test was proposed by Walsh (1901, p. 385), Eichhorn and Voeller (1976, p. 24) and Vogt (1980, p. 68).

[25] Eichhorn and Voeller (1976, p. 28) suggested this test.

T7: *Invariance to proportional changes in current quantities*:

$P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if current period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 1 quantity vector q^1. Vogt (1980, p. 70) was the first to propose this test[26] and his derivation of the test is of some interest. Suppose the quantity index Q satisfies the quantity analogue to the price test T5; i.e., suppose Q satisfies $Q(p^0,p^1,q^0,\lambda q^1) = \lambda Q(p^0,p^1,q^0,q^1)$ for $\lambda > 0$. Then, using the product test (16.17), it can be seen that P must satisfy T7.

T8: *Invariance to proportional changes in base quantities*:[27]

$P(p^0,p^1,\lambda q^0,q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if base period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 0 quantity vector q^0. If the quantity index Q satisfies the following counterpart to T8: $Q(p^0,p^1,\lambda q^0,q^1) = \lambda^{-1}Q(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$, then using equation (16.17), the corresponding price index P must satisfy T8. This argument provides some additional justification for assuming the validity of T8 for the price index function P.

16.41 T7 and T8 together impose the property that the price index P does not depend on the *absolute* magnitudes of the quantity vectors q^0 and q^1.

Invariance and symmetry tests

16.42 The next five tests, T9–T13, are invariance or symmetry tests. Fisher (1922, pp. 62-63, 458-460) and Walsh (1901, p. 105; 1921b, p. 542) seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, pp. 62-63) spoke of fairness but it is clear that he had symmetry properties in mind. It is perhaps unfortunate that he did not realize that there were more symmetry and invariance properties than the ones he proposed; if he had, it is likely that he would have been able to provide an axiomatic characterization for his ideal price index, as is done in paragraphs 16.53 to 16.56. The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed:

T9: *Commodity reversal test* (or invariance to changes in the ordering of commodities):

$P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p^0,p^1,q^0,q^1)$

where p^{t*} denotes a permutation of the components of the vector p^t and q^{t*} denotes the same permutation of the components of q^t for $t = 0,1$. This test is attributable to Fisher (1922, p. 63)[28] and it is one of his three famous reversal tests. The other two are the time reversal test and the factor reversal test, which are considered below.

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[26] Fisher (1911, p. 405) proposed the related test $P(p^0,p^1,q^0,\lambda q^0) = P(p^0,p^1,q^0,q^0) = \sum_{i=1}^{n} p_i^1 q_i^0 \Big/ \sum_{i=1}^{n} p_i^0 q_i^0$.

[27] This test was proposed by Diewert (1992a, p. 216).

[28] "This [test] is so simple as never to have been formulated. It is merely taken for granted and observed instinctively. Any rule for averaging the commodities must be so general as to apply interchangeably to all of the terms averaged". (Fisher (1922, p. 63))

(continued)

16.43 The next test asks that the index be invariant to changes in the units of measurement.

T10: *Invariance to changes in the units of measurement* (commensurability test):

$P(\alpha_1 p_1^0,...,\alpha_n p_n^0;\ \alpha_1 p_1^1,...,\alpha_n p_n^1;\ \alpha_1^{-1} q_1^0,...,\alpha_n^{-1} q_n^0;\ \alpha_1^{-1} q_1^1,...,\alpha_n^{-1} q_n^1) =$
$P(p_1^0,...,p_n^0;\ p_1^1,...,p_n^1;\ q_1^0,...,q_n^0;\ q_1^1,...,q_n^1)$ for all $\alpha_1 > 0,\ \ldots,\ \alpha_n > 0$.

That is, the price index does not change if the units of measurement for each commodity are changed. The concept of this test is attributable to Jevons (1863, p. 23) and the Dutch economist Pierson (1896, p. 131), who criticized several index number formulae for not satisfying this fundamental test. Fisher (1911, p. 411) first called this test the *change of units test*; later, Fisher (1922, p. 420) called it the *commensurability test*.

16.44 The next test asks that the formula be invariant to the period chosen as the base period.

T11: *Time reversal test*: $P(p^0,p^1,q^0,q^1) = 1/P(p^1,p^0,q^1,q^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio, this test will be satisfied (as are all the other tests listed in this section). When the number of goods is greater than one, many commonly used price indices fail this test; e.g., the Laspeyres (1871) price index, P_L defined by equation (15.5) in Chapter 15, and the Paasche (1874) price index, P_P defined by equation (15.6) in Chapter 15, both fail this fundamental test. The concept of the test is attributable to Pierson (1896, p. 128), who was so upset by the fact that many of the commonly used index number formulae did not satisfy this test that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901, p. 368; 1921b, p. 541) and Fisher (1911, p. 534; 1922, p. 64).

16.45 The next two tests are more controversial, since they are not necessarily consistent with the economic approach to index number theory. These tests are, however, quite consistent with the weighted stochastic approach to index number theory, discussed later in this chapter.

T12: *Quantity reversal test* (quantity weights symmetry test):
$P(p^0,p^1,q^0,q^1) = P(p^0,p^1,q^1,q^0)$.

That is, if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities are used to weight the prices in the index number formula, then the period 0 quantities q^0 and the period 1 quantities q^1 must enter the formula in a symmetric or even-handed manner. Funke and Voeller (1978, p. 3) introduced this test; they called it the *weight property*.

16.46 The next test is the analogue to T12 applied to quantity indices:

T13: *Price reversal test* (price weights symmetry test):[29]

[29] This test was proposed by Diewert (1992a, p. 218).

$$\left(\frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0}\right) \Big/ P(p^0, p^1, q^0, q^1) = \left(\frac{\sum_{i=1}^{n} p_i^0 q_i^1}{\sum_{i=1}^{n} p_i^1 q_i^0}\right) \Big/ P(p^1, p^0, q^0, q^1) \qquad (16.18)$$

Thus if we use equation (16.17) to define the quantity index Q in terms of the price index P, then it can be seen that T13 is equivalent to the following property for the associated quantity index Q:

$$Q(p^0, p^1, q^0, q^1) = Q(p^1, p^0, q^0, q^1) \qquad (16.19)$$

That is, if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property T13 implies that these prices enter the quantity index in a symmetric manner.

Mean value tests

16.47 The next three tests, T14–T16, are mean value tests.

T14: *Mean value test for prices*:[30]

$$\min_i(p_i^1 / p_i^0 \ : i = 1,...,n) \le P(p^0, p^1, q^0, q^1) \le \max_i(p_i^1 / p_i^0 \ : i = 1,...,n) \qquad (16.20)$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is supposed to be interpreted as some sort of an average of the n price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfy this test.

16.48 The next test is the analogue to T14 applied to quantity indices:

T15: *Mean value test for quantities*:[31]

$$\min_i(q_i^1 / q_i^0 \ : i = 1,...,n) \le \frac{(V^1/V^0)}{P(p^0, p^1, q^0, q^1)} \le \max_i(q_i^1 / q_i^0 \ : i = 1,...,n) \qquad (16.21)$$

where V^t is the period t value for the aggregate defined by equation (16.1). Using the product test (16.17) to define the quantity index Q in terms of the price index P, it can be seen that T15 is equivalent to the following property for the associated quantity index Q:

$$\min_i(q_i^1/q_i^0 \ : i = 1,...,n) \le Q(p^0, p^1, q^0, q^1) \le \max_i(q_i^1/q_i^0 \ : i = 1,...,n) \qquad (16.22)$$

That is, the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth q_i^1/q_i^0 of the individual quantities.

16.49 In paragraphs 15.18 to 15.32 of Chapter 15, it was argued that it is very reasonable to take an average of the Laspeyres and Paasche price indices as a single "best" measure of overall price change. This point of view can be turned into a test:

T16: *Paasche and Laspeyres bounding test*:[32]

The price index P lies between the Laspeyres and Paasche indices, P_L and P_P, defined by equations (15.5) and (15.6) in Chapter 15.

A test could be proposed where the implicit quantity index Q that corresponds to P via equation (16.17) is to lie between the Laspeyres and Paasche quantity indices, Q_P and Q_L,

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[30] This test seems to have been first proposed by Eichhorn and Voeller (1976, p. 10).

[31] This test was proposed by Diewert (1992a, p. 219).

[32] Bowley (1901, p. 227) and Fisher (1922, p. 403) both endorsed this property for a price index.

defined by equations (15.10) and (15.11) in Chapter 15. However, the resulting test turns out to be equivalent to test T16.

Monotonicity tests

16.50 The final four tests, T17–T20, are monotonicity tests; i.e., how should the price index $P(p^0,p^1,q^0,q^1)$ change as any component of the two price vectors p^0 and p^1 increases or as any component of the two quantity vectors q^0 and q^1 increases?

T17: *Monotonicity in current prices*:

$P(p^0,p^1,q^0,q^1) < P(p^0,p^2,q^0,q^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase, so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1. This property was proposed by Eichhorn and Voeller (1976, p. 23) and it is a very reasonable property for a price index to satisfy.

T18: *Monotonicity in base prices*: $P(p^0,p^1,q^0,q^1) > P(p^2,p^1,q^0,q^1)$ if $p^0 < p^2$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0. This very reasonable property was also proposed by Eichhorn and Voeller (1976, p. 23).

T19: *Monotonicity in current quantities*: if $q^1 < q^2$, then

$$\left(\frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0}\right) \Big/ P(p^0,p^1,q^0,q^1) < \left(\frac{\sum_{i=1}^{n} p_i^1 q_i^2}{\sum_{i=1}^{n} p_i^0 q_i^0}\right) \Big/ P(p^0,p^1,q^0,q^2) \qquad (16.23)$$

T20: *Monotonicity in base quantities*: if $q^0 < q^2$, then

$$\left(\frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0}\right) \Big/ P(p^0,p^1,q^0,q^1) > \left(\frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^2}\right) \Big/ P(p^0,p^1,q^2,q^1). \qquad (16.24)$$

16.51 Let Q be the implicit quantity index that corresponds to P using equation (16.17). Then it is found that T19 translates into the following inequality involving Q:

$$Q(p^0,p^1,q^0,q^1) < Q(p^0,p^1,q^0,q^2) \quad \text{if} \quad q^1 < q^2 \qquad (16.25)$$

That is, if any period 1 quantity increases, then the implicit quantity index Q that corresponds to the price index P must increase. Similarly, we find that T20 translates into:

$$Q(p^0,p^1,q^0,q^1) > Q(p^0,p^1,q^2,q^1) \quad \text{if} \quad q^0 < q^2 \qquad (16.26)$$

That is, if any period 0 quantity increases, then the implicit quantity index Q must decrease. Tests T19 and T20 are attributable to Vogt (1980, p. 70).

16.52 This concludes the listing of tests. The next section offers an answer to the question of whether any index number formula $P(p^0,p^1,q^0,q^1)$ exists that can satisfy all 20 tests.

The Fisher ideal index and the test approach

16.53 It can be shown that the only index number formula $P(p^0,p^1,q^0,q^1)$ which satisfies tests T1-T20 is the Fisher ideal price index P_F defined as the geometric mean of the Laspeyres and Paasche indices:[33]

[33] See Diewert (1992a, p. 221).

$$P_F(p^0,p^1,q^0,q^1) \equiv \left\{P_L(p^0,p^1,q^0,q^1)\ \ P_P(p^0,p^1,q^0,q^1)\right\}^{1/2} \tag{16.27}$$

16.54 It is relatively straightforward to show that the Fisher index satisfies all 20 tests. The more difficult part of the proof is to show that the Fisher index is the *only* index number formula that satisfies these tests. This part of the proof follows from the fact that, if P satisfies the positivity test T1 and the three reversal tests, T11-T13, then P must equal P_F. To see this, rearrange the terms in the statement of test T13 into the following equation:

$$\frac{\sum_{i=1}^{n} p_i^1 q_i^1 \Big/ \sum_{i=1}^{n} p_i^0 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^1 \Big/ \sum_{i=1}^{n} p_i^1 q_i^0} = \frac{P(p^0,p^1,q^0,q^1)}{P(p^1,p^0,q^0,q^1)}$$

$$= \frac{P(p^0,p^1,q^0,q^1)}{P(p^1,p^0,q^1,q^0)} \quad \text{using T12, the quantity reversal test} \tag{16.28}$$

$$= P(p^0,p^1,q^0,q^1)P(p^0,p^1,q^0,q^1) \quad \text{using T11, the time reversal test.}$$

Now take positive square roots of both sides of equation (16.28). It can be seen that the left-hand side of the equation is the Fisher index $P_F(p^0,p^1,q^0,q^1)$ defined by equation (16.27) and the right-hand side is $P(p^0,p^1,q^0,q^1)$. Thus if P satisfies T1, T11, T12 and T13, it must equal the Fisher ideal index P_F.

16.55 The quantity index that corresponds to the Fisher price index using the product test (16.17) is Q_F , the Fisher quantity index, defined by equation (15.14) in Chapter 15.

16.56 It turns out that P_F satisfies yet another test, T21, which was Fisher's (1921, p. 534; 1922, pp. 72-81) third reversal test (the other two being T9 and T11):

T21: *Factor reversal test* (functional form symmetry test):

$$P(p^0,p^1,q^0,q^1)P(q^0,q^1,p^0,p^1) = \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0} \tag{16.29}$$

A justification for this test is the following: if $P(p^0,p^1,q^0,q^1)$ is a good functional form for the price index, then, if the roles of prices and quantities are reversed, $P(q^0,q^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument) and thus the product of the price index $P(p^0,p^1,q^0,q^1)$ and the quantity index $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ ought to equal the value ratio, V^1 / V^0 . The second part of this argument does not seem to be valid, and thus many researchers over the years have objected to the factor reversal test. Nevertheless, if T21 is accepted as a basic test, Funke and Voeller (1978, p. 180) showed that the only index number function $P(p^0,p^1,q^0,q^1)$ which satisfies T1 (positivity), T11 (time reversal test), T12 (quantity reversal test) and T21 (factor reversal test) is the Fisher ideal index P_F defined by equation (16.27). Thus the price reversal test T13 can be replaced by the factor reversal test in order to obtain a minimal set of four tests that lead to the Fisher price index.[34]

[34] Other characterizations of the Fisher price index can be found in Funke and Voeller (1978) and Balk (1985; 1995).

The test performance of other indices

16.57 The Fisher price index P_F satisfies all 20 of the tests T1–T20 listed above. Which tests do other commonly used price indices satisfy? Recall the Laspeyres index P_L defined by equation (15.5), the Paasche index P_P defined by equation (15.6), the Walsh index P_W defined by equation (15.19) and the Törnqvist index P_T defined by equation (15.81) in Chapter 15.

16.58 Straightforward computations show that the Paasche and Laspeyres price indices, P_L and P_P, fail only the three reversal tests, T11, T12 and T13. Since the quantity and price reversal tests, T12 and T13, are somewhat controversial and hence can be discounted, the test performance of P_L and P_P seems at first sight to be quite good. The failure of the time reversal test, T11, is nevertheless a severe limitation associated with the use of these indices.

16.59 The Walsh price index, P_W, fails four tests: T13, the price reversal test; T16, the Paasche and Laspeyres bounding test; T19, the monotonicity in current quantities test; and T20, the monotonicity in base quantities test.

16.60 Finally, the Törnqvist price index P_T fails nine tests: T4 (the fixed basket test), the quantity and price reversal tests T12 and T13, T15 (the mean value test for quantities), T16 (the Paasche and Laspeyres bounding test) and the four monotonicity tests T17 to T20. Thus the Törnqvist index is subject to a rather high failure rate from the viewpoint of this axiomatic approach to index number theory.[35]

16.61 The tentative conclusion that can be drawn from the above results is that, from the viewpoint of this particular bilateral test approach to index numbers, the Fisher ideal price index P_F appears to be "best" since it satisfies all 20 tests. The Paasche and Laspeyres indices are next best if we treat each test as being equally important. Both of these indices, however, fail the very important time reversal test. The remaining two indices, the Walsh and Törnqvist price indices, both satisfy the time reversal test but the Walsh index emerges as being "better" since it passes 16 of the 20 tests whereas the Törnqvist only satisfies 11 tests.[36]

The additivity test

16.62 There is an additional test that many national income accountants regard as very important: the *additivity test*. This is a test or property that is placed on the implicit quantity index $Q(p^0,p^1,q^0,q^1)$ that corresponds to the price index $P(p^0,p^1,q^0,q^1)$ using the product test (16.17). This test states that the implicit quantity index has the following form:

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[35] It is shown in Chapter 19, however, that the Törnqvist index approximates the Fisher index quite closely using "normal" time series data that are subject to relatively smooth trends. Hence, under these circumstances, the Törnqvist index can be regarded as passing the 20 tests to a reasonably high degree of approximation.

[36] This assertion needs to be qualified: there are many other tests that we have not discussed, and price statisticians might hold different opinions regarding the importance of satisfying various sets of tests. Other tests are discussed by Auer (2001; 2002), Eichhorn and Voeller (1976), Balk (1995) and Vogt and Barta (1997), among others. It is shown in paragraphs 16.101 to 16.135 that the Törnqvist index is ideal when considered under a different set of axioms.

$$Q(p^0, p^1, q^0, q^1) = \frac{\sum_{i=1}^{n} p_i^* q_i^1}{\sum_{m=1}^{n} p_m^* q_m^0} \qquad (16.30)$$

where the common across-periods *price* for commodity i, p_i^* for $i = 1,\ldots,n$, can be a function of all $4n$ prices and quantities pertaining to the two periods or situations under consideration, p^0,p^1,q^0,q^1. In the literature on making multilateral comparisons (i.e., comparisons between more than two situations), it is quite common to assume that the quantity comparison between any two regions can be made using the two regional quantity vectors, q^0 and q^1, and a common reference price vector, $p^* \equiv (p_1^*,\ldots,p_n^*)$.[37]

16.63 Obviously, different versions of the additivity test can be obtained if further restrictions are placed on precisely which variables each reference price p_i^* depends. The simplest such restriction is to assume that each p_i^* depends only on the commodity i prices pertaining to each of the two situations under consideration, p_i^0 and p_i^1. If it is further assumed that the functional form for the weighting function is the same for each commodity, so that $p_i^* = m(p_i^0,p_i^1)$ for $i = 1,\ldots,n$, then we are led to the *unequivocal quantity index* postulated by Knibbs (1924, p. 44).

16.64 The theory of the *unequivocal quantity index* (or the *pure quantity index*)[38] parallels the theory of the pure price index outlined in paragraphs 15.24 to 15.32 of Chapter 15. An outline of this theory is given here. Let the pure quantity index Q_K have the following functional form:

$$Q_K(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{n} q_i^1 m(p_i^0, p_i^1)}{\sum_{k=1}^{n} q_k^0 m(p_k^0, p_k^1)} \qquad (16.31)$$

It is assumed that the price vectors p^0 and p^1 are strictly positive and the quantity vectors q^0 and q^1 are non-negative but have at least one positive component.[39] The problem is to determine the functional form for the averaging function m if possible. To do this, it is necessary to impose some tests or properties on the pure quantity index Q_K. As was the case with the pure price index, it is very reasonable to ask that the quantity index satisfy the *time reversal test*:

$$Q_K(p^1, p^0, q^1, q^0) = \frac{1}{Q_K(p^0, p^1, q^0, q^1)} \qquad (16.32)$$

16.65 As was the case with the theory of the unequivocal price index, it can be seen that if the unequivocal quantity index Q_K is to satisfy the time reversal test (16.32), the mean

[37] Hill (1993, p. 395-397) termed such multilateral methods *the block approach* while Diewert (1996a, pp. 250-251) used the term *average price approaches*. Diewert (1999b, p. 19) used the term *additive multilateral system*. For axiomatic approaches to multilateral index number theory, see Balk (1996a; 2001) and Diewert (1999b).

[38] Diewert (2001) used this term.

[39] It is assumed that $m(a,b)$ has the following two properties: $m(a,b)$ is a positive and continuous function, defined for all positive numbers a and b, and $m(a,a) = a$ for all $a > 0$.

function in equation (16.31) must be *symmetric*. It is also asked that Q_K satisfy the following *invariance to proportional changes in current prices test*.

$$Q_K(p^0,\lambda p^1,q^0,q^1) = Q_K(p^0,p^1,q^0,q^1) \text{ for all } p^0,p^1,q^0,q^1 \text{ and all } \lambda > 0. \quad (16.33)$$

16.66 The idea behind this invariance test is this: the quantity index $Q_K(p^0,p^1,q^0,q^1)$ should depend only on the *relative* prices in each period and it should not depend on the amount of inflation between the two periods. Another way to interpret test (16.33) is to look at what the test implies for the corresponding implicit price index, P_{IK}, defined using the product test (16.17). It can be shown that if Q_K satisfies equation (16.33), then the corresponding implicit price index P_{IK} will satisfy test T5 above, the *proportionality in current prices test*. The two tests, (16.32) and (16.33), determine the precise functional form for the pure quantity index Q_K defined by equation (16.31): the *pure quantity index* or Knibbs' *unequivocal quantity index* Q_K must be the Walsh quantity index Q_W[40] defined by:

$$Q_W(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} q_i^1\sqrt{p_i^0 p_i^1}}{\sum_{k=1}^{n} q_k^0\sqrt{p_k^0 p_k^1}} \quad (16.34)$$

16.67 Thus with the addition of two tests, the pure price index P_K must be the Walsh price index P_W defined by equation (15.19) in Chapter 15 and with the addition of the same two tests (but applied to quantity indices instead of price indices), the pure quantity index Q_K must be the Walsh quantity index Q_W defined by equation (16.34). Note, however, that the product of the Walsh price and quantity indices is *not* equal to the expenditure ratio, V^1/V^0. Thus believers in the pure or unequivocal price and quantity index concepts have to choose one of these two concepts; they both cannot apply simultaneously.[41]

16.68 If the quantity index $Q(p^0,p^1,q^0,q^1)$ satisfies the additivity test (16.30) for some price weights p_i^*, then the percentage change in the quantity aggregate, $Q(p^0,p^1,q^0,q^1) - 1$, can be rewritten as follows:

$$Q(p^0,p^1,q^0,q^1) - 1 = \frac{\sum_{i=1}^{n} p_i^* q_i^1}{\sum_{m=1}^{n} p_m^* q_m^0} - 1 = \frac{\sum_{i=1}^{n} p_i^* q_i^1 - \sum_{m=1}^{n} p_m^* q_m^0}{\sum_{m=1}^{n} p_m^* q_m^0} = \sum_{i=1}^{n} w_i (q_i^1 - q_i^0) \quad (16.35)$$

where the *weight* for commodity *i*, w_i, is defined as

$$w_i \equiv \frac{p_i^*}{\sum_{m=1}^{n} p_m^* q_m^0}; \quad i = 1,\ldots,n \quad (16.36)$$

Note that the change in commodity *i* going from situation 0 to situation 1 is $q_i^1 - q_i^0$. Thus the *i*th term on the right-hand side of equation (16.35) is the contribution of the change in commodity *i* to the overall percentage change in the aggregate going from period 0 to 1. Business analysts often want statistical agencies to provide decompositions such as equation (16.35) so that they can decompose the overall change in an aggregate into sector-specific

[40] This is the quantity index that corresponds to the price index 8 defined by Walsh (1921a, p. 101).

[41] Knibbs (1924) did not notice this point.

components of change.[42] Thus there is a demand on the part of users for additive quantity indices.

16.69 For the Walsh quantity index defined by equation (16.34), the *i*th weight is

$$w_{W_i} \equiv \frac{\sqrt{p_i^0 p_i^1}}{\sum_{m=1}^{n} q_m^0 \sqrt{p_m^0 p_m^1}}; \quad i = 1,...,n \tag{16.37}$$

Thus the Walsh quantity index Q_W has a percentage decomposition into component changes of the form of equation (16.35), where the weights are defined by equation (16.37).
16.70 It turns out that the Fisher quantity index Q_F, defined by equation (15.14) in Chapter 15, also has an additive percentage change decomposition of the form given by equation (16.35).[43] The *i*th weight w_{F_i} for this Fisher decomposition is rather complicated and depends on the Fisher quantity index $Q_F(p^0,p^1,q^0,q^1)$ as follows:[44]

$$w_{F_i} \equiv \frac{w_i^0 + (Q_F)^2 w_i^1}{1 + Q_F}; \; i = 1,...,n \tag{16.38}$$

where Q_F is the value of the Fisher quantity index, $Q_F(p^0,p^1,q^0,q^1)$, and the period *t* normalized price for commodity *i*, w_i^t, is defined as the period *i* price p_i^t divided by the period *t* expenditure on the aggregate:

$$w_i^t \equiv \frac{p_i^t}{\sum_{m=1}^{n} p_m^t q_m^t}; \quad t = 0,1\,; \quad i = 1,\ldots,n \tag{16.39}$$

16.71 Using the weights w_{F_i} defined by equations (16.38) and (16.39), the following exact decomposition is obtained for the Fisher ideal quantity index:

$$Q_F(p^0, p^1, q^0, q^1) - 1 = \sum_{i=1}^{n} w_{F_i}(q_i^1 - q_i^0) \tag{16.40}$$

Thus the Fisher quantity index has an additive percentage change decomposition.[45]

16.72 Because of the symmetric nature of the Fisher price and quantity indices, it can be seen that the Fisher price index P_F defined by equation (16.27) also has the following additive percentage change decomposition:

[42] Business and government analysts also often demand an analogous decomposition of the change in price aggregate into sector-specific components that add up.

[43] The Fisher quantity index also has an additive decomposition of the type defined by equation (16.30) attributable to Van Ijzeren (1987, p. 6). The *i*th reference price p_i* is defined as $p_i^* \equiv [(1/2)p_i^0 + (1/2)p_i^1]/P_F\left(p^0 p^1 q^0 q^1\right)$ for $i = 1,\ldots,n$ and where P_F is the Fisher price index. This decomposition was also independently derived by Dikhanov (1997). The Van Ijzeren decomposition for the Fisher quantity index is currently being used by the US Bureau of Economic Analysis; see Moulton and Seskin (1999, p. 16) and Ehemann, Katz and Moulton (2002).

[44] This decomposition was obtained by Diewert (2002a) and Reinsdorf, Diewert and Ehemann (2002). For an economic interpretation of this decomposition, see Diewert (2002a).

[45] To verify the exactness of the decomposition, substitute equation (16.38) into equation (16.40) and solve the resulting equation for Q_F. It is found that the solution is equal to Q_F defined by equation (15.14) in Chapter 15.

$$P_F(p^0,p^1,q^0,q^1)-1=\sum_{i=1}^{n} v_{F_i}(p_i^1-p_i^0) \qquad (16.41)$$

where the commodity *i* weight v_{F_i} is defined as

$$v_{F_i} \equiv \frac{v_i^0+(P_F)^2 v_i^1}{1+P_F};\ i=1,\ldots,n \qquad (16.42)$$

where P_F is the value of the Fisher price index, $P_F(p^0,p^1,q^0,q^1)$, and the period *t* normalized quantity for commodity *i*, v_i^t, is defined as the period *i* quantity q_i^t divided by the period *t* expenditure on the aggregate:

$$v_i^t \equiv \frac{q_i^t}{\sum_{m=1}^{n} p_m^t q_m^t};\quad t = 0,1\ ;\quad i = 1,\ldots,n \qquad (16.43)$$

16.73 The above results show that the Fisher price and quantity indices have exact additive decompositions into components that give the contribution to the overall change in the price (or quantity) index of the change in each price (or quantity).

The stochastic approach to price indices

The early unweighted stochastic approach

16.74 The stochastic approach to the determination of the price index can be traced back to the work of Jevons (1863; 1865) and Edgeworth (1888) over 100 years ago.[46] The basic idea behind the (unweighted) stochastic approach is that each price relative, p_i^1/p_i^0 for $i = 1,2,\ldots,n$ can be regarded as an estimate of a common inflation rate α between periods 0 and 1.[47]
It is assumed that

$$\frac{p_i^1}{p_i^0}=\alpha+\varepsilon_i;\ i=1,2,\ldots,n \qquad (16.44)$$

where α is the common inflation rate and the ε_i are random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for α is the Carli (1764) price index P_C defined as

$$P_C(p^0,p^1)\equiv\sum_{i=1}^{n}\frac{1}{n}\frac{p_i^1}{p_i^0} \qquad (16.45)$$

A drawback of the Carli price index is that it does not satisfy the time reversal test, i.e., $P_C(p^1,p^0) \neq 1/ P_C(p^0,p^1)$.[48]

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[46] For references to the literature, see Diewert (1993a, pp. 37-38; 1995a; 1995b).

[47] "In drawing our averages the independent fluctuations will more or less destroy each other; the one required variation of gold will remain undiminished" (Jevons (1863, p. 26)).

[48] In fact, Fisher (1922, p. 66) noted that $P_C(p^0,p^1)P_C(p^1,p^0) \geq 1$ unless the period 1 price vector p^1 is proportional to the period 0 price vector p^0; i.e., Fisher showed that the Carli index has a definite upward bias. He urged statistical agencies not to use this formula. Walsh (1901, pp. 331, 530) also discovered this result for the case $n = 2$.

16.75 Now change the stochastic specification and assume that the logarithm of each price relative, $\ln(p_i^1/p_i^0)$, is an unbiased estimate of the logarithm of the inflation rate between periods 0 and 1, β say. The counterpart to equation (16.44) is:

$$\ln\left(\frac{p_i^1}{p_i^0}\right) = \beta + \varepsilon_i; \quad i = 1,2,...,n \qquad (16.46)$$

where $\beta \equiv \ln\alpha$ and the ε_i are independently distributed random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for β is the logarithm of the geometric mean of the price relatives. Hence the corresponding estimate for the common inflation rate α[49] is the Jevons (1865) price index P_J defined as follows:

$$P_J(p^0, p^1) \equiv \prod_{i=1}^{n} \sqrt[n]{\frac{p_i^1}{p_i^0}} \qquad (16.47)$$

16.76 The Jevons price index P_J does satisfy the time reversal test and hence is much more satisfactory than the Carli index P_C. Both the Jevons and Carli price indices nevertheless suffer from a fatal flaw: each price relative p_i^1/p_i^0 is regarded as being equally important and is given an equal weight in the index number formulae (16.45) and (16.47). John Maynard Keynes was particularly critical of this unweighted stochastic approach to index number theory.[50] He directed the following criticism towards this approach, which was vigorously advocated by Edgeworth (1923):

> Nevertheless I venture to maintain that such ideas, which I have endeavoured to expound above as fairly and as plausibly as I can, are root-and-branch erroneous. The “errors of observation”, the “faulty shots aimed at a single bull’s eye” conception of the index number of prices, Edgeworth’s “objective mean variation of general prices”, is the result of confusion of thought. There is no bull’s eye. There is no moving but unique centre, to be called the general price level or the objective mean variation of general prices, round which are scattered the moving price levels of individual things. There are all the various, quite definite, conceptions of price levels of composite commodities appropriate for various purposes and inquiries which have been scheduled above, and many others too. There is nothing else. Jevons was pursuing a mirage.
>
> What is the flaw in the argument? In the first place it assumed that the fluctuations of individual prices round the “mean” are “random” in the sense required by the theory of the combination of independent observations. In this theory the divergence of one “observation” from the true position is assumed to have no influence on the divergences of other “observations”. But in the case of prices, a movement in the price of one commodity necessarily influences the movement in the prices of other commodities, whilst the magnitudes of these compensatory movements depend on the magnitude of the change in expenditure on the first commodity as compared with the importance of the expenditure on the commodities secondarily affected. Thus, instead of “independence”, there is between the “errors” in the

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[49] Greenlees (1999) pointed out that although $(1/n)\sum_{i=1}^{n}\ln(p_i^1/p_i^0)$ is an unbiased estimator for β, the corresponding exponential of this estimator, P_J defined by equation (16.47), will generally not be an unbiased estimator for α under our stochastic assumptions. To see this, let $x_i = \ln p_i^1/p_i^0$. Taking expectations, we have: $Ex_i = \beta = \ln\alpha$. Define the positive, convex function f of one variable x by $f(x) \equiv e^x$. By Jensen’s (1906) inequality, $Ef(x) \geq f(Ex)$. Letting x equal the random variable x_i, this inequality becomes: $E(p_i^1/p_i^0) = Ef(x_i) \geq f(Ex_i) = f(\beta) = e^\beta = e^{\ln\alpha} = \alpha$. Thus for each n, $E(p_i^1/p_i^0) \geq \alpha$, and it can be seen that the Jevons price index will generally have an upward bias under the usual stochastic assumptions.

[50] Walsh (1901, p. 83) also stressed the importance of proper weighting according to the economic importance of the commodities in the periods being compared: “But to assign uneven weighting with approximation to the relative sizes, either over a long series of years or for every period separately, would not require much additional trouble; and even a rough procedure of this sort would yield results far superior to those yielded by even weighting. It is especially absurd to refrain from using roughly reckoned uneven weighting on the ground that it is not accurate, and instead to use even weighting, which is much more inaccurate.”

> successive "observations" what some writers on probability have called "connexity", or, as Lexis expressed it, there is "sub-normal dispersion".
>
> We cannot, therefore, proceed further until we have enunciated the appropriate law of connexity. But the law of connexity cannot be enunciated without reference to the relative importance of the commodities affected—which brings us back to the problem that we have been trying to avoid, of weighting the items of a composite commodity (Keynes (1930, pp. 76-77)).

The main point Keynes seemed to be making in the above quotation is that prices in the economy are not independently distributed from each other and from quantities. In current macroeconomic terminology, Keynes can be interpreted as saying that a macroeconomic shock will be distributed across all prices and quantities in the economy through the normal interaction between supply and demand; i.e., through the workings of the general equilibrium system. Thus Keynes seemed to be leaning towards the economic approach to index number theory (even before it was developed to any great extent), where quantity movements are functionally related to price movements. A second point that Keynes made in the above quotation is that there is no such thing as *the* inflation rate; there are only price changes that pertain to well-specified sets of commodities or transactions; i.e., the domain of definition of the price index must be carefully specified.[51] A final point that Keynes made is that price movements must be weighted by their economic importance; i.e., by quantities or expenditures.

16.77 In addition to the above theoretical criticisms, Keynes also made the following strong empirical attack on Edgeworth's unweighted stochastic approach:

> The Jevons–Edgeworth "objective mean variation of general prices", or "indefinite" standard, has generally been identified, by those who were not as alive as Edgeworth himself was to the subtleties of the case, with the purchasing power of money—if only for the excellent reason that it was difficult to visualise it as anything else. And since any respectable index number, however weighted, which covered a fairly large number of commodities could, in accordance with the argument, be regarded as a fair approximation to the indefinite standard, it seemed natural to regard any such index as a fair approximation to the purchasing power of money also.
>
> Finally, the conclusion that all the standards "come to much the same thing in the end" has been reinforced "inductively" by the fact that rival index numbers (all of them, however, of the wholesale type) have shown a considerable measure of agreement with one another in spite of their different compositions … On the contrary, the tables given above (pp. 53, 55) supply strong presumptive evidence that over long period as well as over short period the movements of the wholesale and of the consumption standards respectively are capable of being widely divergent (Keynes (1930, pp. 80-81)).

In the above quotation, Keynes noted that the proponents of the unweighted stochastic approach to price change measurement were comforted by the fact that all of the then existing (unweighted) indices of wholesale prices showed broadly similar movements. Keynes showed empirically, however, that his wholesale price indices moved quite differently from his consumer price indices.

16.78 In order to overcome the above criticisms of the unweighted stochastic approach to index numbers, it is necessary to:

- have a definite domain of definition for the index number;

[51] See paragraphs 15.7 to 15.17 in Chapter 15 for additional discussion on this point.

- weight the price relatives by their economic importance.[52]

Alternative methods of weighting are discussed in the following sections.

The weighted stochastic approach

16.79 Walsh (1901, pp. 88-89) seems to have been the first index number theorist to point out that a sensible stochastic approach to measuring price change means that individual price relatives should be weighted according to their economic importance or their *transactions value* in the two periods under consideration:

> It might seem at first sight as if simply every price quotation were a single item, and since every commodity (any kind of commodity) has one price-quotation attached to it, it would seem as if price-variations of every kind of commodity were the single item in question. This is the way the question struck the first inquirers into price-variations, wherefore they used simple averaging with even weighting. But a price-quotation is the quotation of the price of a generic name for many articles; and one such generic name covers a few articles, and another covers many. … A single price-quotation, therefore, may be the quotation of the price of a hundred, a thousand, or a million dollar's worths, of the articles that make up the commodity named. Its weight in the averaging, therefore, ought to be according to these money-unit's worth (Walsh (1921a, pp. 82-83)).

But Walsh did not give a convincing argument on exactly how these economic weights should be determined.

16.80 Henri Theil (1967, pp. 136-137) proposed a solution to the lack of weighting in the Jevons index, P_J defined by equation (16.47). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the *i*th price relative is equal to $s_i^0 \equiv p_i^0 q_i^0 \big/ \sum_{k=1}^n p_k^0 q_k^0$, the period 0 expenditure share for commodity *i*. Then the overall mean (period 0 weighted) logarithmic price change is $\sum_{i=1}^n s_i^0 \ln(p_i^1 / p_i^0)$.[53] Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) logarithmic price change of $\sum_{i=1}^n s_i^1 \ln(p_i^1 / p_i^0)$.[54]

16.81 Each of these measures of overall logarithmic price change seems equally valid, so we could argue for taking a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change. Theil[55] argued that a "nice" symmetric

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[52] Walsh (1901, pp. 82-90; 1921a, pp. 82-83) also objected to the lack of weighting in the unweighted stochastic approach to index number theory.

[53] In Chapter 19, this index is called the *geometric Laspeyres index*, P_{GL}. Vartia (1978, p. 272) referred to this index as the *logarithmic Laspeyres index*. Yet another name for the index is the *base weighted geometric index*.

[54] In Chapter 19, this index is called the *geometric Paasche index*, P_{GP}. Vartia (1978, p. 272) referred to this index as the *logarithmic Paasche index*. Yet another name for the index is the *current period weighted geometric index*.

[55] "The price index number defined in (1.8) and (1.9) uses the *n* individual logarithmic price differences as the basic ingredients. They are combined linearly by means of a two-stage random selection procedure: First, we give each region the same chance ½ of being selected, and second, we give each dollar spent in the selected region the same chance ($1/m_a$ or $1/m_b$) of being drawn" (Theil (1967, p. 138)). The indexes (1.8) and (1.9) are the geometric Laspeyres and Paasche indexes.

index number formula can be obtained if the probability of selection for the *n*th price relative is made equal to the arithmetic average of the period 0 and 1 expenditure shares for commodity *n*.
Using these probabilities of selection, Theil's final measure of overall logarithmic price change was

$$\ln P_T(p^0, p^1, q^0, q^1) \equiv \sum_{i=1}^{n} \frac{1}{2}(s_i^0 + s_i^1) \ln\left(\frac{p_i^1}{p_i^0}\right) \qquad (16.48)$$

Note that the index P_T defined by equation (16.48) is equal to the Törnqvist index defined by equation (15.81) in Chapter 15.

16.82 A statistical interpretation of the right-hand side of equation (16.48) can be given. Define the *i*th logarithmic price ratio r_i by:

$$r_i \equiv \ln\left(\frac{p_i^1}{p_i^0}\right) \quad \text{for } i = 1,...,n \qquad (16.49)$$

Now define the discrete random variable, *R* say, as the random variable which can take on the values r_i with probabilities $\rho_i \equiv (1/2)[s_i^0 + s_i^1]$ for $i = 1,\ldots,n$. Note that, since each set of expenditure shares, s_i^0 and s_i^1, sums to one over *i*, the probabilities ρ_i will also sum to one. It can be seen that the expected value of the discrete random variable *R* is

$$\begin{aligned} E[R] &\equiv \sum_{i=1}^{n} \rho_i r_i = \sum_{i=1}^{n} \frac{1}{2}(s_i^0 + s_i^1) \ln\left(\frac{p_i^1}{p_i^0}\right) \\ &= \ln P_T(p^0, p^1, q^0, q^1). \end{aligned} \qquad (16.50)$$

Thus the logarithm of the index P_T can be interpreted as *the expected value of the distribution of the logarithmic price ratios* in the domain of definition under consideration, where the *n* discrete price ratios in this domain of definition are weighted according to Theil's probability weights, $\rho_i \equiv (1/2)[s_i^0 + s_i^1]$ for $i = 1,\ldots,n$.

16.83 Taking antilogs of both sides of equation (16.48), the Törnqvist (1936; 1937) Theil price index, P_T, is obtained.[56] This index number formula has a number of good properties. In particular, P_T satisfies the proportionality in current prices test T5 and the time reversal test T11, discussed above. These two tests can be used to justify Theil's (arithmetic) method of forming an average of the two sets of expenditure shares in order to obtain his probability weights, $\rho_i \equiv (1/2)[s_i^0 + s_i^1]$ for $i = 1,\ldots,n$. Consider the following *symmetric mean class of logarithmic index number formulae*:

$$\ln P_S(p^0, p^1, q^0, q^1) \equiv \sum_{i=1}^{n} m(s_i^0, s_i^1) \ln\left(\frac{p_i^1}{p_i^0}\right) \qquad (16.51)$$

where $m(s_i^0,s_i^1)$ is a positive function of the period 0 and 1 expenditure shares on commodity *i*, s_i^0 and s_i^1 respectively. In order for P_S to satisfy the time reversal test, it is necessary that the function *m* be symmetric. Then it can be shown[57] that for P_S to satisfy test T5, *m* must be the arithmetic mean. This provides a reasonably strong justification for Theil's choice of the mean function.

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[56] The sampling bias problem studied by Greenlees (1999) (see footnote 49 above) does not occur in the present context because there is no sampling involved in definition (16.50): the sum of the $p_i^t q_i^t$ over *i* for each period *t* is assumed to equal the value aggregate V^t for period *t*.

[57] See Diewert (2000) and Balk and Diewert (2001).

16.84 The stochastic approach of Theil has another “nice” symmetry property. Instead of considering the distribution of the logarithmic price ratios $r_i = \ln p_i^1/p_i^0$, we could also consider the distribution of the logarithms of the *reciprocals* of the price ratios, say:

$$t_i \equiv \ln \frac{p_i^0}{p_i^1} = \ln\left(\frac{p_i^1}{p_i^0}\right)^{-1} = -\ln \frac{p_i^1}{p_i^0} = -r_i \quad \text{for } i = 1,\ldots,n \qquad (16.52)$$

The symmetric probability, $\rho_i \equiv (1/2)[\, s_i^0 + s_i^1]$, can still be associated with the *i*th reciprocal logarithmic price ratio t_i for $i = 1,\ldots,n$. Now define the discrete random variable, T say, as the random variable which can take on the values t_i with probabilities $\rho_i \equiv (1/2)[\, s_i^0 + s_i^1]$ for $i = 1,\ldots,n$. It can be seen that the expected value of the discrete random variable T is

$$\begin{aligned} E[T] &\equiv -\sum_{i=1}^{n} \rho_i r_i \\ &= -\sum_{i=1}^{n} \rho_i t_i \quad \text{using (16.52)} \qquad (16.53) \\ &= -E[R] \quad \text{using (16.50)} \\ &= -\ln P_T(p^0, p^1, q^0, q^1). \end{aligned}$$

Thus it can be seen that the distribution of the random variable T is equal to the distribution of the random variable minus R. Hence it does not matter whether the distribution of the original logarithmic price ratios, $r_i \equiv \ln p_i^1/p_i^0$, is considered or the distribution of their logarithmic reciprocals, $t_i \equiv \ln p_i^0/p_i^1$, is considered: essentially the same stochastic theory is obtained.

16.85 It is possible to consider weighted stochastic approaches to index number theory where the distribution of the price ratios, p_i^1/p_i^0, is considered rather than the distribution of the logarithmic price ratios, $\ln p_i^1/p_i^0$. Thus, again following in the footsteps of Theil, suppose that price relatives are drawn at random in such a way that each dollar of expenditure in the *base period* has an equal chance of being selected. Then the probability that the *i*th price relative will be drawn is equal to s_i^0, the period 0 expenditure share for commodity i. Thus the overall mean (period 0 weighted) price change is:

$$P_L(p^0, p^1, q^0, q^1) = \sum_{i=1}^{n} s_i^0 \frac{p_i^1}{p_i^0} \qquad (16.54)$$

which turns out to be the Laspeyres price index, P_L. This stochastic approach is the natural one for studying *sampling problems* associated with implementing a Laspeyres price index.

16.86 Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) price change equal to:

$$P_{PAL}(p^0, p^1, q^0, q^1) = \sum_{i=1}^{n} s_i^1 \frac{p_i^1}{p_i^0} \qquad (16.55)$$

This is known as the Palgrave (1886) index number formula.[58]

[58] It is formula number 9 in Fisher’s (1922, p. 466) listing of index number formulae.

16.87 It can be verified that neither the Laspeyres nor Palgrave price indices satisfy the time reversal test, T11. Thus, again following in the footsteps of Theil, it might be attempted to obtain a formula that satisfied the time reversal test by taking a symmetric average of the two sets of shares. Thus consider the following class of *symmetric mean index number formulae*:

$$P_m(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^{n} m(s_i^0,s_i^1)\frac{p_i^1}{p_i^0} \qquad (16.56)$$

where $m(s_i^0,s_i^1)$ is a symmetric function of the period 0 and 1 expenditure shares for commodity i, s_i^0 and s_i^1 respectively. In order to interpret the right hand-side of equation (16.56) as an expected value of the price ratios p_i^1/p_i^0, it is necessary that

$$\sum_{i=1}^{n} m(s_i^0,s_i^1) = 1 \qquad (16.57)$$

In order to satisfy equation (16.57), however, m must be the arithmetic mean.[59] With this choice of m, equation (16.56) becomes the following (unnamed) index number formula, P_u:

$$P_u(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^{n} \frac{1}{2}(s_i^0+s_i^1)\frac{p_i^1}{p_i^0} \qquad (16.58)$$

Unfortunately, the unnamed index P_u does not satisfy the time reversal test either.[60]

16.88 Instead of considering the distribution of the price ratios, p_i^1/p_i^0, the distribution of the *reciprocals* of these price ratios could be considered. The counterparts to the asymmetric indices defined earlier by equations (16.54) and (16.55) are now $\sum_{i=1}^{n} s_i^0(p_i^0/p_i^1)$ and $\sum_{i=1}^{n} s_i^1(p_i^0/p_i^1)$, respectively. These are (stochastic) price indices going *backwards* from period 1 to 0. In order to make these indices comparable with other previous forward-looking indices, take the reciprocals of these indices (which leads to harmonic averages) and the following two indices are obtained:

$$P_{HL}(p^0,p^1,q^0,q^1) \equiv \frac{1}{\sum_{i=1}^{n} s_i^0 \frac{p_i^0}{p_i^1}} \qquad (16.59)$$

$$P_{HP}(p^0,p^1,q^0,q^1) \equiv \frac{1}{\sum_{i=1}^{n} s_i^1 \frac{p_i^0}{p_i^1}} = \frac{1}{\sum_{i=1}^{n} s_i^1 \left(\frac{p_i^1}{p_i^0}\right)^{-1}} = P_P(p^0,p^1,q^0,q^1) \qquad (16.60)$$

using equation (15.9) in Chapter 15. Thus the reciprocal stochastic price index defined by equation (16.60) turns out to equal the fixed basket Paasche price index, P_P. This stochastic approach is the natural one for studying sampling problems associated with implementing a Paasche price index. The other asymmetrically weighted reciprocal stochastic price index

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[59] For a proof of this assertion, see Balk and Diewert (2001).

[60] In fact, this index suffers from the same upward bias as the Carli index in that $P_u(p^0,p^1,q^0,q^1)P_u(p^1,p^0,q^1,q^0) \geq 1$. To prove this, note that the previous inequality is equivalent to $[P_u(p^1,p^0,q^1,q^0)]^{-1} \leq P_u(p^0,p^1,q^0,q^1)$ and this inequality follows from the fact that a weighted harmonic mean of n positive numbers is equal or less than the corresponding weighted arithmetic mean; see Hardy, Littlewood and Pólya (1934, p. 26).

defined by the formula (16.59) has no author's name associated with it but it was noted by Fisher (1922, p. 467) as his index number formula 13. Vartia (1978, p. 272) called this index *the harmonic Laspeyres index* and his terminology will be used.

16.89 Now consider the class of symmetrically weighted reciprocal price indices defined as:

$$P_{mr}(p^0, p^1, q^0, q^1) \equiv \frac{1}{\sum_{i=1}^{n} m(s_i^0, s_i^1)\left(\frac{p_i^1}{p_i^0}\right)^{-1}} \tag{16.61}$$

where, as usual, $m(s_i^0, s_i^1)$ is a homogeneous symmetric mean of the period 0 and 1 expenditure shares on commodity *i*. However, none of the indices defined by equations (16.59) to (16.61) satisfies the time reversal test.

16.90 The fact that Theil's index number formula P_T satisfies the time reversal test leads to a preference for Theil's index as the "best" weighted stochastic approach.

16.91 The main features of the weighted stochastic approach to index number theory can be summarized as follows. It is first necessary to pick two periods and a transactions domain of definition. As usual, each value transaction for each of the *n* commodities in the domain of definition is split up into price and quantity components. Then, assuming there are no new commodities or no disappearing commodities, there are *n* price relatives p_i^1/p_i^0 pertaining to the two situations under consideration along with the corresponding 2*n* expenditure shares. The weighted stochastic approach just assumes that these *n* relative prices, or some transformation of these price relatives, $f(p_i^1/p_i^0)$, have a discrete statistical distribution, where the *i*th probability, $\rho_i = m(s_i^0, s_i^1)$, is a function of the expenditure shares pertaining to commodity *i* in the two situations under consideration, s_i^0 and s_i^1. Different price indices result, depending on how the functions *f* and *m* are chosen. In Theil's approach, the transformation function *f* is the natural logarithm and the mean function *m* is the simple unweighted arithmetic mean.

16.92 There is a third aspect to the weighted stochastic approach to index number theory: it has to be decided what *single number* best summarizes the distribution of the *n* (possibly transformed) price relatives. In the above analysis, the *mean* of the discrete distribution was chosen as the "best" summary measure for the distribution of the (possibly transformed) price relatives; but other measures are possible. In particular, the *weighted median* or various *trimmed means* are often suggested as the "best" measure of central tendency because these measures minimize the influence of outliers. Detailed discussion of these alternative measures of central tendency is, however, beyond the scope of this chapter. Additional material on stochastic approaches to index number theory and references to the literature can be found in Clements and Izan (1981; 1987), Selvanathan and Rao (1994), Diewert (1995b), Cecchetti (1997) and Wynne (1997; 1999).

16.93 Instead of taking the above stochastic approach to index number theory, it is possible to take the same raw data that is used in this approach but use an axiomatic approach. Thus, in the following section, the price index is regarded as a value-weighted function of the *n* price relatives and the test approach to index number theory is used in order to determine the functional form for the price index. Put another way, the axiomatic approach in the next section looks at the *properties* of alternative descriptive statistics that aggregate the individual price relatives (weighted by their economic importance) into summary measures of price change in an attempt to find the "best" summary measure of price change. Thus the axiomatic approach pursued below can be viewed as a branch of the theory of descriptive statistics.

The second axiomatic approach to bilateral price indices
The basic framework and some preliminary tests

16.94 As mentioned in paragraphs 16.1 to 16.10, one of Walsh's approaches to index number theory was an attempt to determine the "best" weighted average of the price relatives, r_i.[61] This is equivalent to using an axiomatic approach to try and determine the "best" index of the form $P(r,v^0,v^1)$, where v^0 and v^1 are the vectors of expenditures on the n commodities during periods 0 and 1.[62] Initially, rather than starting with indices of the form $P(r,v^0,v^1)$, indices of the form $P(p^0,p^1,v^0,v^1)$ will be considered, since this framework will be more comparable to the first bilateral axiomatic framework taken in paragraphs 16.30 to 16.73. As will be seen below, if the invariance to changes in the units of measurement test is imposed on an index of the form $P(p^0,p^1,v^0,v^1)$, then $P(p^0,p^1,v^0,v^1)$ can be written in the form $P(r,v^0,v^1)$.

16.95 Recall that the product test (16.17) was used to define the quantity index $Q(p^0,p^1,q^0,q^1) \equiv V^1/V^0P(p^0,p^1,q^0,q^1)$ that corresponded to the bilateral price index $P(p^0,p^1,q^0,q^1)$. A similar product test holds in the present framework; i.e., given that the functional form for the price index $P(p^0,p^1,v^0,v^1)$ has been determined, then the corresponding *implicit quantity index* can be defined in terms of P as follows:

$$Q(p^0,p^1,v^0,v^1) \equiv \frac{\sum_{i=1}^{n} v_i^1}{\left(\sum_{i=1}^{n} v_i^0\right) P(p^0,p^1,v^0,v^1)} \qquad (16.62)$$

16.96 In paragraphs 16.30 to 16.73, the price and quantity indices $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ were determined *jointly*; i.e., not only were axioms imposed on $P(p^0,p^1,q^0,q^1)$ but they were also imposed on $Q(p^0,p^1,q^0,q^1)$ and the product test (16.17) was used to translate these tests on Q into tests on P. In this section, this approach will not be followed: only tests on $P(p^0,p^1,v^0,v^1)$ will be used in order to determine the "best" price index of this

[61] Fisher also took this point of view when describing his approach to index number theory:

> An index number of the prices of a number of commodities is an average of their price relatives. This definition has, for concreteness, been expressed in terms of prices. But in like manner, an index number can be calculated for wages, for quantities of goods imported or exported, and, in fact, for any subject matter involving divergent changes of a group of magnitudes. Again, this definition has been expressed in terms of time. But an index number can be applied with equal propriety to comparisons between two places or, in fact, to comparisons between the magnitudes of a group of elements under any one set of circumstances and their magnitudes under another set of circumstances (Fisher (1922, p. 3)).

In setting up his axiomatic approach, Fisher imposed axioms on the price and quantity indices written as functions of the two price vectors, p^0 and p^1, and the two quantity vectors, q^0 and q^1; i.e., he did not write his price index in the form $P(r,v^0,v^1)$ and impose axioms on indices of this type. Of course, in the end, his ideal price index turned out to be the geometric mean of the Laspeyres and Paasche price indices and, as was seen in Chapter 15, each of these indices can be written as expenditure share weighted averages of the n price relatives, $r_i \equiv p_i^1/p_i^0$.

[62] Chapter 3 in Vartia (1976) considered a variant of this axiomatic approach.

form. Thus there is a parallel theory for quantity indices of the form $Q(q^0,q^1,v^0,v^1)$, where it is attempted to find the "best" value weighted average of the quantity relatives, q_i^1/q_i^0.[63]

16.97 For the most part, the tests which will be imposed on the price index $P(p^0,p^1,v^0,v^1)$ in this section are counterparts to the tests that were imposed on the price index $P(p^0,p^1,q^0,q^1)$ in paragraphs 16.30 to 16.73. It will be assumed that every component of each price and value vector is positive; i.e., $p^t >> 0_n$ and $v^t >> 0_n$ for $t = 0,1$. If it is desired to set $v^0 = v^1$, the common expenditure vector is denoted by v; if it is desired to set $p^0 = p^1$, the common price vector is denoted by p.

16.98 The first two tests are straightforward counterparts to the corresponding tests in paragraph 16.34.

T1: *Positivity*: $P(p^0,p^1,v^0,v^1) > 0$

T2: *Continuity*: $P(p^0,p^1,v^0,v^1)$ is a continuous function of its arguments

T3: *Identity or constant prices test*: $P(p,p,v^0,v^1) = 1$

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the value vectors are. Note that the two value vectors are allowed to be different in the above test.

Homogeneity tests

16.99 The following four tests restrict the behaviour of the price index P as the scale of any one of the four vectors p^0,p^1,v^0,v^1 changes.

T4: *Proportionality in current prices*

:$P(p^0,\lambda p^1,v^0,v^1) = \lambda P(p^0,p^1,v^0,v^1)$ for $\lambda > 0$

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. This test is the counterpart to test T5 in paragraph 16.37.

16.100 In the next test, instead of multiplying all period 1 prices by the same number, all period 0 prices are multiplied by the number λ.

T5: *Inverse proportionality in base period prices*:

$P(\lambda p^0,p^1,v^0,v^1) = \lambda^{-1}P(p^0,p^1,v^0,v^1)$ for $\lambda > 0$

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0. This test is the counterpart to test T6 in paragraph 16.39.

16.101 The following two homogeneity tests can also be regarded as invariance tests.

T6: *Invariance to proportional changes in current period values*:

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[63] It turns out that the price index that corresponds to this "best" quantity index, defined as $P^*(q^0,q^1,v^0,v^1) \equiv \sum_{i=1}^{n} v_i^1 \Big/ \left[\sum_{i=1}^{n} v_i^0 Q(q^0,q^1,v^0,v^1)\right]$, will not equal the "best" price index, $P(p^0,p^1,v^0,v^1)$. Thus the axiomatic approach used here generates *separate* "best" price and quantity indices whose product does not equal the value ratio in general. This is a disadvantage of the second axiomatic approach to bilateral indices compared to the first approach studied above.

$P(p^0,p^1,v^0,\lambda v^1) = P(p^0,p^1,v^0,v^1)$ for all $\lambda > 0$

That is, if current period values are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree zero in the components of the period 1 value vector v^1.

T7: *Invariance to proportional changes in base period values*:

$P(p^0,p^1,\lambda v^0,v^1) = P(p^0,p^1,v^0,v^1)$ for all $\lambda > 0$

That is, if base period values are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree zero in the components of the period 0 value vector v^0.

16.102 T6 and T7 together impose the property that the price index P does not depend on the *absolute* magnitudes of the value vectors v^0 and v^1. Using test T6 with $\lambda = 1/\sum_{i=1}^{n} v_i^1$ and using test T7 with $\lambda = 1/\sum_{i=1}^{n} v_i^0$, it can be seen that P has the following property:

$$P(p^0, p^1, v^0, v^1) = P(p^0, p^1, s^0, s^1) \qquad (16.63)$$

where s^0 and s^1 are the vectors of expenditure shares for periods 0 and 1; i.e., the *i*th component of s^t is $s_i^t \equiv v_i^t / \sum_{k=1}^{n} v_k^t$ for $t = 0,1$. Thus the tests T6 and T7 imply that the price index function P is a function of the two price vectors p^0 and p^1 and the two vectors of expenditure shares, s^0 and s^1.

16.103 Walsh (1901, p. 104) suggested the spirit of tests T6 and T7 as the following quotation indicates: "What we are seeking is to average the variations in the exchange value of one given total sum of money in relation to the several classes of goods, to which several variations [i.e., the price relatives] must be assigned weights proportional to the relative sizes of the classes. Hence the relative sizes of the classes at both the periods must be considered."

16.104 Walsh also realized that weighting the *i*th price relative r_i by the arithmetic mean of the value weights in the two periods under consideration, $(1/2)[v_i^0 + v_i^1]$ would give too much weight to the expenditures of the period that had the highest level of prices:

> At first sight it might be thought sufficient to add up the weights of every class at the two periods and to divide by two. This would give the (arithmetic) mean size of every class over the two periods together. But such an operation is manifestly wrong. In the first place, the sizes of the classes at each period are reckoned in the money of the period, and if it happens that the exchange value of money has fallen, or prices in general have risen, greater influence upon the result would be given to the weighting of the second period; or if prices in general have fallen, greater influence would be given to the weighting of the first period. Or in a comparison between two countries, greater influence would be given to the weighting of the country with the higher level of prices. But it is plain that *the one period, or the one country, is as important, in our comparison between them, as the other, and the weighting in the averaging of their weights should really be even* (Walsh (1901, pp. 104-105)).

16.105 As a solution to the above weighting problem, Walsh (1901, p. 202; 1921a, p. 97) proposed the following *geometric price index*:

$$P_{GW}(p^0, p^1, v^0, v^1) \equiv \prod_{i=1}^{n} \left(\frac{p_i^1}{p_i^0} \right)^{w(i)} \qquad (16.64)$$

where the *i*th weight in the above formula was defined as

$$w(i) \equiv \frac{(v_i^0 v_i^1)^{1/2}}{\sum_{k=1}^{n}(v_k^0 v_k^1)^{1/2}} = \frac{(s_i^0 s_i^1)^{1/2}}{\sum_{k=1}^{n}(s_k^0 s_k^1)^{1/2}} \qquad i = 1,...,n. \qquad (16.65)$$

The second equation in (16.65) shows that Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ can also be written as a function of the expenditure share vectors, s^0 and s^1; i.e., $P_{GW}(p^0,p^1,v^0,v^1)$ is homogeneous of degree zero in the components of the value vectors v^0 and v^1 and so $P_{GW}(p^0,p^1,v^0,v^1) = P_{GW}(p^0,p^1,s^0,s^1)$. Thus Walsh came very close to deriving the Törnqvist–Theil index defined earlier by equation (16.48).[64]

Invariance and symmetry tests

16.106 The next five tests are *invariance* or *symmetry tests* and four of them are direct counterparts to similar tests in paragraphs 16.42 to 16.46 above. The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed.

T8: *Commodity reversal test* (or invariance to changes in the ordering of commodities):

$P(p^{0}*,p^{1}*,v^{0}*,v^{1}*) = P(p^0,p^1,v^0,v^1)$

where p^t* denotes a permutation of the components of the vector p^t and v^t* denotes the same permutation of the components of v^t for $t = 0,1$. The term "commodity reversal test" is due to Fisher (1922; p. 63) but Walter Lane suggested a more appropriate name for the test might be the "commodity permutation test".

16.107 The next test asks that the index be invariant to changes in the units of measurement.

T9: *Invariance to changes in the units of measurement* (commensurability test):

$P(\alpha_1 p_1^0,...,\alpha_n p_n^0;\ \alpha_1 p_1^1,...,\alpha_n p_n^1;\ v_1^0,...,v_n^0;\ v_1^1,...,v_n^1) =$

$P(p_1^0,...,p_n^0;\ p_1^1,...,p_n^1;\ v_1^0,...,v_n^0;\ v_1^1,...,v_n^1)$ for all $\alpha_1 > 0, \ldots, \alpha_n > 0$

That is, the price index does not change if the units of measurement for each commodity are changed. Note that the expenditure on commodity i during period t, v_i^t, does not change if the unit by which commodity i is measured changes.

16.108 The last test has a very important implication. Let $\alpha_1 = 1/p_1^0, \ldots, \alpha_n = 1/p_n^0$ and substitute these values for the α_i into the definition of the test. The following equation is obtained:

$$P(p^0, p^1, v^0, v^1) = P(1_n, r, v^0, v^1) \equiv P^*(r, v^0, v^1) \qquad (16.66)$$

where 1_n is a vector of ones of dimension n and r is a vector of the price relatives; i.e., the ith component of r is $r_i \equiv p_i^1/p_i^0$. Thus, if the commensurability test T9 is satisfied, then the price index $P(p^0,p^1,v^0,v^1)$, which is a function of $4n$ variables, can be written as a function of $3n$ variables, $P^*(r, v^0,v^1)$, where r is the vector of price relatives and $P^*(r, v^0,v^1)$ is defined as $P(1_n,r,v^0,v^1)$.

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[64] Walsh's index could be derived using the same arguments as Theil, except that the geometric average of the expenditure shares $(s_i^0 s_i^1)^{1/2}$ could be taken as a preliminary probability weight for the ith logarithmic price relative, $\ln r_i$. These preliminary weights are then normalized to add up to unity by dividing by their sum. It is evident that Walsh's geometric price index will closely approximate Theil's index using normal time series data. More formally, regarding both indices as functions of p^0,p^1,v^0,v^1, it can be shown that $P_W(p^0,p^1,v^0,v^1)$ approximates $P_T(p^0,p^1,v^0,v^1)$ to the second order around an equal price (i.e., $p^0 = p^1$) and quantity (i.e., $q^0 = q^1$) point.

16.109 The next test asks that the formula be invariant to the period chosen as the base period.

T10: *Time reversal test*: $P(p^0,p^1,v^0,v^1) = 1/P(p^1,p^0,v^1,v^0)$

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one commodity case when the price index is simply the single price ratio, this test will be satisfied (as are all the other tests listed in this section).

16.110 The next test is a variant of the circularity test, introduced in paragraphs 15.76 to 15.97 of Chapter 15.[65]

T11: *Transitivity in prices for fixed value weights*:

$P(p^0,p^1,v^r,v^s)P(p^1,p^2,v^r,v^s) = P(p^0,p^2,v^r,v^s)$

In this test, the expenditure weighting vectors, v^r and v^s, are held constant while making all price comparisons. Given that these weights are held constant, however, the test asks that the product of the index going from period 0 to 1, $P(p^0,p^1,v^r,v^s)$, times the index going from period 1 to 2, $P(p^1,p^2,v^r,v^s)$, should equal the direct index that compares the prices of period 2 with those of period 0, $P(p^0,p^2,v^r,v^s)$. Obviously, this test is a many-commodity counterpart to a property that holds for a single price relative.

16.111 The final test in this section captures the idea that the value weights should enter the index number formula in a symmetric manner.

T12: *Value weights symmetry test*:

$P(p^0,p^1,v^0,v^1) = P(p^0,p^1,v^1,v^0)$

That is, if the expenditure vectors for the two periods are interchanged, then the price index remains invariant. This property means that, if values are used to weight the prices in the index number formula, then the period 0 values v^0 and the period 1 values v^1 must enter the formula in a symmetric or even-handed manner.

A mean value test

16.112 The next test is a *mean value test*.

T13: *Mean value test for prices*:

$$\min_i (p_i^1/p_i^0 : i=1,\dots,n) \le P(p^0,p^1,v^0,v^1) \le \max_i (p_i^1/p_i^0 : i=1,\dots,n) \qquad (16.67)$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is to be interpreted as an average of the n price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfy this test.

Monotonicity tests

16.113 The next two tests in this section are *monotonicity tests*; i.e., how should the price index $P(p^0,p^1,v^0,v^1)$ change as any component of the two price vectors p^0 and p^1 increases.

T14: *Monotonicity in current prices*:

$P(p^0,p^1,v^0,v^1) < P(p^0,p^2,v^0,v^1)$ if $p^1 < p^2$

That is, if some period 1 price increases, then the price index must increase (holding the value vectors fixed), so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1 for fixed p^0, v^0 and v^1.

[65] See equation (15.77) in Chapter 15.

T15: *Monotonicity in base prices*:

$P(p^0,p^1,v^0,v^1) > P(p^2,p^1,v^0,v^1)$ if $p^0 < p^2$

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0 for fixed p^1, v^0 and v^1.

Weighting tests

16.114 The above tests are not sufficient to determine the functional form of the price index; for example, it can be shown that both Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by equation (16.65) and the Törnqvist–Theil index $P_T(p^0,p^1,v^0,v^1)$ defined by equation (16.48) satisfy all of the above axioms. Thus, at least one more test will be required in order to determine the functional form for the price index $P(p^0,p^1,v^0,v^1)$.

16.115 The tests proposed thus far do not specify exactly how the expenditure share vectors s^0 and s^1 are to be used in order to weight, say, the first price relative, p_1^1/p_1^0. The next test says that only the expenditure shares s_1^0 and s_1^1 pertaining to the first commodity are to be used in order to weight the prices that correspond to commodity 1, p_1^1 and p_1^0.

T16. *Own share price weighting*:

$$P(p_1^0,1,...,1\,;\,p_1^1,1,...,1\,;\,v^0,v^1) = f\left(p_1^0,p_1^1,\left[v_1^0\Big/\sum_{k=1}^{n}v_k^0\right],\left[v_1^1\Big/\sum_{k=1}^{n}v_k^1\right]\right) \qquad (16.68)$$

Note that $v_1^t/\sum_{k=1}^{n}v_k^t$ equals s_1^t, the expenditure share for commodity 1 in period t. The above test says that if all the prices are set equal to 1 except the prices for commodity 1 in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity 1 and the two expenditure shares for commodity 1. The axiom says that a function of $2 + 2n$ variables is actually only a function of four variables.[66]

16.116 Of course, if test T16 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property:

$$P(1,...,1,p_i^0,1,...,1\,;\,1,...,1,p_i^1,1,...,1\,;\,v^0\,;\,v^1) = f\left(p_i^0,p_i^1,\left[v_i^0\Big/\sum_{k=1}^{n}v_k^0\right],\left[v_i^1\Big/\sum_{k=1}^{n}v_k^1\right]\right) \quad i = 1,...,n. \qquad (16.69)$$

Equation (16.69) says that, if all the prices are set equal to 1 except the prices for commodity i in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity i and the two expenditure shares for commodity i.

16.117 The final test that also involves the weighting of prices is the following one:

T17: *Irrelevance of price change with tiny value weights*:

$$P(p_1^0,1,\ldots,1\,;\,p_1^1,1,\ldots,1\,;\,0,v_2^0,\ldots,v_n^0\,;\,0,v_2^1,\ldots,v_n^1) = 1 \qquad (16.70)$$

[66] In the economics literature, axioms of this type are known as separability axioms.

The test T17 says that, if all the prices are set equal to 1 except the prices for commodity 1 in the two periods, and the expenditures on commodity 1 are zero in the two periods but the expenditures on the other commodities are arbitrarily given, then the index is equal to 1.[67] Thus, roughly speaking, if the value weights for commodity 1 are tiny, then it does not matter what the price of commodity 1 is during the two periods.

16.118 Of course, if test T17 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property: for $i = 1,\ldots,n$:

$$P(1,\ldots,1, p_i^0, 1,\ldots,1\ ;\ 1,\ldots,1, p_i^1, 1,\ldots,1\ ;\ v_1^0,\ldots,0,\ldots,v_n^0\ ;\ v_1^1,\ldots,0,\ldots,v_n^1) = 1 \qquad (16.71)$$

Equation (16.71) says that, if all the prices are set equal to 1 except the prices for commodity i in the two periods, and the expenditures on commodity i are 0 during the two periods but the other expenditures in the two periods are arbitrarily given, then the index is equal to 1.

16.119 This completes the listing of tests for the approach to bilateral index number theory based on the weighted average of price relatives. It turns out that the above tests are sufficient to imply a specific functional form for the price index, as seen in the next section.

The Törnqvist–Theil price index and the second test approach to bilateral indices

16.120 In Appendix 16.1 to this chapter, it is shown that, if the number of commodities n exceeds two and the bilateral price index function $P(p^0,p^1,v^0,v^1)$ satisfies the 17 axioms listed above, then P must be the Törnqvist–Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined by equation (16.48).[68] Thus the 17 properties or tests listed in paragraphs 16.94 to 16.129 provide an axiomatic characterization of the Törnqvist–Theil price index, just as the 20 tests listed in paragraphs 16.30 to 16.73 provided an axiomatic characterization of the Fisher ideal price index.

16.121 Obviously, there is a parallel axiomatic theory for quantity indices of the form $Q(q^0,q^1,v^0,v^1)$ that depend on the two quantity vectors for periods 0 and 1, q^0 and q^1, as well as on the corresponding two expenditure vectors, v^0 and v^1. Thus, if $Q(q^0,q^1,v^0,v^1)$ satisfies the quantity counterparts to tests T1 to T17, then Q must be equal to the Törnqvist–Theil quantity index $Q_T(q^0,q^1,v^0,v^1)$ defined, as follows:

$$\ln Q_T(q^0,q^1,v^0,v^1) \equiv \sum_{i=1}^{n} \frac{1}{2}(s_i^0 + s_i^1)\ln\left(\frac{q_i^1}{q_i^0}\right) \qquad (16.72)$$

where as usual, the period t expenditure share on commodity i, s_i^t, is defined as $v_i^t / \sum_{k=1}^{n} v_k^t$ for $i = 1,\ldots,n$ and $t = 0,1$.

[67] Strictly speaking, since all prices and values are required to be positive, the left-hand side of equation (16.70) should be replaced by the limit as the commodity 1 values, v_1^0 and v_1^1, approach 0.

[68] The Törnqvist–Theil price index satisfies all 17 tests, but the proof in Appendix 16.1 does not use all these tests to establish the result in the opposite direction: tests 5, 13, 15 and one of 10 or 12 were not required in order to show that an index satisfying the remaining tests must be the Törnqvist–Theil price index. For alternative characterizations of the Törnqvist–Theil price index, see Balk and Diewert (2001) and Hillinger (2002).

16.122 Unfortunately, the implicit Törnqvist–Theil price index, $P_{IT}(q^0,q^1,v^0,v^1)$ that corresponds to the Törnqvist–Theil quantity index Q_T, defined by equation (16.72) using the product test, is not equal to the direct Törnqvist–Theil price index $P_T(p^0,p^1,v^0,v^1)$, defined by equation (16.48). The product test equation that defines P_{IT} in the present context is given by the following equation:

$$P_{IT}(q^0,q^1,v^0,v^1) \equiv \frac{\sum_{i=1}^{n} v_i^1}{\left(\sum_{i=1}^{n} v_i^0\right) Q_T(q^0,q^1,v^0,v^1)} \qquad (16.73)$$

The fact that the direct Törnqvist–Theil price index P_T is not in general equal to the implicit Törnqvist–Theil price index P_{IT} ,defined by equation (16.73), is something of a disadvantage compared to the axiomatic approach outlined in paragraphs 16.30 to 16.73, which led to the Fisher ideal price and quantity indices being considered "best". Using the Fisher approach meant that it was not necessary to decide whether the aim was to find a "best" price index or a "best" quantity index: the theory outlined in paragraphs 16.30 to 16.73 determined both indices simultaneously. In the Törnqvist–Theil approach outlined in this section, however, it is necessary to choose between a "best" price index or a "best" quantity index.[69]

16.123 Other tests are of course possible. A counterpart to Test T16 in paragraph 16.50, the Paasche and Laspeyres bounding test, is the following *geometric Paasche and Laspeyres bounding test*:

$$\begin{aligned} &P_{GL}(p^0,p^1,v^0,v^1) \le P(p^0,p^1,v^0,v^1) \le P_{GP}(p^0,p^1,v^0,v^1) \text{ or} \\ &P_{GP}(p^0,p^1,v^0,v^1) \le P(p^0,p^1,v^0,v^1) \le P_{GL}(p^0,p^1,v^0,v^1) \end{aligned} \qquad (16.74)$$

where the logarithms of the geometric Laspeyres and geometric Paasche price indices, P_{GL} and P_{GP}, are defined as follows:

$$\ln P_{GL}(p^0,p^1,v^0,v^1) \equiv \sum_{i=1}^{n} s_i^0 \ln\left(\frac{p_i^1}{p_i^0}\right) \qquad (16.75)$$

$$\ln P_{GP}(p^0,p^1,v^0,v^1) \equiv \sum_{i=1}^{n} s_i^1 \ln\left(\frac{p_i^1}{p_i^0}\right) \qquad (16.76)$$

As usual, the period t expenditure share on commodity i, s_i^t, is defined as $v_1^t / \sum_{k=1}^{n} v_k^t$ for i = 1,…,n and t = 0,1. It can be shown that the Törnqvist–Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined by equation (16.48) satisfies this test, but the geometric Walsh price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by equation (16.65) does not. The geometric Paasche and Laspeyres bounding test was not included as a primary test in this section because it was not known a priori what form of averaging of the price relatives (e. g., geometric or arithmetic or harmonic) would turn out to be appropriate in this test framework. The test (16.74) is an appropriate one if it has been decided that geometric averaging of the price relatives is the appropriate framework, since the geometric Paasche and Laspeyres indices correspond to "extreme" forms of value weighting in the context of geometric averaging and it is natural to require that the "best" price index lies between these extreme indices.

[69] Hillinger (2002) suggested taking the geometric mean of the direct and implicit Törnqvist–Theil price indices in order to resolve this conflict. Unfortunately, the resulting index is not "best" for either set of axioms that were suggested in this section.

16.124 Walsh (1901, p. 408) pointed out a problem with his geometric price index defined by equation (16.65), which also applies to the Törnqvist–Theil price index, $P_T(p^0,p^1,v^0,v^1)$, defined by equation (16.48): these geometric type indices do not give the "right" answer when the quantity vectors are constant (or proportional) over the two periods. In this case, Walsh thought that the "right" answer must be the Lowe index, which is the ratio of the costs of purchasing the constant basket during the two periods. Put another way, the geometric indices P_{GW} and P_T do not satisfy the fixed basket test T4 in paragraph 16.35. What then was the argument that led Walsh to define his geometric average type index P_{GW}? It turns out that he was led to this type of index by considering another test, which will now be explained.

16.125 Walsh (1901, pp. 228-231) derived his test by considering the following very simple framework. Let there be only two commodities in the index and suppose that the expenditure share on each commodity is equal in each of the two periods under consideration. The price index under these conditions is equal to $P(p_1^0,p_2^0;p_1^1,p_2^1;v_1^0,v_2^0;v_1^1,v_2^1) = P^*(r_1,r_2;1/2,1/2;1/2,1/2) \equiv m(r_1,r_2)$, where $m(r_1,r_2)$ is a symmetric mean of the two price relatives, $r_1 \equiv p_1^1/p_1^0$ and $r_2 \equiv p_2^1/p_2^0$.[70] In this framework, Walsh then proposed the following *price relative reciprocal test*:

$$m(r_1, r_1^{-1}) = 1 \qquad (16.77)$$

Thus, if the value weighting for the two commodities is equal over the two periods and the second price relative is the reciprocal of the first price relative r_1, then Walsh (1901, p. 230) argued that the overall price index under these circumstances ought to equal one, since the relative fall in one price is exactly counterbalanced by a rise in the other and both commodities have the same expenditures in each period. He found that the geometric mean satisfied this test perfectly but the arithmetic mean led to index values greater than one (provided that r_1 was not equal to one) and the harmonic mean led to index values that were less than one, a situation which was not at all satisfactory.[71] Thus he was led to some form of geometric averaging of the price relatives in one of his approaches to index number theory.

16.126 A generalization of Walsh's result is easy to obtain. Suppose that the mean function, $m(r_1,r_2)$, satisfies Walsh's reciprocal test (16.77) and, in addition, m is a homogeneous mean, so that it satisfies the following property for all $r_1 > 0$, $r_2 > 0$ and $\lambda > 0$:

$$m(\lambda r_1, \lambda r_2) = \lambda m(r_1, r_2) \qquad (16.78)$$

Let $r_1 > 0$, $r_2 > 0$. Then

$$\begin{aligned} m(r_1, r_2) &= \left(\frac{r_1}{r_1}\right) m(r_1, r_2) \\ &= r_1 m\left(\frac{r_1}{r_1}, \frac{r_2}{r_1}\right) \qquad \text{using (16.78) with } \lambda = \frac{1}{r_1} \qquad (16.79) \\ &= r_1 m \geq \left(1, \frac{r_2}{r_1}\right) = r_1 f\left(\frac{r_2}{r_1}\right) \end{aligned}$$

where the function of one (positive) variable $f(z)$ is defined as

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[70] Walsh considered only the cases where m was the arithmetic, geometric and harmonic means of r_1 and r_2.

[71] "This tendency of the arithmetic and harmonic solutions to run into the ground or to fly into the air by their excessive demands is clear indication of their falsity" (Walsh (1901, p. 231)).

$f(z) \equiv m(1, z)$ (16.80)

Using equation (16.77):

$$1 = m(r_1, r_1^{-1})$$

$$= \left(\frac{r_1}{r_1}\right) m(r_1, r_1^{-1}) \quad (16.81)$$

$$= r_1 m(1, r_1^{-2}) \qquad \text{using (16.78) with } \lambda = \frac{1}{r_1}$$

Using equation (16.80), equation (16.81) can be rearranged in the following form:

$$f(r_1^{-2}) = r_1^{-1} \quad (16.82)$$

Letting $z \equiv r_1^{-2}$ so that $z^{1/2} = r_1^{-1}$, equation (16.82) becomes:

$$f(z) = z^{1/2} \quad (16.83)$$

Now substitute equation (16.83) into equation (16.79) and the functional form for the mean function $m(r_1,r_2)$ is determined:

$$m(r_1, r_2) = r_1 f\left(\frac{r_2}{r_1}\right) = r_1 \left(\frac{r_2}{r_1}\right)^{1/2} = r_1^{1/2} r_2^{1/2} \quad (16.84)$$

Thus, the geometric mean of the two price relatives is the only homogeneous mean that will satisfy Walsh's price relative reciprocal test.

16.127 There is one additional test that should be mentioned. Fisher (1911; p. 401) introduced this test in his first book that dealt with the test approach to index number theory. He called it the *test of determinateness as to prices* and described it as follows: "A price index should not be rendered zero, infinity, or indeterminate by an individual price becoming zero. Thus, if any commodity should in 1910 be a glut on the market, becoming a 'free good', that fact ought not to render the index number for 1910 zero." In the present context, this test could be interpreted as the following one: if any single price p_i^0 or p_i^1 tends to zero, then the price index $P(p^0,p^1,v^0,v^1)$ should not tend to zero or plus infinity. However, with this interpretation of the test, which regards the values v_i^t as remaining constant as the p_i^0 or p_i^1 tends to zero, none of the commonly used index number formulae would satisfy this test. Hence this test should be interpreted as a test that applies to price indices $P(p^0,p^1,q^0,q^1)$ of the type studied in paragraphs 16.30 to 16.73, which is how Fisher intended the test to apply. Thus, Fisher's price determinateness test should be interpreted as follows: if any single price p_i^0 or p_i^1 tends to zero, then the price index $P(p^0,p^1,q^0,q^1)$ should not tend to zero or plus infinity. With this interpretation of the test, it can be verified that Laspeyres, Paasche and Fisher indices satisfy this test but the Törnqvist–Theil price index does not. Thus, when using the Törnqvist–Theil price index, care must be taken to bound the prices away from zero in order to avoid a meaningless index number value.

16.128 Walsh was aware that geometric average type indices such as the Törnqvist-Theil price index P_T or Walsh's geometric price index P_{GW} defined by equation (16.64) become somewhat unstable[72] as individual price relatives become very large or small:

> Hence in practice the geometric average is not likely to depart much from the truth. Still, we have seen that when the classes [i.e., expenditures] are very unequal and the price variations are very great, this average may deflect considerably (Walsh (1901, p. 373)).

[72] That is, the index may approach zero or plus infinity.

> In the cases of moderate inequality in the sizes of the classes and of excessive variation in one of the prices, there seems to be a tendency on the part of the geometric method to deviate by itself, becoming untrustworthy, while the other two methods keep fairly close together (Walsh (1901, p. 404)).

16.129 Weighing all the arguments and tests presented above, it seems that there may be a slight preference for the use of the Fisher ideal price index as a suitable target index for a statistical agency, but, of course, opinions may differ on which set of axioms is the most appropriate to use in practice.

The test properties of the Lowe and Young indices

16.130 The Young and Lowe indices were defined in Chapter 15. In the present section, the axiomatic properties of these indices with respect to their price arguments are developed.[73]

16.131 Let $q^b \equiv [q_1{}^b,...,q_n{}^b]$ and $p^b \equiv [p_1{}^b,...,p_n{}^b]$ denote the quantity and price vectors pertaining to some base year. The corresponding *base year expenditure shares* can be defined in the usual way as

$$s_i^b \equiv \frac{p_i^b q_i^b}{\sum_{k=1}^{n} p_k^b q_k^b} \qquad i = 1,...,n \qquad (16.85)$$

Let $s^b \equiv [s_1{}^b,...,s_n{}^b]$ denote the vector of base year expenditure shares. The Young (1812) price index between periods 0 and t is defined as follows:

$$P_Y(p^0, p^t, s^b) \equiv \sum_{i=1}^{n} s_i^b \left(\frac{p_i^t}{p_i^0}\right) \qquad (16.86)$$

The Lowe (1823, p. 316) price index[74] between periods 0 and t is defined as follows:

$$P_{Lo}(p^0, p^t, q^b) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{k=1}^{n} p_k^0 q_k^b} = \frac{\sum_{i=1}^{n} s_i^b \left(\frac{p_i^t}{p_i^b}\right)}{\sum_{k=1}^{n} s_k^b \left(\frac{p_k^0}{p_k^b}\right)} \qquad (16.87)$$

16.132 Drawing on the axioms listed above in this chapter, 12 desirable axioms for price indices of the form $P(p^0,p^1)$ are listed below. The period 0 and t price vectors, p^0 and p^t, are presumed to have strictly positive components.

T1: *Positivity*: $P(p^0,p^t) > 0$ if all prices are positive
T2: *Continuity*: $P(p^0,p^t)$ is a continuous function of prices
T3: *Identity test*: $P(p^0,p^0) = 1$
T4: *Homogeneity test for period* t *prices*: $P(p^0,\lambda p^t) = \lambda P(p^0,p^t)$ for all $\lambda > 0$
T5: *Homogeneity test for period* 0 *prices*: $P(\lambda p^0,p^t) = \lambda^{-1}P(p^0,p^t)$ for all $\lambda > 0$

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[73] Baldwin (1990, p. 255) worked out a few of the axiomatic properties of the Lowe index.

[74] This index number formula is also precisely Bean and Stine's (1924, p. 31) Type A index number formula. Walsh (1901, p. 539) initially mistakenly attributed Lowe's formula to G. Poulett Scrope (1833), who wrote *Principles of political economy* in 1833 and suggested Lowe's formula without acknowledging Lowe's priority. But in his discussion of Fisher's (1921) paper, Walsh (1921b, p. 543-544) corrects his mistake on assigning Lowe's formula:

> What index number should you then use? It should be this: $\sum q\, p_1 / \sum q\, p_0$. This is the method used by Lowe within a year or two of one hundred years ago. In my [1901] book, I called it Scope's index number; but it should be called Lowe's. Note that in it are used quantities neither of a base year nor of a subsequent year. The quantities used should be rough estimates of what the quantities were throughout the period or epoch.

T6: *Commodity reversal test*: $P(p^0,p^t) = P(p^{0*},p^{t*})$ where p^{0*} and p^{t*} denote the same permutation of the components of the price vectors p^0 and p^t[75]

T7: *Invariance to changes in the units of measurement (commensurability test)*

T8: *Time reversal test*: $P(p^t,p^0) = 1/P(p^0,p^t)$

T9: *Circularity or transitivity test*: $P(p^0,p^2) = P(p^0,p^1)P(p^1,p^2)$

T10: *Mean value test*: $\min\{p_i^t/p_i^0 : i = 1,\ldots,n\} \le P(p^0,p^t) \le \max\{p_i^t/p_i^0 : i = 1,\ldots,n\}$

T11: *Monotonicity test with respect to period* t *prices*: $P(p^0,p^t) < P(p^0,p^{t*})$ if $p^t < p^{t*}$

T12: *Monotonicity test with respect to period* 0 *prices*: $P(p^0,p^t) > P(p^{0*},p^t)$ if $p^0 < p^{0*}$

16.133 It is straightforward to show that the Lowe index defined by equation (16.87) satisfies all 12 of the axioms or tests listed above. Hence the Lowe index has very good axiomatic properties with respect to its price variables.[76]

16.134 It is straightforward to show that the Young index defined by equation (16.86) satisfies 10 of the 12 axioms, failing the time reversal test T8 and the circularity test T9. Thus the axiomatic properties of the Young index are definitely inferior to those of the Lowe index.

16.135 In place of the Young index P_Y defined by (16.86), it possible to define the Geometric Young index (or the weighted Jevons index) as follows:

$$P_{GY}(p^0,p^t,s^b) \equiv \prod_{i=1}^{n}\left[\frac{p_i^t}{p_i^0}\right]^{s_i^b} . \qquad (16.88)$$

This index satisfies all 12 tests so it is as good as the Lowe index with respect to its axiomatic properties.

Appendix 16.1 Proof of the optimality of the Törnqvist–Theil price index in the second bilateral test approach

The tests (T1, T2, etc.) mentioned in this appendix are those presented in paragraphs 16.98 to 16.119.

1. Define $r_i \equiv p_i^1/p_i^0$ for $i = 1,\ldots,n$. Using T1, T9 and equation (16.66), $P(p^0,p^1,v^0,v^1) = P^*(r, v^0,v^1)$. Using T6, T7 and equation (16.63):

$$P(p^0,p^1,v^0,v^1) = P^*(r,s^0,s^1) \qquad (A16.1.1)$$

where s^t is the period t expenditure share vector for $t = 0,1$.

2. Let $x \equiv (x_1,\ldots,x_n)$ and $y \equiv (y_1,\ldots,y_n)$ be strictly positive vectors. The transitivity test T11 and equation (A16.1.1) imply that the function P^* has the following property:

$$P^*(x;s^0,s^1)P^*(y;s^0,s^1) = P^*(x_1y_1,\ldots,x_ny_n;s^0,s^1) \qquad (A16.1.2)$$

3. Using test T1, $P^*(r,s^0,s^1) > 0$ and using test T14, $P^*(r, s^0,s^1)$ is strictly increasing in the components of r. The identity test T3 implies that

$$P^*(1_n,s^0,s^1) = 1 \qquad (A16.1.3)$$

[75] In applying this test to the Lowe and Young indices, it is assumed that the base year quantity vector q^b and the base year share vector s^b are subject to the same permutation.

[76] From the discussion in Chapter 15, it will be recalled that the main problem with the Lowe index occurs if the quantity weight vector q^b is not representative of the quantities that were purchased during the time interval between periods 0 and 1.

where 1_n is a vector of ones of dimension n. Using a result attributable to Eichhorn (1978, p. 66), it can be seen that these properties of P^* are sufficient to imply that there exist positive functions $\alpha_i(s^0,s^1)$ for $i = 1,\ldots,n$ such that P^* has the following representation:

$$\ln P^*(r,s^0,s^1) = \sum_{i=1}^{n}\alpha_i(s^0,s^1)\ln r_i \qquad \text{(A16.1.4)}$$

4. The continuity test T2 implies that the positive functions $\alpha_i(s^0,s^1)$ are continuous. For $\lambda > 0$, the linear homogeneity test T4 implies that

$$\begin{aligned}\ln P^*(\lambda r,s^0,s^1) &= \ln\lambda + \ln P^*(r,s^0,s^1)\\ &= \sum_{i=1}^{n}\alpha_i(s^0,s^1)\ln\lambda r_i && \text{using (A16.1.4)}\\ &= \sum_{i=1}^{n}\alpha_i(s^0,s^1)\ln\lambda + \sum_{i=1}^{n}\alpha_i(s^0,s^1)\ln r_i && \text{(A16.1.5)}\\ &= \sum_{i=1}^{n}\alpha_i(s^0,s^1)\ln\lambda + \ln P^*(r,s^0,s^1) && \text{using (A16.1.4).}\end{aligned}$$

Equating the right-hand sides of the first and last lines in equation (A16.1.5) shows that the functions $\alpha_i(s^0,s^1)$ must satisfy the following restriction:

$$\sum_{i=1}^{n}\alpha_i(s^0,s^1) = 1 \qquad \text{(A16.1.6)}$$

for all strictly positive vectors s^0 and s^1.

5. Using the weighting test T16 and the commodity reversal test T8, equations (16.69) hold. Equation (16.69) combined with the commensurability test T9 implies that P^* satisfies the following equation:

$$P^*(1,\ldots,1,r_i,1,\ldots,1;\,s^0,s^1) = f(1,r_i,s_i^0,s_i^1); \qquad i=1,\ldots,n \qquad \text{(A16.1.7)}$$

for all $r_i > 0$ where f is the function defined in test T16.

6. Substitute equation (A16.1.7) into equation (A16.1.4) in order to obtain the following system of equations:

$$\ln P^*(1,\ldots,1,r_i,1,\ldots,1;\,s^0,s^1) = \ln f(1,r_i,s^0,s^1) = \alpha_i(s^0,s^1)\ln r_i; \qquad i=1,\ldots,n. \quad \text{(A16.1.8)}$$

But equation (A16.1.8) implies that the positive continuous function of $2n$ variables $\alpha_i(s^0,s^1)$ is constant with respect to all of its arguments except s_i^0 and s_i^1 and this property holds for each i. Thus each $\alpha_i(s^0,s^1)$ can be replaced by the positive continuous function of two variables $\beta_i(s_i^0,s_i^1)$ for $i = 1,\ldots,n$.[77] Now replace the $\alpha_i(s^0,s^1)$ in equation (A16.1.4) by the $\beta_i(s_i^0,s_i^1)$ for $i = 1,\ldots,n$ and the following representation for P^* is obtained:

$$\ln P^*(r,s^0,s^1) = \sum_{i=1}^{n}\beta_i(s_i^0,s_i^1)\ln r_i. \qquad \text{(A16.1.9)}$$

7. Equation (A16.1.6) implies that the functions $\beta_i(s_i^0,s_i^1)$ also satisfy the following restrictions:

$$\sum_{i=1}^{n}s_i^0 = 1; \text{ and } \sum_{i=1}^{n}s_i^1 = 1 \text{ implies } \sum_{i=1}^{n}\beta_i(s_i^0,s_i^1) = 1 \qquad \text{(A16.1.10)}$$

[77] More explicitly, $\beta_1(s_1^0,s_1^1) \equiv \alpha_1(s_1^0,1,\ldots,1;s_1^1,1,\ldots,1)$ and so on. That is, in defining $\beta_1(s_1^0,s_1^1)$, the function $\alpha_1(s_1^0,1,\ldots,1;s_1^1,1,\ldots,1)$ is used where all components of the vectors s^0 and s^1 except the first are set equal to an arbitrary positive number such as 1.

8. Assume that the weighting test T17 holds and substitute equation (16.71) into equation (A16.1.9) in order to obtain the following equation:

$$\beta_i(0,0)\ln\left(\frac{p_i^1}{p_i^0}\right) = 0\,; \qquad i = 1,\ldots,n \qquad \text{(A16.1.11)}$$

Since the p_i^1 and p_i^0 can be arbitrary positive numbers, it can be seen that equation (A16.1.11) implies

$$\beta_i(0,0) = 0\ ; \qquad i = 1,\ldots,n \qquad \text{(A16.1.12)}$$

9. Assume that the number of commodities n is equal to or greater than 3. Using equations (A16.1.10) and (A16.1.12), Theorem 2 in Aczél (1987, p. 8) can be applied and the following functional form for each of the $\beta_i(s_i^0,s_i^1)$ is obtained:

$$\beta_i(s_i^0, s_i^1) = \gamma\, s_i^0 + (1-\gamma)s_i^1; \qquad i = 1,\ldots,n \qquad \text{(A16.1.13)}$$

where γ is a positive number satisfying $0 < \gamma < 1$.

10. Finally, the time reversal test T10 *or* the quantity weights symmetry test T12 can be used to show that γ must equal ½. Substituting this value for γ back into equation (A16.1.13) and then substituting that equation back into equation (A16.1.9), the functional form for P^* and hence P is determined as

$$\ln P(p^0, p^1, v^0, v^1) = \ln P^*(r, s^0, s^1) = \sum_{i=1}^{n} \frac{1}{2}(s_i^0 + s_i^1)\ln\left(\frac{p_i^1}{p_i^0}\right). \qquad \text{(A16.1.14)}$$

17 THE ECONOMIC APPROACH TO INDEX NUMBER THEORY: THE SINGLE-HOUSEHOLD CASE

Introduction

17.1 This chapter and the next cover the economic approach to index number theory. This chapter considers the case of a *single* household, while the following chapter deals with the case of *many* households. A brief outline of the contents of the present chapter follows.

17.2 In paragraphs 17.9 to 17.17, the theory of the cost of living index for a single consumer or household is presented. This theory was originally developed by the Russian economist, A.A. Konüs (1924). The relationship between the (unobservable) true cost of living index and the observable Laspeyres and Paasche indices will be explained. It should be noted that, in the economic approach to index number theory, it is assumed that households regard the observed price data as given, while the quantity data are regarded as solutions to various economic optimization problems. Many price statisticians find the assumptions made in the economic approach to be somewhat implausible. Perhaps the best way to regard the assumptions made in the economic approach is that these assumptions simply formalize the fact that consumers tend to purchase more of a commodity if its price falls relative to other prices.

17.3 In paragraphs 17.18 to 17.26, the preferences of the consumer are restricted compared to the completely general case treated in paragraphs 17.9 to 17.17. In paragraphs 17.18 to 17.26, it is assumed that the function that represents the consumer's preferences over alternative combinations of commodities is homogeneous of degree one. This assumption means that each indifference surface (the set of commodity bundles that give the consumer the same satisfaction or utility) is a radial blow-up of a single indifference surface. With this extra assumption, the theory of the true cost of living simplifies, as will be seen.

17.4 In the sections starting with paragraphs 17.27, 17.33 and 17.44, it is shown that the Fisher, Walsh and Törnqvist price indices (which emerge as being "best" in the various non-economic approaches) are also among the "best" in the economic approach to index number theory. In these sections, the preference function of the single household will be further restricted compared to the assumptions on preferences made in the previous two sections. Specific functional forms for the consumer's utility function are assumed and it turns out that, with each of these specific assumptions, the consumer's true cost of living index can be exactly calculated using observable price and quantity data. Each of the three specific functional forms for the consumer's utility function has the property that it can approximate an arbitrary linearly homogeneous function to the second order; i.e., in economics terminology, each of these three functional forms is *flexible*. Hence, using the terminology introduced by Diewert (1976), the Fisher, Walsh and Törnqvist price indices are examples of *superlative* index number formulae.

17.5 In paragraphs 17.50 to 17.54, it is shown that the Fisher, Walsh and Törnqvist price indices approximate each other very closely using "normal" time series data. This is a very convenient result since these three index number formulae repeatedly show up as being "best" in all the approaches to index number theory. Hence this approximation result implies that it normally will not matter which of these three indices is chosen as the preferred target index for a consumer price index (CPI).

17.6 The Paasche and Laspeyres price indices have a very convenient mathematical property: they are *consistent in aggregation*. For example, if the Laspeyres formula is used to construct sub-indices for, say, food or clothing, then these sub-index values can be treated as sub-aggregate price relatives and, using the expenditure shares on these sub-aggregates, the Laspeyres formula can be applied again to form a two-stage Laspeyres price index. Consistency in aggregation means that this two-stage index is equal to the corresponding single-stage index. In paragraphs 17.55 to 17.60, it is shown that the superlative indices derived in the earlier sections are not exactly consistent in aggregation but are approximately consistent in aggregation.

17.7 In paragraphs 17.61 to 17.64, a very interesting index number formula is derived: the Lloyd (1975) and Moulton (1996a) price index. This index number formula makes use of the same information that is required in order to calculate a Laspeyres index (namely, base period expenditure shares, base period prices and current period prices), plus one other parameter (the elasticity of substitution between commodities). If information on this extra parameter can be obtained, then the resulting index can largely eliminate substitution bias and it can be calculated using basically the same information that is required to obtain the Laspeyres index.

17.8 The section starting with paragraph 17.65 considers the problem of defining a true cost of living index when the consumer has annual preferences over commodities but faces monthly (or quarterly) prices. This section attempts to provide an economic foundation for the Lowe index studied in Chapter 15. It also provides an introduction to the problems associated with the existence of seasonal commodities, which are considered at more length in Chapter 22. The final section deals with situations where there may be a zero price for a commodity in one period, but where the price is non-zero in the other period.

The Konüs cost of living index and observable bounds

17.9 This section deals with the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, Konüs (1924). This theory relies on the assumption of *optimizing behaviour* on the part of economic agents (consumers or producers). Thus, given a vector of commodity prices p^t that the household faces in a given time period t, it is assumed that the corresponding observed quantity vector q^t is the solution to a cost minimization problem that involves the consumer's preference or utility function f.[1] Thus in contrast to the axiomatic approach to index number theory, the economic approach does not assume that the two quantity vectors q^0 and q^1 are independent of the two price vectors p^0 and p^1. In the economic approach, the period 0 quantity vector q^0 is determined by the consumer's preference function f and the period 0 vector of prices p^0 that the consumer faces, and the period 1 quantity vector q^1 is determined by the consumer's preference function f and the period 1 vector of prices p^1.

17.10 The economic approach to index number theory assumes that "the" consumer has well-defined *preferences* over different combinations of the n consumer commodities or

[1] For a description of the economic theory of the input and output price indices, see Balk (1998a). In the economic theory of the output price index, q^t is assumed to be the solution to a revenue maximization problem involving the output price vector p^t.

items.[2] Each combination of items can be represented by a positive quantity vector $q \equiv [q_1,\ldots,q_n]$. The consumer's preferences over alternative possible consumption vectors, q, are assumed to be representable by a continuous, non-decreasing and concave[3] utility function f. Thus if $f(q^1) > f(q^0)$, then the consumer prefers the consumption vector q^1 to q^0. It is further assumed that the consumer minimizes the cost of achieving the period t utility level $u^t \equiv f(q^t)$ for periods $t = 0,1$. Thus we assume that the observed period t consumption vector q^t solves the following period t cost minimization problem:

$$\begin{aligned} C(u^t, p^t) &\equiv \min_q \left\{ \sum_{i=1}^{n} p_i^t q_i : f(q) = u^t \equiv f(q^t) \right\} \\ &= \sum_{i=1}^{n} p_i^t q_i^t \quad \text{for} \quad t = 0,1 \end{aligned} \qquad (17.1)$$

The period t price vector for the n commodities under consideration that the consumer faces is p^t. Note that the solution to the cost or expenditure minimization problem (17.1) for a general utility level u and general vector of commodity prices p defines the *consumer's cost function*, $C(u, p)$. The cost function will be used below in order to define the consumer's *cost of living price index*.

17.11 The Konüs (1924) family of *true cost of living indices* pertaining to two periods where the consumer faces the strictly positive price vectors $p^0 \equiv (p_1^0,\ldots,p_n^0)$ and $p^1 \equiv (p_1^1,\ldots,p_n^1)$ in periods 0 and 1, respectively, is defined as the ratio of the minimum costs of achieving the same utility level $u \equiv f(q)$ where $q \equiv (q_1,\ldots,q_n)$ is a positive reference quantity vector:

$$P_K(p^0, p^1, q) \equiv \frac{C(f(q), p^1)}{C(f(q), p^0)} \qquad (17.2)$$

Note that definition (17.2) defines a family of price indices, because there is one such index for each reference quantity vector q chosen.

17.12 It is natural to choose two specific reference quantity vectors q in definition (17.2): the observed base period quantity vector q^0 and the current period quantity vector q^1. The first of these two choices leads to the following *Laspeyres–Konüs true cost of living index*:

[2] In this chapter, these preferences are assumed to be invariant over time, while in the following chapter, this assumption is relaxed (one of the environmental variables could be a time variable that shifts tastes).

[3] Note that f is concave if and only if $f(\lambda q^1 + (1-\lambda)q^2) \geq \lambda f(q^1) + (1-\lambda)f(q^2)$ for all $0 \leq \lambda \leq 1$ and all $q^1 >> 0_n$ and $q^2 >> 0_n$. Note also that $q \geq 0_N$ means that each component of the N-dimensional vector q is non-negative, $q >> 0_n$ means that each component of q is positive and $q > 0_n$ means that $q \geq 0_n$ but $q \neq 0_n$; i.e., q is non-negative but at least one component is positive.

$$P_K(p^0,p^1,q^0) \equiv \frac{C(f(q^0),p^1)}{C(f(q^0),p^0)}$$

$$= \frac{C(f(q^0),p^1)}{\sum_{i=1}^{n} p_i^0 q_i^0} \quad \text{using (17.1) for } t=0 \tag{17.3}$$

$$= \frac{\min_q \left\{ \sum_{i=1}^{n} p_i^1 q_i : f(q) = f(q^0) \right\}}{\sum_{i=1}^{n} p_i^0 q_i^0}$$

using the definition of the cost minimization problem that defines C(f(q^0), p^1

$$\leq \frac{\sum_{i=1}^{n} p_i^1 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0}$$

since $q^0 \equiv (q^0{}_{1,\ldots}\, q^0{}_n)$ is feasible for the minimization problem

$$= P_L(p^0,p^1,q^0,q^1)$$

where P_L is the Laspeyres price index. Thus the (unobservable) Laspeyres–Konüs true cost of living index is bounded from above by the observable Laspeyres price index.[4]

17.13 The second of the two natural choices for a reference quantity vector q in definition (17.2) leads to the following *Paasche–Konüs true cost of living index*:

$$P_K(p^0,p^1,q^1) \equiv \frac{C(f(q^1),p^1)}{C(f(q^1),p^0)}$$

$$= \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{C(f(q^1),p^0)} \quad \text{using (17.1) for } t=1$$

$$= \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\min_q \left\{ \sum_{i=1}^{n} p_i^0 q_i : f(q) = f(q^1) \right\}}$$

using the definition of the cost minimization problem that defines $C(f(q^0),p^0)$ (17.4)

$$\geq \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^1} \quad \text{since } q^1 \equiv (q_1^1,\ldots,q_n^1) \text{ is feasible for the minimization problem and thus}$$

$$C(f(q^1),p^0) \leq \sum_{i=1}^{n} p_i^0 q_i^1 \text{ and hence } \frac{1}{C(f(q^1),p^0)} \geq \frac{1}{\sum_{i=1}^{n} p_i^0 q_i^1}$$

$$= P_P(p^0,p^1,q^0,q^1)$$

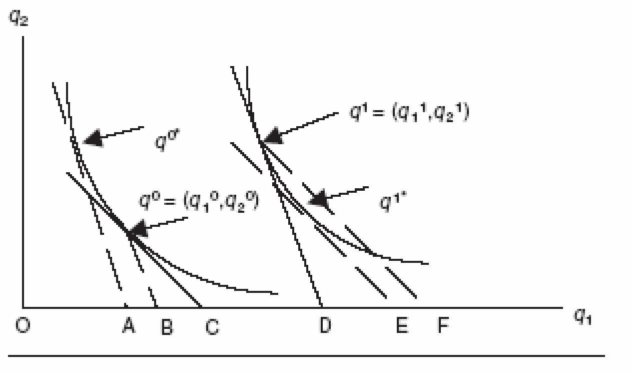
[4] This inequality was first obtained by Konüs (1924; 1939, p. 17). See also Pollak (1983).

where P_P is the Paasche price index. Thus the (unobservable) Paasche–Konüs true cost of living index is bounded from below by the observable Paasche price index.[5]

17.14 It is possible to illustrate the two inequalities (17.3) and (17.4) if there are only two commodities; see Figure 17.1. The solution to the period 0 cost minimization problem is the vector q^0. The straight line C represents the consumer's period 0 budget constraint, the set of quantity points q_1, q_2 such that $p_1^0 q_1 + p_2^0 q_2 = p_1^0 q_1^0 + p_2^0 q_2^0$. The curved line through q^0 is the consumer's period 0 indifference curve, the set of points q_1, q_2 such that $f(q_1, q_2) = f(q_1^0, q_2^0)$; i.e., it is the set of consumption vectors that give the same utility as the observed period 0 consumption vector q^0. The solution to the period 1 cost minimization problem is the vector q^1. The straight line D represents the consumer's period 1 budget constraint, the set of quantity points q_1, q_2 such that $p_1^1 q_1 + p_2^1 q_2 = p_1^1 q_1^1 + p_2^1 q_2^1$. The curved line through q^1 is the consumer's period 1 indifference curve, the set of points q_1, q_2 such that $f(q_1, q_2) = f(q_1^1, q_2^1)$; i.e., it is the set of consumption vectors that give the same utility as the observed period 1 consumption vector q^1. The point q^{0*} solves the hypothetical problem of minimizing the cost of achieving the base period utility level $u^0 \equiv f(q^0)$ when facing the period 1 price vector $p^1 = (p_1^1, p_2^1)$. Thus we have $C[u^0, p^1] = p_1^1 q_1^{0*} + p_2^1 q_2^{0*}$ and the dashed line A is the corresponding isocost line $p_1^1 q_1 + p_2^1 q_2 = C[u^0, p^1]$. Note that the hypothetical cost line A is parallel to the actual period 1 cost line D. From equation (17.3), the Laspeyres–Konüs true index is $C[u^0, p^1] / [p_1^0 q_1^0 + p_2^0 q_2^0]$, while the ordinary Laspeyres index is $[p_1^1 q_1^0 + p_2^1 q_2^0] / [p_1^0 q_1^0 + p_2^0 q_2^0]$. Since the denominators for these two indices are the same, the difference between the indices is attributable to the differences in their numerators. In Figure 17.1, this difference in the numerators is expressed by the fact that the cost line through A lies below the parallel cost line through B. Now if the consumer's indifference curve through the observed period 0 consumption vector q^0 were L-shaped with vertex at q^0, then the consumer would not change his or her consumption pattern in response to a change in the relative prices of the two commodities while keeping a fixed standard of living. In this case, the hypothetical vector q^{0*} would coincide with q^0, the dashed line through A would coincide with the dashed line through B and the true Laspeyres–Konüs index would coincide with the ordinary Laspeyres index. However, L-shaped indifference curves are not generally consistent with consumer behaviour; i.e., when the price of a commodity decreases, consumers generally demand more of it. Thus, in the general case, there will be a gap between the points A and B. The magnitude of this gap represents the amount of *substitution bias* between the true index and the corresponding Laspeyres index; i.e., the Laspeyres index will generally be greater than the corresponding true cost of living index, $P_K(p^0, p^1, q^0)$.

Figure 17.1 The Laspeyres and Paasche bounds to the true cost of living

[5] This inequality is attributable to Konüs (1924; 1939, p. 19); see also Pollak (1983).



17.15 Figure 17.1 can also be used to illustrate the inequality (17.4). First note that the dashed lines through E and *F* are parallel to the period 0 isocost line through C. The point q^{1*} solves the hypothetical problem of minimizing the cost of achieving the current period utility level $u^1 \equiv f(q^1)$ when facing the period 0 price vector $p^0 = (p_1^0,p_2^0)$. Thus we have $C[u^1,p^0] = p_1^0 q_1^{1*} + p_2^0 q_2^{1*}$. From equation (17.4), the Paasche–Konüs true index is $[p_1^1 q_1^1 + p_2^1 q_2^1] / C[u^1,p^0]$, while the ordinary Paasche index is $[p_1^1 q_1^1 + p_2^1 q_2^1] / [p_1^0 q_1^1 + p_2^0 q_2^1]$. Since the numerators for these two indices are the same, the difference between the indices is attributable to the differences in their denominators. In Figure 17.1, this difference in the denominators is expressed by the fact that the cost line through E lies below the parallel cost line through F. The magnitude of this difference represents the amount of substitution bias between the true index and the corresponding Paasche index; i.e., the Paasche index will generally be less than the corresponding true cost of living index, $P_K(p^0, p^1, q^1)$. Note that this inequality goes in the opposite direction to the previous inequality between the two Laspeyres indices. The reason for this change in direction is attributable to the fact that one set of differences between the two indices takes place in the numerators of the indices (the Laspeyres inequalities), while the other set takes place in the denominators of the indices (the Paasche inequalities).

17.16 The bound (17.3) on the Laspeyres–Konüs true cost of living index $P_K(p^0, p^1, q^0)$ using the base period level of utility as the living standard is *one-sided*, as is the bound (17.4) on the Paasche–Konüs true cost of living index $P_K(p^0, p^1, q^1)$ using the *current period* level of utility as the living standard. In a remarkable result, Konüs (1924; 1939, p. 20) showed that there exists an intermediate consumption vector q^* that is on the straight line joining the base period consumption vector q^0 and the current period consumption vector q^1 such that the corresponding (unobservable) true cost of living index $P_K(p^0, p^1, q^*)$ is between the observable Laspeyres and Paasche indices, P_L and P_P.[6] Thus we have the existence of a number λ^* between 0 and 1 such that

$$P_L \le P_K(p^0, p^1, \lambda^* q^0 + (1-\lambda^*)q^1) \le P_P \text{ or } P_P \le P_K(p^0, p^1, \lambda^* q^0 + (1-\lambda^*)q^1) \le P_L. \quad (17.5)$$

The inequalities (17.5) are of some practical importance. If the observable (in principle) Paasche and Laspeyres indices are not too far apart, then taking a symmetric average of these indices should provide a good approximation to a true cost of living index where the reference standard of living is somewhere between the base and current period living standards. To determine the precise symmetric average of the Paasche and Laspeyres indices, appeal can be made to the results in paragraphs 15.18 to 15.32 in Chapter 15, and the

[6] For more recent applications of the Konüs method of proof, see Diewert (1983a, p. 191) for an application to the consumer context and Diewert (1983b, pp. 1059-1061) for an application to the producer context.

geometric mean of the Paasche and Laspeyres indices can be justified as being the "best" average, which is the Fisher price index. Thus the Fisher ideal price index receives a fairly strong justification as a good approximation to an unobservable theoretical cost of living index.

17.17 The bounds (17.3)–(17.5) are the best that can be obtained on true cost of living indices without making further assumptions. Further assumptions are made below on the class of utility functions that describe the consumer's tastes for the n commodities under consideration. With these extra assumptions, the consumer's true cost of living can be determined exactly.

The true cost of living index when preferences are homothetic

17.18 Up to now, the consumer's preference function f did not have to satisfy any particular homogeneity assumption. For the remainder of this section, it is assumed that f is (positively*) linearly homogeneous*.[7] In the economics literature, this is known as the assumption of *homothetic preferences*.[8] This assumption is not strictly justified from the viewpoint of actual economic behaviour, but it leads to economic price indices that are independent of the consumer's standard of living.[9] Under this assumption, the consumer's expenditure or cost function, $C(u, p)$ defined by equation (17.1), decomposes as follows. For positive commodity prices $p >> 0_N$ and a positive utility level u, then, using the definition of C as the minimum cost of achieving the given utility level u, the following equalities can be derived:

[7] The linear homogeneity property means that f satisfies the following property: $f(\lambda q) = \lambda f(q)$ for all $\lambda > 0$ and all $q >> 0_n$. This assumption is fairly restrictive in the consumer context. It implies that each indifference curve is a radial projection of the unit utility indifference curve. It also implies that all income elasticities of demand are unity, which is contradicted by empirical evidence.

[8] More precisely, Shephard (1953) defined a homothetic function to be a monotonic transformation of a linearly homogeneous function. However, if a consumer's utility function is homothetic, it can always be rescaled to be linearly homogeneous without changing consumer behaviour. Hence, the homothetic preferences assumption can simply be identified with the linear homogeneity assumption.

[9] This particular branch of the economic approach to index number theory is attributable to Shephard (1953; 1970) and Samuelson and Swamy (1974). Shephard in particular realized the importance of the homotheticity assumption in conjunction with separability assumptions in justifying the existence of subindices of the overall cost of living index. It should be noted that, if the consumer's change in real income or utility between the two periods under consideration is not too large, then assuming that the consumer has homothetic preferences will lead to a true cost of living index which is very close to Laspeyres–Konüs and Paasche–Konüs true cost of living indices defined by equations (17.3) and (17.4). Another way of justifying the homothetic preferences assumption is to use equation (17.49), which justifies the use of the superlative Törnqvist–Theil index P_T in the context of non-homothetic preferences. Since P_T is usually numerically close to other superlative indices that are derived using the homothetic preferences assumption, it can be seen that the assumption of homotheticity will usually not be empirically misleading in the index number context.

$$
\begin{aligned}
C(u,p) &\equiv \min_q \left\{ \sum_{i=1}^{n} p_i q_i : f(q_1,...,q_n) \geq u \right\} \\
&= \min_q \left\{ \sum_{i=1}^{n} p_i q_i : \frac{1}{u} f(q_1,...,q_n) \geq 1 \right\} \text{ dividing by } u > 0 \\
&= \min_q \left\{ \sum_{i=1}^{n} p_i q_i : f(\frac{q_1}{u},...,\frac{q_n}{u}) \geq 1 \right\} \text{ using the linear homogeneity of } f \\
&= u \min_q \left\{ \sum_{i=1}^{n} \frac{p_i q_i}{u} : f(\frac{q_1}{u},...,\frac{q_n}{u}) \geq 1 \right\} \qquad (17.6) \\
&= u \min_z \left\{ \sum_{i=1}^{n} p_i z_i : f(z_1,...,z_n) \geq 1 \right\} \text{ letting } z_i = \frac{q_i}{u} \\
&= uC(1,p) \qquad \text{using definition (17.1)} \\
&= uc(p)
\end{aligned}
$$

where $c(p) \equiv C(1,p)$ is the *unit cost function* that corresponds to f.[10] It can be shown that the unit cost function $c(p)$ satisfies the same regularity conditions that f satisfies; i.e., $c(p)$ is positive, concave and (positively) linearly homogeneous for positive price vectors.[11] Substituting equation (17.6) into equation (17.1) and using $u^t = f(q^t)$ leads to the following equation:

$$\sum_{i=1}^{n} p_i^t q_i^t = c(p^t) f(q^t) \quad \text{for } t = 0,1 \qquad (17.7)$$

Thus, under the linear homogeneity assumption on the utility function f, observed period t expenditure on the n commodities is equal to the period t unit cost $c(p^t)$ of achieving one unit of utility times the period t utility level, $f(q^t)$. Obviously, the period t unit cost, $c(p^t)$, can be identified as the period t price level P^t and the period t level of utility, $f(q^t)$, as the period t quantity level Q^t.[12]

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[10] Economists will recognize the producer theory counterpart to the result $C(u,p) = uc(p)$: if a producer's production function f is subject to constant returns to scale, then the corresponding total cost function $C(u,p)$ is equal to the product of the output level u times the unit cost $c(p)$.

[11] Obviously, the utility function f determines the consumer's cost function $C(u,p)$ as the solution to the cost minimization problem in the first line of equation (17.6). Then the unit cost function $c(p)$ is defined as $C(1,p)$. Thus f determines c. But we can also use c to determine f under appropriate regularity conditions. In the economics literature, this is known as *duality theory*. For additional material on duality theory and the properties of f and c, see Samuelson (1953), Shephard (1953) and Diewert (1974a; 1993b, pp. 107-123).

[12] There is also a producer theory interpretation of the above theory; i.e., let f be the producer's (constant returns to scale) production function, let p be a vector of input prices that the producer faces, let q be an input vector and let $u = f(q)$ be the maximum output that can be produced using the input vector q. $C(u,p) \equiv \min_q \{ \sum_{i=1}^{n} p_i q_i : f(q) \geq u \}$ is the producer's cost function in this case and $c(p^t)$ can be identified as the period t input price level, while $f(q^t)$ is the period t aggregate input level.

17.19 The linear homogeneity assumption on the consumer's preference function f leads to a simplification for the family of Konüs true cost of living indices, $P_K(p^0, p^1, q)$, defined by equation (17.2). Using this definition for an arbitrary reference quantity vector q:

$$
\begin{aligned}
P_K(p^0, p^1, q) &\equiv \frac{C(f(q), p^1)}{C(f(q), p^0)} \\
&= \frac{c(p^1) f(q)}{c(p^0) f(q)} \quad \text{using (17.6) twice} \qquad (17.8) \\
&= \frac{c(p^1)}{c(p^0)}
\end{aligned}
$$

Thus under the homothetic preferences assumption, the entire family of Konüs true cost of living indices collapses to a single index, $c(p^1)/c(p^0)$, the ratio of the minimum costs of achieving unit utility level when the consumer faces period 1 and 0 prices respectively. Put another way, under the homothetic preferences assumption, $P_K(p^0, p^1, q)$ is independent of the reference quantity vector q.

17.20 If the Konüs true cost of living index defined by the right-hand side of equation (17.8) is used as the price index concept, then the corresponding implicit quantity index defined using the product test (i.e., the product of the price index times the quantity index is equal to the value ratio) has the following form:

$$
\begin{aligned}
Q(p^0, p^1, q^0, q^1) &\equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^t q_i^t P_K(p^0, p^1, q)} \\
&= \frac{c(p^1) f(q^1)}{c(p^0) f(q^0) P_K(p^0, p^1, q)} \quad \text{using (17.7) twice} \qquad (17.9) \\
&= \frac{c(p^1) f(q^1)}{c(p^0) f(q^0) \{c(p^1)/c(p^0)\}} \quad \text{using (17.8)} \\
&= \frac{f(q^1)}{f(q^0)}
\end{aligned}
$$

Thus, under the homothetic preferences assumption, the implicit quantity index that corresponds to the true cost of living price index $c(p^1)/c(p^0)$ is the utility ratio $f(q^1)/f(q^0)$. Since the utility function is assumed to be homogeneous of degree one, this is the natural definition for a quantity index.

17.21 In subsequent material, two additional results from economic theory will be needed: Wold's Identity and Shephard's Lemma. Wold's (1944, pp. 69-71; 1953, p. 145) Identity is the following result. Assuming that the consumer satisfies the cost minimization assumptions (17.1) for periods 0 and 1 and that the utility function f is differentiable at the observed quantity vectors q^0 and q^1, it can be shown[13] that the following equation holds:

[13] To prove this, consider the first-order necessary conditions for the strictly positive vector q^t to solve the period t cost minimization problem. The conditions of Lagrange with respect to the vector of q variables are: $p^t = \lambda^t \nabla f(q^t)$, where λ^t is the optimal Lagrange multiplier and $\nabla f(q^t)$ is the vector of first-order partial derivatives of f evaluated at q^t. Note that this system of equations is the price equals a constant times marginal utility equations that are familiar to economists. Now take the inner product of both sides of this equation with respect

(continued)

$$\frac{p_i^t}{\sum_{k=1}^{n} p_k^t q_k^t} = \frac{\dfrac{\partial f(q^t)}{\partial q_i}}{\sum_{k=1}^{n} q_k^t \dfrac{\partial f(q^t)}{\partial q_k}} \quad \text{for } t = 0,1 \text{ and } i = 1,\ldots,\text{n} \tag{17.10}$$

where $\partial f(q^t)/\partial q_i$ denotes the partial derivative of the utility function f with respect to the ith quantity q_i, evaluated at the period t quantity vector q^t.

17.22 If the homothetic preferences assumption is made and it is assumed that the utility function is linearly homogeneous, then Wold's Identity can be simplified into an equation that will prove to be very useful:[14]

$$\frac{p_i^t}{\sum_{k=1}^{n} p_k^t q_k^t} = \frac{\partial f(q^t)/\partial q_i}{f(q^t)} \quad \text{for } t = 0,1 \text{ and } i = 1,\ldots,\text{n} \tag{17.11}$$

17.23 Shephard's (1953, p. 11) Lemma is the following result. Consider the period t cost minimization problem defined by equation (17.1). If the cost function $C(u, p)$ is differentiable with respect to the components of the price vector p, then the period t quantity vector q^t is equal to the vector of first-order partial derivatives of the cost function with respect to the components of p:

$$q_i^t = \frac{\partial C(u^t, p^t)}{\partial p_i} \quad \text{for } i = 1,\ldots,n \text{ and } t = 0,1 \tag{17.12}$$

17.24 To explain why equation (17.12) holds, consider the following argument. Because it is assumed that the observed period t quantity vector q^t solves the cost minimization problem defined by $C(u^t, p^t)$, then q^t must be feasible for this problem so it must be the case that $f(q^t) = u^t$. Thus, q^t is a feasible solution for the following cost-minimization problem where the general price vector p has replaced the specific period t price vector p^t:

$$C(u^t, p) \equiv \min_q \left\{ \sum_{i=1}^{n} p_i q_i : f(q_1,\ldots,q_n) \geq u^t \right\} \leq \sum_{i=1}^{n} p_i q_i^t \tag{17.13}$$

where the inequality follows from the fact that $q^t \equiv (q_1^t,\ldots,q_n^t)$ is a feasible (but usually not optimal) solution for the cost minimization problem in equation (17.13). Now define for each strictly positive price vector p the function $g(p)$ as follows:

$$g(p) \equiv \sum_{i=1}^{n} p_i q_i^t - C(u^t, p) \tag{17.14}$$

where, as usual, $p \equiv (p_1,\ldots,p_n)$. Using equations (17.13) and (17.1), it can be seen that $g(p)$ is minimized (over all strictly positive price vectors p) at $p = p^t$. Thus the first-order necessary conditions for minimizing a differentiable function of n variables hold, which simplify to equation (17.12).

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to the period t quantity vector q^t and solve the resulting equation for λ^t. Substitute this solution back into the vector equation $p^t = \lambda^t \nabla f(q^t)$ and equation (17.10) is obtained.

[14] Differentiate both sides of the equation $f(\lambda q) = \lambda f(q)$ with respect to λ, and then evaluate the resulting equation at $\lambda = 1$. The equation $\sum_{i=1}^{n} f_i(q) q_i = f(q)$ is obtained where $f_i(q) \equiv \partial f(q)/\partial q_i$.

17.25 If the homothetic preferences assumption is made and it is assumed that the utility function is linearly homogeneous, then using equation (17.6), Shephard's Lemma (17.12) becomes:

$$q_i^t = u^t \frac{\partial c(p^t)}{\partial p_i} \text{ for } i = 1,...,n \text{ and } t = 0,1 \tag{17.15}$$

Combining equations (17.15) and (17.7), the following equation is obtained:

$$\frac{q_i^t}{\sum_{k=1}^{n} p_k^t q_k^t} = \frac{\partial c(p^t)}{\partial p_i} \bigg/ c(p^t) \quad \text{for } i = 1,...,n \quad \text{and } t = 0,1 \tag{17.16}$$

17.26 Note the symmetry of equation (17.16) with equation (17.11). It is these two equations that will be used in subsequent material in this chapter.

Superlative indices: The Fisher ideal index

17.27 Suppose the consumer has the following homogeneous quadratic utility function:

$$f(q_1,...,q_n) \equiv \sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} q_i q_k}, \text{where } a_{ik} = a_{ki} \text{ for all } i \text{ and } k \tag{17.17}$$

Differentiating $f(q)$ defined by equation (17.17) with respect to q_i yields the following equation:

$$\begin{aligned} f_i(q) &= \frac{1}{2} \frac{2\sum_{k=1}^{n} a_{ik} q_k}{\sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} q_j q_k}} \quad \text{for } i = 1,...,n \\ &= \frac{\sum_{k=1}^{n} a_{ik} q_k}{f(q)} \end{aligned} \tag{17.18}$$

where $f_i(q) \equiv \partial f(q)/\partial q_i$. In order to obtain the first equation in (17.18), it is necessary to use the symmetry conditions, $a_{ik} = a_{ki}$. Now evaluate the second equation in (17.18) at the observed period t quantity vector $q^t \equiv (q_1^t,\ldots,q_n^t)$ and divide both sides of the resulting equation by $f(q^t)$. The following equations are obtained:

$$\frac{f_i(q^t)}{f(q^t)} = \frac{\sum_{k=1}^{n} a_{ik} q_k^t}{\{f(q^t)\}^2} \text{ for } t = 0,1 \text{ and } i = 1,...,n \tag{17.19}$$

Assume cost-minimizing behaviour for the consumer in periods 0 and 1. Since the utility function f defined by equation (17.17) is linearly homogeneous and differentiable, equation (17.11) will hold. Now recall the definition of the Fisher ideal quantity index, Q_F, defined earlier in Chapter 15:

$$Q_F(p^0,p^1,q^0,q^1)=\sqrt{\frac{\sum_{i=1}^{n}p_i^0q_i^1}{\sum_{k=1}^{n}p_k^0q_k^0}}\sqrt{\frac{\sum_{i=1}^{n}p_i^1q_i^1}{\sum_{k=1}^{n}p_k^1q_k^0}}$$

$$=\sqrt{\sum_{i=1}^{n}f_i(q^0)\frac{q_i^1}{f(q^0)}}\sqrt{\frac{\sum_{i=1}^{n}p_i^1q_i^1}{\sum_{k=1}^{n}p_k^1q_k^0}}\quad \text{using equation (17.11) for } t=0$$

$$=\sqrt{\sum_{i=1}^{n}f_i(q^0)\frac{q_i^1}{f(q^0)}}\Bigg/\sqrt{\frac{\sum_{k=1}^{n}p_k^1q_k^0}{\sum_{i=1}^{n}p_i^1q_i^1}} \qquad (17.20)$$

$$=\sqrt{\sum_{i=1}^{n}f_i(q^0)\frac{q_i^1}{f(q^0)}}\Bigg/\sqrt{\sum_{i=1}^{n}f_i(q^1)\frac{q_i^0}{f(q^1)}}\quad \text{using equation (17.11) for } t=1$$

$$=\sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}q_k^0\frac{q_i^1}{\{f(q^0)\}^2}}\Bigg/\sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}q_k^1\frac{q_i^0}{\{f(q^1)\}^2}}\quad \text{using equation (17.19)}$$

$$=\sqrt{\frac{1}{\{f(q^0)\}^2}}\Bigg/\sqrt{\frac{1}{\{f(q^1)\}^2}}\quad \text{using equation(17.17) and canceling terms}$$

$$=\frac{f(q^1)}{f(q^0)}$$

Thus under the assumption that the consumer engages in cost minimizing behaviour during periods 0 and 1 and has preferences over the *n* commodities that correspond to the utility function defined by equation (17.17), the Fisher ideal quantity index Q_F is exactly equal to the true quantity index, $f(q^1)/f(q^0)$.[15]

17.28 As was noted in paragraphs 15.18 to 15.23 of Chapter 15, the price index that corresponds to the Fisher quantity index Q_F using the product test (15.3) is the Fisher price index P_F, defined by equation (15.12). Let $c(p)$ be the unit cost function that corresponds to the homogeneous quadratic utility function f defined by equation (17.17). Then using equations (17.16) and (17.20), it can be seen that

$$P_F(p^0,p^1,q^0,q^1)=\frac{c(p^1)}{c(p^0)} \qquad (17.21)$$

Thus, under the assumption that the consumer engages in cost minimizing behaviour during periods 0 and 1 and has preferences over the *n* commodities that correspond to the utility

[15] For the early history of this result, see Diewert (1976, p. 184).

function defined by equation (17.17), the Fisher ideal price index P_F is exactly equal to the true price index, $c(p^1)/c(p^0)$.

17.29 A twice continuously differentiable function $f(q)$ of n variables $q \equiv (q_1,\ldots,q_n)$ can provide a *second-order approximation* to another such function $f^*(q)$ around the point q^*, if the level and all the first-order and second-order partial derivatives of the two functions coincide at q^*. It can be shown[16] that the homogeneous quadratic function f defined by equation (17.17) can provide a second-order approximation to an arbitrary f^* around any (strictly positive) point q^* in the class of linearly homogeneous functions. Thus the homogeneous quadratic functional form defined by equation (17.17) is a *flexible functional form*.[17.] Diewert (1976, p. 117) termed an index number formula $Q(p^0, p^1, q^0, q^1)$ that was exactly equal to the true quantity index $f(q^1)/f(q^0)$ (where f is a flexible functional form) *a superlative index number formula*.[18] Equation (17.20) and the fact that the homogeneous quadratic function f defined by equation (17.17) is a flexible functional form show that the Fisher ideal quantity index Q_F defined by equation (15.14) is a superlative index number formula. Since the Fisher ideal price index P_F satisfies equation (17.21), where $c(p)$ is the unit cost function that is generated by the homogeneous quadratic utility function, P_F is also called a superlative index number formula.

17.30 It is possible to show that the Fisher ideal price index is a superlative index number formula by a different route. Instead of starting with the assumption that the consumer's utility function is the homogeneous quadratic function defined by equation (17.17), it is possible to start with the assumption that the consumer's unit cost function is a homogeneous quadratic.[19] Thus, suppose that the consumer has the following unit cost function:

$$c(p_1,\ldots,p_n) \equiv \sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}\, p_i\, p_k} \quad \text{where } b_{ik} = b_{ki} \ \text{ for all } i \text{ and } k \qquad (17.22)$$

Differentiating $c(p)$ defined by equation (17.22) with respect to p_i yields the following equations:

16 See Diewert (1976, p. 130) and let the parameter r equal 2.

17 Diewert (1974a, p. 133) introduced this term into the economics literature.

18 Fisher (1922, p. 247) used the term superlative to describe the Fisher ideal price index. Thus, Diewert adopted Fisher's terminology but attempted to give some precision to Fisher's definition of superlativeness. Fisher defined an index number formula to be superlative if it approximated the corresponding Fisher ideal results using his data set.

19 Given the consumer's unit cost function $c(p)$, Diewert (1974a, p. 112) showed that the corresponding utility function $f(q)$ can be defined as follows: for a strictly positive quantity vector q, $f(q) \equiv 1/\max_p \{\sum_{i=1}^{n} p_i q_i : c(p) = 1\}$.

$$c_i(p) = \frac{1}{2}\frac{2\sum_{k=1}^{n} b_{ik}p_k}{\sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk}p_j p_k}} \quad \text{for } i = 1,...,n$$

$$= \frac{\sum_{k=1}^{n} b_{ik}p_k}{c(p)} \tag{17.23}$$

where $c_i(p) \equiv \partial c(p^t)/\partial p_i$. In order to obtain the first equation in (17.23), it is necessary to use the symmetry conditions. Now evaluate the second equation in (17.23) at the observed period t price vector $p^t \equiv (p_1^t,\ldots,p_n^t)$ and divide both sides of the resulting equation by $c(p^t)$. The following equation is obtained:

$$\frac{c_i(p^t)}{c(p^t)} = \frac{\sum_{k=1}^{n} b_{ik}p_k^t}{\{c(p^t)\}^2} \quad \text{for } t = 0,1 \text{ and } i = 1,...,n \tag{17.24}$$

As cost-minimizing behaviour for the consumer in periods 0 and 1 is being assumed and, since the unit cost function c defined by equation (17.22) is differentiable, equations (17.16) will hold. Now recall the definition of the Fisher ideal price index, P_F, given by equation (15.12) in Chapter 15:

$$P_F(p^0,p^1,q^0,q^1) = \sqrt{\frac{\sum_{i=1}^{n} p_i^1 q_i^0}{\sum_{k=1}^{n} p_k^0 q_k^0}}\sqrt{\frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{k=1}^{n} p_k^0 q_k^1}}$$

$$= \sqrt{\sum_{i=1}^{n} p_i^1 \frac{c_i(p^0)}{c(p^0)}}\sqrt{\frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{k=1}^{n} p_k^0 q_k^1}} \quad \text{using equation (17.16) for } t = 0$$

$$= \sqrt{\sum_{i=1}^{n} p_i^1 \frac{c_i(p^0)}{c(p^0)}} \Bigg/ \sqrt{\frac{\sum_{k=1}^{n} p_k^0 q_k^1}{\sum_{i=1}^{n} p_i^1 q_i^1}} \tag{17.25}$$

$$= \sqrt{\sum_{i=1}^{n} p_i^1 \frac{c_i(p^0)}{c(p^0)}} \Bigg/ \sqrt{\sum_{i=1}^{n} p_i^0 \frac{c_i(p^1)}{c(p^1)}} \quad \text{using equation (17.16) for } t = 1$$

$$= \sqrt{\frac{1}{\{c(p^0)\}^2}} \Bigg/ \sqrt{\frac{1}{\{c(p^1)\}^2}} \quad \text{using equation (17.22) and cancelling terms}$$

$$= \frac{c(p^1)}{c(p^0)}.$$

Thus, under the assumption that the consumer engages in cost-minimizing behaviour during periods 0 and 1 and has preferences over the n commodities that correspond to the unit cost function defined by equation (17.22), the Fisher ideal price index P_F is exactly equal to the true price index, $c(p^1)/c(p^0)$.[20]

17.31 Since the homogeneous quadratic unit cost function $c(p)$ defined by equation (17.22) is also a flexible functional form, the fact that the Fisher ideal price index P_F exactly equals the true price index $c(p^1)/c(p^0)$ means that P_F is a superlative index number formula.[21]

17.32 Suppose that the b_{ik} coefficients in equation (17.22) satisfy the following restrictions:

$$b_{ik} = b_i b_k \quad \text{for } i,k = 1,\dots,n \tag{17.26}$$

where the n numbers b_i are non-negative. In this special case of equation (17.22), it can be seen that the unit cost function simplifies as follows:

$$\begin{aligned} c(p_1,\dots,p_n) &\equiv \sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n} b_i b_k\, p_i\, p_k} \\ &= \sqrt{\sum_{i=1}^{n} b_i\, p_i \sum_{k=1}^{n} b_k\, p_k} = \sum_{i=1}^{n} b_i\, p_i \end{aligned} \tag{17.27}$$

Substituting equation (17.27) into Shephard's Lemma (17.15) yields the following expressions for the period t quantity vectors, q^t:

$$q_i^t = u^t \frac{\partial c(p^t)}{\partial p_i} = b_i u^t \quad i = 1,\dots,n;\ t = 0,1 \tag{17.28}$$

Thus if the consumer has the preferences that correspond to the unit cost function defined by equation (17.22) where the b_{ik} satisfy the restrictions (17.26), then the period 0 and 1 quantity vectors are equal to a multiple of the vector $b \equiv (b_1,\dots,b_n)$; i.e., $q^0 = b\,u^0$ and $q^1 = b\,u^1$. Under these assumptions, the Fisher, Paasche and Laspeyres indices, P_F, P_P and P_L, all coincide. The preferences which correspond to the unit cost function defined by equation (17.27) are, however, not consistent with normal consumer behaviour since they imply that the consumer will not substitute away from more expensive commodities to cheaper commodities if relative prices change going from period 0 to 1.

Quadratic mean of order *r* superlative indices

17.33 It turns out that there are many other superlative index number formulae; i.e., there exist many quantity indices $Q(p^0, p^1, q^0, q^1)$ that are exactly equal to $f(q^1)/f(q^0)$ and many price indices $P(p^0, p^1, q^0, q^1)$ that are exactly equal to $c(p^1)/c(p^0)$, where the aggregator function f or the unit cost function c is a flexible functional form. Two families of superlative indices are defined below.

17.34 Suppose the consumer has the following quadratic mean of order r utility function.[22]

[20] This result was obtained by Diewert (1976, pp. 133-134).

[21] Note that it has been shown that the Fisher index P_F is exact for the preferences defined by equation (17.17), as well as the preferences that are dual to the unit cost function defined by equation (17.22). These two classes of preferences do not coincide in general. However, if the n by n symmetric matrix A of the a_{ik} has an inverse, then it can be shown that the n by n matrix B of the b_{ik} will equal A^{-1}.

[22] The terminology is attributable to Diewert (1976, p. 129).

$$f^r(q_1,...,q_n) \equiv \sqrt[r]{\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} q_i^{r/2} q_k^{r/2}} \qquad (17.29)$$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976, p. 130) showed that the utility function f^r defined by equation (17.29) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when $r = 2$, f^r equals the homogeneous quadratic function defined by equation (17.17).

17.35 Define the quadratic mean of order r quantity index Q^r by:

$$Q^r(p^0, p^1, q^0, q^1) \equiv \frac{\sqrt[r]{\sum_{i=1}^{n} s_i^0 (q_i^1/q_i^0)^{r/2}}}{\sqrt[r]{\sum_{i=1}^{n} s_i^1 (q_i^1/q_i^0)^{-r/2}}} \qquad (17.30)$$

where $s_i^t \equiv p_i^t q_i^t \Big/ \sum_{k=1}^{n} p_k^t q_k^t$ is the period t expenditure share for commodity i as usual.

17.36 Using exactly the same techniques as were used in paragraphs 17.27 to 17.32, it can be shown that Q^r is exact for the aggregator function f^r defined by equation (17.29); i.e., the following exact relationship between the quantity index Q^r and the utility function f^r holds:

$$Q^r(p^0, p^1, q^0, q^1) = \frac{f^r(q^1)}{f^r(q^0)} \qquad (17.31)$$

Thus under the assumption that the consumer engages in cost-minimizing behaviour during periods 0 and 1 and has preferences over the n commodities that correspond to the utility function defined by equation (17.29), the quadratic mean of order r quantity index Q_F is exactly equal to the true quantity index, $f^r(q^1)/f^r(q^0)$.[23] Since Q^r is exact for f^r and f^r is a flexible functional form, it can be seen that the quadratic mean of order r quantity index Q^r is a superlative index for each $r \neq 0$. Thus there is an infinite number of superlative quantity indices.

17.37 For each quantity index Q^r, the product test (15.3) in Chapter 15 can be used in order to define the corresponding implicit quadratic mean of order r price index P^r*:

$$P^{r*}(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0 Q^r(p^0, p^1, q^0, q^1)} = \frac{c^{r*}(p^1)}{c^{r*}(p^0)} \qquad (17.32)$$

where c^r* is the unit cost function that corresponds to the aggregator function f^r defined by equation (17.29). For each $r \neq 0$, the implicit quadratic mean of order r price index P^r* is also a superlative index.

17.38 When $r = 2$, Q^r defined by equation (17.30) simplifies to Q_F, the Fisher ideal quantity index, and P^r* defined by equation (17.32) simplifies to P_F, the Fisher ideal price index. When $r = 1$, Q^r defined by equation (17.30) simplifies to:

[23] See Diewert (1976, p. 130).

$$Q^1(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} s_i^0 \sqrt{\frac{q_i^1}{q_i^0}}}{\sum_{i=1}^{n} s_i^1 \sqrt{\frac{q_i^0}{q_i^1}}} = \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0} \frac{\sum_{i=1}^{n} p_i^0 q_i^0 \sqrt{\frac{q_i^1}{q_i^0}}}{\sum_{i=1}^{n} p_i^1 q_i^1 \sqrt{\frac{q_i^0}{q_i^1}}}$$

$$= \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0} \frac{\sum_{i=1}^{n} p_i^0 \sqrt{q_i^0 q_i^1}}{\sum_{i=1}^{n} p_i^1 \sqrt{q_i^0 q_i^1}}$$

$$= \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0} \Bigg/ \frac{\sum_{i=1}^{n} p_i^1 \sqrt{q_i^0 q_i^1}}{\sum_{i=1}^{n} p_i^0 \sqrt{q_i^0 q_i^1}}$$

$$= \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0} / P_W(p^0,p^1,q^0,q^1) \qquad (17.33)$$

where P_W is the Walsh price index defined previously by equation (15.19) in Chapter 15. Thus P^{1*} is equal to P_W, the Walsh price index, and hence it is also a superlative price index.

17.39 Suppose the consumer has the following quadratic mean of order r unit cost function:[24]

$$c^r(p_1,\ldots,p_n) \equiv \sqrt[r]{\sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}\, p_i^{r/2}\, p_k^{r/2}} \qquad (17.34)$$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all i and k, and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976, p. 130) showed that the unit cost function c^r defined by equation (17.34) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when $r = 2$, c^r equals the homogeneous quadratic function defined by equation (17.22).

17.40 Define the quadratic mean of order r price index P^r by:

$$P^r(p^0,p^1,q^0,q^1) \equiv \frac{\sqrt[r]{\sum_{i=1}^{n} s_i^0 \left(\frac{p_i^1}{p_i^0}\right)^{r/2}}}{\sqrt[r]{\sum_{i=1}^{n} s_i^1 \left(\frac{p_i^1}{p_i^0}\right)^{-r/2}}} \qquad (17.35)$$

where $s_i^t \equiv p_i^t q_i^t \Big/ \sum_{k=1}^{n} p_k^t q_k^t$ is the period t expenditure share for commodity i as usual.

[24] This terminology is attributable to Diewert (1976, p. 130), this unit cost function being first defined by Denny (1974).

17.41 Using exactly the same techniques as were used in paragraphs 17.27 to 17.32, it can be shown that P^r is exact for the aggregator function defined by equation (17.34); i.e., the following exact relationship between the index number formula P^r and the unit cost function c^r holds:

$$P^r(p^0, p^1, q^0, q^1) = \frac{c^r(p^1)}{c^r(p^0)} \qquad (17.36)$$

Thus, under the assumption that the consumer engages in cost-minimizing behaviour during periods 0 and 1, and has preferences over the n commodities that correspond to the unit cost function defined by equation (17.34), the quadratic mean of order r price index P^r is exactly equal to the true price index, $c^r(p^1)/c^r(p^0)$.[25] Since P^r is exact for c^r and c^r is a flexible functional form, it can be seen that the quadratic mean of order r price index P^r is a superlative index for each $r \neq 0$. Thus there are an infinite number of superlative price indices.

17.42 For each price index P^r, the product test (15.3) in Chapter 15 can be used in order to define the corresponding implicit quadratic mean of order r quantity index Q^r*:

$$Q^{r*}(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0 P^r(p^0, p^1, q^0, q^1)} = \frac{f^{r*}(p^1)}{f^{r*}(p^0)} \qquad (17.37)$$

where f^{r}* is the aggregator function that corresponds to the unit cost function c^r defined by equation (17.34).[26] For each $r \neq 0$, the implicit quadratic mean of order r quantity index Q^r* is also a superlative index.

17.43 When $r = 2$, P^r defined by equation (17.35) simplifies to P_F, the Fisher ideal price index, and Q^r* defined by equation (17.37) simplifies to Q_F, the Fisher ideal quantity index. When $r = 1$, P^r defined by equation (17.35) simplifies to:

[25] See Diewert (1976, pp. 133-134).

[26] The function f^r* can be defined by using c^r as follows: $f^r*(q) \equiv 1/\max_p \{ \sum_{i=1}^{n} p_i q_i : c^r(p) = 1 \}$.

$$P^1(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^n s_i^0 \sqrt{\frac{p_i^1}{p_i^0}}}{\sum_{i=1}^n s_i^1 \sqrt{\frac{p_i^0}{p_i^1}}} = \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} \frac{\sum_{i=1}^n p_i^0 q_i^0 \sqrt{\frac{p_i^1}{p_i^0}}}{\sum_{i=1}^n p_i^1 q_i^1 \sqrt{\frac{p_i^0}{p_i^1}}}$$

$$= \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} \frac{\sum_{i=1}^n q_i^0 \sqrt{p_i^0 p_i^1}}{\sum_{i=1}^n q_i^1 \sqrt{p_i^0 p_i^1}}$$

$$= \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} \Bigg/ \frac{\sum_{i=1}^n q_i^1 \sqrt{p_i^0 p_i^1}}{\sum_{i=1}^n q_i^0 \sqrt{p_i^0 p_i^1}}$$

$$= \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} / Q_W(p^0,p^1,q^0,q^1) \qquad (17.38)$$

where Q_W is the Walsh quantity index defined previously in footnote 30 of Chapter 15. Thus Q^{1*} is equal to Q_W, the Walsh quantity index, and hence it is also a superlative quantity index.

Superlative indices: The Törnqvist index

17.44 In this section, the same assumptions that were made on the consumer in paragraphs 17.9 to 17.17 are made here. In particular, it is not assumed that the consumer's utility function f is necessarily linearly homogeneous as in paragraphs 17.18 to 17.43.

17.45 Before the main result is derived, a preliminary result is required. Suppose the function of n variables, $f(z_1,\ldots,z_n) \equiv f(z)$, is quadratic; i.e.,

$$f(z_1,\ldots,z_n) \equiv a_0 + \sum_{i=1}^n a_i z_i + \frac{1}{2}\sum_{i=1}^n \sum_{k=1}^n a_{ik} z_i z_k \text{ and } \quad a_{ik} = a_{ki} \quad \text{for all } i \text{ and } k \qquad (17.39)$$

where the a_i and the a_{ik} are constants. Let $f_i(z)$ denote the first-order partial derivative of f evaluated at z with respect to the ith component of z, z_i. Let $f_{ik}(z)$ denote the second-order partial derivative of f with respect to z_i and z_k. Then it is well known that the second-order Taylor series approximation to a quadratic function is exact; i.e., if f is defined by equation (17.39), then for any two points, z^0 and z^1, the following equation holds:

$$f(z^1) - f(z^0) = \sum_{i=1}^n f_i(z^0)\{z_i^1 - z_i^0\} + \frac{1}{2}\sum_{i=1}^n \sum_{k=1}^n f_{ik}(z^0)\{z_i^1 - z_i^0\}\{z_k^1 - z_k^0\} \qquad (17.40)$$

It is less well known that an average of two first-order Taylor series approximations to a quadratic function is also exact; i.e., if f is defined by equation (17.39) above, then for any two points, z^0 and z^1, the following equation holds:[27]

[27] The proof of this and the foregoing relation is by straightforward verification.

$$f(z^1) - f(z^0) = \frac{1}{2}\sum_{i=1}^{n}\{f_i(z^0) + f_i(z^1)\}\{z_i^1 - z_i^0\} \qquad (17.41)$$

Diewert (1976, p. 118) and Lau (1979) showed that equation (17.41) characterized a quadratic function and called the equation the *quadratic approximation lemma*. In this chapter, equation (17.41) will be called the *quadratic identity*.

17.46 Suppose that the consumer's *cost function*[28] $C(u,p)$, has the following *translog functional form*:[29]

$$\ln C(u,p) \equiv a_0 + \sum_{i=1}^{n} a_i \ln p_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} \ln p_i \ln p_k + b_0 \ln u + \sum_{i=1}^{n} b_i \ln p_i \ln u + \frac{1}{2} b_{00} (\ln u)^2 \qquad (17.42)$$

where ln is the natural logarithm function and the parameters a_i, a_{ik}, and b_i satisfy the following restrictions:

$$a_{ik} = a_{ki},\ \sum_{i=1}^{n} a_i = 1,\ \sum_{i=1}^{n} b_i = 0 \text{ and } \sum_{k=1}^{n} a_{ik} = 0 \text{ for } i,k = 1,\ldots,n \qquad (17.43)$$

These parameter restrictions ensure that $C(u,p)$ defined by equation (17.42) is linearly homogeneous in p, a property that a cost function must have. It can be shown that the translog cost function defined by equation (17.42) can provide a second-order Taylor series approximation to an arbitrary cost function.[30]

17.47 Assume that the consumer has preferences that correspond to the translog cost function and that the consumer engages in cost-minimizing behaviour during periods 0 and 1. Let p^0 and p^1 be the period 0 and 1 observed price vectors, and let q^0 and q^1 be the period 0 and 1 observed quantity vectors. These assumptions imply:

$$C(u^0, p^0) = \sum_{i=1}^{n} p_i^0 q_i^0 \text{ and } C(u^1, p^1) = \sum_{i=1}^{n} p_i^1 q_i^1 \qquad (17.44)$$

where C is the translog cost function defined above. Now apply Shephard's Lemma, equation (17.12), and the following equation results:

$$\begin{aligned} q_i^t &= \frac{\partial C(u^t, p^t)}{\partial p_i} \text{ for } i = 1,\ldots,n \text{ and } t = 0,1 \\ &= \frac{C(u^t, p^t)}{p_i^t}\frac{\partial \ln C(u^t, p^t)}{\partial \ln p_i} \end{aligned} \qquad (17.45)$$

Now use equation (17.44) to replace $C(u^t, p^t)$ in equation (17.45). After some cross multiplication, this becomes the following:

$$\frac{p_i^t q_i^t}{\sum_{k=1}^{n} p_k^t q_k^t} \equiv s_i^t = \frac{\partial \ln C(u^t, p^t)}{\partial \ln p_i} \text{ for } i = 1,\ldots,n \text{ and } t = 0,1 \qquad (17.46)$$

or

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[28] The consumer's cost function was defined by equation (17.6) above.

[29] Christensen, Jorgenson and Lau (1971) introduced this function into the economics literature.

[30] It can also be shown that, if all the $b_i = 0$ and $b_{00} = 0$, then $C(u,p) = uC(1,p) \equiv uc(p)$; i.e., with these additional restrictions on the parameters of the general translog cost function, homothetic preferences are the result of these restrictions. Note that it is also assumed that utility u is scaled so that u is always positive.

$$s_i^t = a_i + \sum_{k=1}^{n} a_{ik} \ln p_k^t + b_i \ln u^t \text{ for } i = 1,\ldots,n \text{ and } t = 0,1 \qquad (17.47)$$

where s_i^t is the period t expenditure share on commodity i.

17.48 Define the geometric average of the period 0 and 1 utility levels as u^*; i.e., define

$$u^* \equiv \sqrt{u^0 u^1} \qquad (17.48)$$

Now observe that the right-hand side of the equation that defines the natural logarithm of the translog cost function, equation (17.42), is a quadratic function of the variables $z_i \equiv \ln p_i$ if utility is held constant at the level u^*. Hence the quadratic identity (17.41) can be applied, and the following equation is obtained:

$$\begin{aligned}
&\ln C(u^*, p^1) - \ln C(u^*, p^0) \\
&= \frac{1}{2}\sum_{i=1}^{n}\left\{\frac{\partial \ln C(u^*, p^0)}{\partial \ln p_i} + \frac{\partial \ln C(u^*, p^1)}{\partial \ln p_i}\right\}\left\{\ln p_i^1 - \ln p_i^0\right\} \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(a_i + \sum_{k=1}^{n} a_{ik} \ln p_k^0 + b_i \ln u^* + a_i + \sum_{k=1}^{n} a_{ik} \ln p_k^1 + b_i \ln u^*\right)\left(\ln p_i^1 - \ln p_i^0\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(a_i + \sum_{k=1}^{n} a_{ik} \ln p_k^0 + b_i \ln \sqrt{u^0 u^1} + a_i + \sum_{k=1}^{n} a_{ik} \ln p_k^1 + b_i \ln \sqrt{u^0 u^1}\right)\left(\ln p_i^1 - \ln p_i^0\right) \qquad (17.49) \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(a_i + \sum_{k=1}^{n} a_{ik} \ln p_k^0 + b_i \ln u^0 + a_i + \sum_{k=1}^{n} a_{ik} \ln p_k^1 + b_i \ln u^1\right)\left(\ln p_i^1 - \ln p_i^0\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}\left\{\frac{\partial \ln C(u^0, p^0)}{\partial \ln p_i} + \frac{\partial \ln C(u^1, p^1)}{\partial \ln p_i}\right\}\left(\ln p_i^1 - \ln p_i^0\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(s_i^0 + s_i^1\right)\left(\ln p_i^1 - \ln p_i^0\right) \qquad \text{using (17.46).}
\end{aligned}$$

The last equation in (17.49) can be recognized as the logarithm of the Törnqvist–Theil index number formula P_T, defined earlier by equation (15.81) in Chapter 15. Hence, exponentiating both sides of equation (17.49) yields the following equality between the true cost of living between periods 0 and 1, evaluated at the intermediate utility level u^* and the observable Törnqvist–Theil index P_T:[31]

$$\frac{C(u^*, p^1)}{C(u^*, p^0)} = P_T(p^0, p^1, q^0, q^1) \qquad (17.50)$$

Since the translog cost function which appears on the left-hand side of equation (17.49) is a flexible functional form, the Törnqvist–Theil price index P_T is also a superlative index.

17.49 It is somewhat mysterious how a ratio of unobservable cost functions of the form appearing on the left-hand side of the above equation can be exactly estimated by an observable index number formula. The key to this mystery is the assumption of cost-minimizing behaviour and the quadratic identity (17.41), along with the fact that derivatives of cost functions are equal to quantities, as specified by Shephard's Lemma. In fact, all the exact index number results derived in paragraphs 17.27 to 17.43 can be derived using

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[31] This result is attributable to Diewert (1976, p. 122).

transformations of the quadratic identity along with Shephard's Lemma (or Wold's Identity).[32] Fortunately, for most empirical applications, assuming that the consumer has (transformed) quadratic preferences will be an adequate assumption, so the results presented in paragraphs 17.27 to 17.49 are quite useful to index number practitioners who are willing to adopt the economic approach to index number theory.[33] Essentially, the economic approach to index number theory provides a strong justification for the use of the Fisher price index P_F defined by equation (15.12), the Törnqvist–Theil price index P_T defined by equation (15.81), the implicit quadratic mean of order r price indices P^{r*} defined by equation (17.32) (when r = 1, this index is the Walsh price index defined by equation (15.19) in Chapter 15) and the quadratic mean of order r price indices P^r defined by equation (17.35). In the next section, we ask if it matters which one of these formulae is chosen as "best".

The approximation properties of superlative indices

17.50 The results of paragraphs 17.27 to 17.49 provide price statisticians with a large number of index number formulae which appear to be equally good from the viewpoint of the economic approach to index number theory. Two questions arise as a consequence of these results:

- Does it matter which of these formulae is chosen?
- If it does matter, which formula should be chosen?

17.51 With respect to the first question, Diewert (1978, p. 888) showed that all of the superlative index number formulae listed in paragraphs 17.27 to 17.49 approximate each other to the second order around any point where the two price vectors, p^0 and p^1, are equal and where the two quantity vectors, q^0 and q^1, are equal. In particular, this means that the following equalities are valid for all r and s not equal to 0, provided that $p^0 = p^1$ and $q^0 = q^1$.[34]

32 See Diewert (2002a).

33 If, however, consumer preferences are non-homothetic and the change in utility is substantial between the two situations being compared, then it may be desirable to compute separately the Laspeyres–Konüs and Paasche–Konüs true cost of living indices defined by equations (17.3) and (17.4), $C(u^0,p^1)/C(u^0,p^0)$ and $C(u^1,p^1)/C(u^1,p^0)$, respectively. In order to do this, it would be necessary to use econometrics and estimate empirically the consumer's cost or expenditure function.

34 To prove the equalities in equations (17.51) to (17.56), simply differentiate the various index number formulae and evaluate the derivatives at $p^0 = p^1$ and $q^0 = q^1$. Actually, equations (17.51) to (17.56) are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$; i.e., provided that the period 1 price vector is proportional to the period 0 price vector and that the period 1 quantity vector is proportional to the period 0 quantity vector.

$$P_T(p^0,p^1,q^0,q^1) = P^r(p^0,p^1,q^0,q^1) = P^{s*}(p^0,p^1,q^0,q^1) \qquad (17.51)$$

$$\frac{\partial P_T(p^0,p^1,q^0,q^1)}{\partial p_i^t} = \frac{\partial P^r(p^0,p^1,q^0,q^1)}{\partial p_i^t} = \frac{\partial P^{s*}(p^0,p^1,q^0,q^1)}{\partial p_i^t} \text{ for } i=1,\ldots,n \text{ and } t=0,1 \qquad (17.52)$$

$$\frac{\partial P_T(p^0,p^1,q^0,q^1)}{\partial q_i^t} = \frac{\partial P^r(p^0,p^1,q^0,q^1)}{\partial q_i^t} = \frac{\partial P^{s*}(p^0,p^1,q^0,q^1)}{\partial q_i^t} \text{ for } i=1,\ldots,n \text{ and } t=0,1 \qquad (17.53)$$

$$\frac{\partial^2 P_T(p^0,p^1,q^0,q^1)}{\partial p_i^t \partial p_k^t} = \frac{\partial^2 P^r(p^0,p^1,q^0,q^1)}{\partial p_i^t \partial p_k^t} = \frac{\partial^2 P^{s*}(p^0,p^1,q^0,q^1)}{\partial p_i^t \partial p_k^t} \text{ for } i,k=1,\ldots,n \text{ and } t=0,1 \qquad (17.54)$$

$$\frac{\partial^2 P_T(p^0,p^1,q^0,q^1)}{\partial p_i^t \partial q_k^t} = \frac{\partial^2 P^r(p^0,p^1,q^0,q^1)}{\partial p_i^t \partial q_k^t} = \frac{\partial^2 P^{s*}(p^0,p^1,q^0,q^1)}{\partial p_i^t \partial q_k^t} \text{ for } i,k=1,\ldots,n \text{ and } t=0,1 \qquad (17.55)$$

$$\frac{\partial^2 P_T(p^0,p^1,q^0,q^1)}{\partial q_i^t \partial q_k^t} = \frac{\partial^2 P^r(p^0,p^1,q^0,q^1)}{\partial q_i^t \partial q_k^t} = \frac{\partial^2 P^{s*}(p^0,p^1,q^0,q^1)}{\partial q_i^t \partial q_k^t} \text{ for } i,k=1,\ldots,n \text{ for } t=0,1 \qquad (17.56)$$

where the Törnqvist–Theil price index P_T is defined by equation (15.81), the implicit quadratic mean of order r price index P^r* is defined by equation (17.32) and the quadratic mean of order r price index P^r is defined by equation (17.35). Using the results in the previous paragraph, Diewert (1978, p. 884) concluded that “all superlative indices closely approximate each other”.

17.52 The above conclusion is, however, not true even though the equations (17.51) to (17.56) are true. The problem is that the quadratic mean of order r price indices P^r and the implicit quadratic mean of order s price indices P^s* are (continuous) functions of the parameters r and s respectively. Hence, as r and s become very large in magnitude, the indices P^r and P^s* can differ substantially from, say, $P^2 = P_F$, the Fisher ideal index. In fact, using definition (17.35) and the limiting properties of means of order r,[35] Robert Hill (2002, p. 7) showed that P^r has the following limit as r approaches plus or minus infinity:

$$\lim_{r\to+\infty} P^r(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^r(p^0,p^1,q^0,q^1) = \sqrt{\min_i\left(\frac{p_i^1}{p_i^0}\right)\max_i\left(\frac{p_i^1}{p_i^0}\right)} \qquad (17.57)$$

Using Hill’s method of analysis, it can be shown that the implicit quadratic mean of order r price index has the following limit as r approaches plus or minus infinity:

$$\lim_{r\to+\infty} P^{r*}(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^{r*}(p^0,p^1,q^0,q^1)$$

$$= \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0 \sqrt{\min_i\left(\frac{p_i^1}{p_i^0}\right)\max_i\left(\frac{p_i^1}{p_i^0}\right)}} \qquad (17.58)$$

[35] See Hardy, Littlewood and Polya (1934).

Thus for r large in magnitude, P^r and P^{r*} can differ substantially from P_T, P^1, $P^{1*} = P_W$ (the Walsh price index) and $P^2 = P^{2*} = P_F$ (the Fisher ideal index).[36]

17.53 Although Hill's theoretical and empirical results demonstrate conclusively that not all superlative indices will necessarily closely approximate each other, there is still the question of how well the more commonly used superlative indices will approximate each other. All the commonly used superlative indices, P^r and P^{r*}, fall into the interval $0 \leq r \leq 2$.[37] Hill (2002, p. 16) summarized how far apart the Törnqvist and Fisher indices were, making all possible bilateral comparisons between any two data points for his time series data set as follows:

> The superlative spread S(0,2) is also of interest since, in practice, Törnqvist (r = 0) and Fisher (r = 2) are by far the two most widely used superlative indexes. In all 153 bilateral comparisons, $S(0,2)$ is less than the Paasche–Laspeyres spread and on average, the superlative spread is only 0.1 per cent. It is because attention, until now, has focussed almost exclusively on superlative indexes in the range $0 \leq r \leq 2$ that a general misperception has persisted in the index number literature that all superlative indexes approximate each other closely.

Thus, for Hill's time series data set covering 64 components of United States gross domestic product from 1977 to 1994 and making all possible bilateral comparisons between any two years, the Fisher and Törnqvist price indices differed by only 0.1 per cent on average. This close correspondence is consistent with the results of other empirical studies using annual time series data.[38] Additional evidence on this topic may be found in Chapter 19.

17.54 In the earlier chapters of this manual, it is found that several index number formulae seem "best" when viewed from various perspectives. Thus the Fisher ideal index $P_F = P^2 = P^{2*}$ defined by equation (15.12) seemed to be best from one axiomatic viewpoint, the Törnqvist–Theil price index P_T defined by equation (15.81) seems to be best from another axiomatic perspective, as well as from the stochastic viewpoint, and the Walsh index P_W defined by equation (15.19) (which is equal to the implicit quadratic mean of order r price indices P^{r*} defined by equation (17.32) when $r = 1$) seems to be best from the viewpoint of the "pure" price index. The results presented in this section indicate that for "normal" time series data, these three indices will give virtually the same answer. To determine precisely which one of these three indices to use as a theoretical target or actual index, the statistical agency will have to decide which approach to bilateral index number theory is most consistent with its goals. For most practical purposes, however, it will not matter which of these three indices is chosen as a theoretical target index for making price comparisons between two periods.

Superlative indices and two-stage aggregation

17.55 Most statistical agencies use the Laspeyres formula to aggregate prices in two stages. At the first stage of aggregation, the Laspeyres formula is used to aggregate components of the overall index (e.g., food, clothing, services); then at the second stage of aggregation, these component sub-indices are further combined into the overall index. The following question

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[36] Hill (2000) documents this for two data sets. His time series data consist of annual expenditure and quantity data for 64 components of United States gross domestic product from 1977 to 1994. For this data set, Hill (2000, p. 16) found that "superlative indexes can differ by more than a factor of two (i.e., by more than 100 per cent), even though Fisher and Törnqvist never differ by more than 0.6 per cent".

[37] Diewert (1980, p. 451) showed that the Törnqvist index P_T is a limiting case of $P^{r,}$ as r tends to 0.

[38] See, for example, Diewert (1978, p. 894) or Fisher (1922), which is reproduced in Diewert (1976, p. 135).

then naturally arises: does the index computed in two stages coincide with the index computed in a single stage? Initially, this question is addressed in the context of the Laspeyres formula.[39]

17.56 Suppose that the price and quantity data for period t, p^t and q^t, can be written in terms of M subvectors as follows:

$$p^t = \left(p^{t_1}, p^{t_2}, ..., p^{t_M}\right) \text{and } q^t = \left(q^{t_1}, q^{t_2}, ..., q^{t_M}\right) \text{for } t = 0{,}1 \qquad (17.59)$$

where the dimensionality of the subvectors p^{t_m} and q^{t_m} is N_m for $m = 1,2,\ldots,M$ with the sum of the dimensions N_m equal to n. These subvectors correspond to the price and quantity data for subcomponents of the consumer price index for period t. Now construct sub-indices for each of these components going from period 0 to 1. For the base period, set the price for each of these subcomponents, say P_m^0 for $m = 1,2,\ldots M$, equal to 1 and set the corresponding base period subcomponent quantities, say Q_m^0 for $m = 1,2,\ldots,M$, equal to the base period value of consumption for that subcomponent for $m = 1,2,\ldots,M$:

$$P_m^0 \equiv 1 \text{ and } Q_m^0 \equiv \sum_{i=1}^{N_m} p_i^{0_m} q_i^{0_m} \quad \text{for } m = 1,2,...,M \qquad (17.60)$$

Now use the Laspeyres formula in order to construct a period 1 price for each subcomponent, say P_m^1 for $m = 1,2,\ldots,M$, of the CPI. Since the dimensionality of the subcomponent vectors, p^{t_m} and q^{t_m} ,differs from the dimensionality of the complete period t vectors of prices and quantities, p^t and q^t, it is necessary to use different symbols for these subcomponent Laspeyres indices, say P_L^m for $m = 1,2,\ldots,M$. Thus the period 1 subcomponent prices are defined as follows:

$$P_m^1 \equiv P_L^m(p^{0_m}, p^{1_m}, q^{0_m}, q^{1_m}) \equiv \frac{\sum_{i=1}^{N_m} p_i^{1_m} q_i^{0_m}}{\sum_{i=1}^{N_m} p_i^{0_m} q_i^{0_m}} \quad \textit{for } m = 1,2,...,M \qquad (17.61)$$

Once the period 1 prices for the M sub-indices have been defined by equation (17.61), then corresponding subcomponent period 1 quantities Q_m^1 for $m = 1,2,\ldots,M$ can be defined by deflating the period 1 subcomponent values $\sum_{i=1}^{N_m} p_i^{1_m} q_i^{1_m}$ by the prices P_m^1

$$Q_m^1 \equiv \frac{\sum_{i=1}^{N_m} p_i^{1_m} q_i^{1_m}}{P_m^1} \quad \textit{for } m = 1,2,...,M \qquad (17.62)$$

Now define subcomponent price and quantity vectors for each period $t = 0{,}1$ using equations (17.60) to (17.62). Thus define the period 0 and 1 subcomponent price vectors P^0 and P^1 as follows:

$$P^0 = (P_1^0, P_2^0, ..., P_M^0) \equiv 1_M \text{ and } P^1 = (P_1^1, P_2^1, ..., P_M^1) \qquad (17.63)$$

where 1_M denotes a vector of ones of dimension M and the components of P^1 are defined by equation (17.61). The period 0 and 1 subcomponent quantity vectors Q^0 and Q^1 are defined as follows:

[39] Much of the material in this section is adapted from Diewert (1978) and Alterman, Diewert and Feenstra (1999). See also Balk (1996b) for a discussion of alternative definitions for the two-stage aggregation concept and references to the literature on this topic.

$$Q^0 = (Q_1^0, Q_2^0, ..., Q_M^0) \text{ and } Q^1 = (Q_1^1, Q_2^1, ..., Q_M^1) \qquad (17.64)$$

where the components of Q^0 are defined in equation (17.60) and the components of Q^1 are defined by equation (17.62). The price and quantity vectors in equations (17.63) and (17.64) represent the results of the first-stage aggregation. Now use these vectors as inputs into the second-stage aggregation problem; i.e., apply the Laspeyres price index formula, using the information in equations (17.63) and (17.64) as inputs into the index number formula. Since the price and quantity vectors that are inputs into this second-stage aggregation problem have dimension M instead of the single-stage formula which utilized vectors of dimension n, a different symbol is required for the new Laspeyres index: this is chosen to be P_L*. Thus the Laspeyres price index computed in two stages can be denoted as $P_L*(P^0,P^1,Q^0,Q^1)$. Now ask whether this two-stage Laspeyres index equals the corresponding single-stage index P_L that was studied in the previous sections of this chapter; i.e., ask whether

$$P_L^*(P^0, P^1, Q^0, Q^1) = P_L(p^0, p^1, q^0, q^1) \qquad (17.65)$$

If the Laspeyres formula is used at each stage of each aggregation, the answer to the above question is yes: straightforward calculations show that the Laspeyres index calculated in two stages equals the Laspeyres index calculated in one stage.

17.57 Now suppose that the Fisher or Törnqvist formula is used at each stage of the aggregation. That is, in equation (17.61), suppose that the Laspeyres formula $P_L{}^m(p^{0m}, p^{1m}, q^{0m}, q^{1m})$ is replaced by the Fisher formula $P_F{}^m(p^{0m}, p^{1m}, q^{0m}, q^{1m})$ or by the Törnqvist formula $P_T{}^m(p^{0m}, p^{1m}, q^{0m}, q^{1m})$; and in equation (17.65), suppose that $P_L*(P^0,P^1,Q^0,Q^1)$ is replaced by P_F* (or by P_T*) and $P_L(p^0,p^1,q^0,q^1)$ is replaced by P_F (or by P_T). Then is it the case that counterparts are obtained to the two-stage aggregation result for the Laspeyres formula, equation (17.65)? The answer is no; it can be shown that, in general,

$$P_F^*(P^0, P^1, Q^0, Q^1) \neq P_F(p^0, p^1, q^0, q^1) \quad \text{and} \quad P_T^*(P^0, P^1, Q^0, Q^1) \neq P_T(p^0, p^1, q^0, q^1) \qquad (17.66)$$

Similarly, it can be shown that the quadratic mean of order r index number formula P^r defined by equation (17.35) and the implicit quadratic mean of order r index number formula P^r* defined by equation (17.32) are also not consistent in aggregation.

17.58 Nevertheless, even though the Fisher and Törnqvist formulae are not exactly consistent in aggregation, it can be shown that these formulae are approximately consistent in aggregation. More specifically, it can be shown that the two-stage Fisher formula P_F* and the single-stage Fisher formula P_F in the inequality (17.66), both regarded as functions of the $4n$ variables in the vectors p^0, p^1, q^0, q^1, approximate each other to the second order around a point where the two price vectors are equal (so that $p^0 = p^1$) and where the two quantity vectors are equal (so that $q^0 = q^1$), and a similar result holds for the two-stage and single-stage Törnqvist indices in equation (17.66).[40] As was seen in the previous section, the single-stage Fisher and Törnqvist indices have a similar approximation property, so all four indices in the inequality (17.66) approximate each other to the second order around an equal (or proportional) price and quantity point. Thus for normal time series data, single-stage and two-

[40] See Diewert (1978, p. 889). In other words, a string of equalities similar to equations (17.51) to (17.56) hold between the two-stage indices and their single-stage counterparts. In fact, these equalities are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$.

stage Fisher and Törnqvist indices will usually be numerically very close. This result is illustrated in Chapter 19 for an artificial data set.[41]

17.59 Similar approximate consistency in aggregation results (to the results for the Fisher and Törnqvist formulae explained in the previous paragraph) can be derived for the quadratic mean of order r indices, P^r, and for the implicit quadratic mean of order r indices, P^{r*}; see Diewert (1978, p. 889). Nevertheless, the results of Hill (2002) again imply that the second-order approximation property of the single-stage quadratic mean of order r index P^r to its two-stage counterpart will break down as r approaches either plus or minus infinity. To see this, consider a simple example where there are only four commodities in total. Let the first price ratio p_1^1/p_1^0 be equal to the positive number a, let the second two price ratios p_i^1/p_i^0 equal b and let the last price ratio p_4^1/p_4^0 equal c, where we assume $a < c$ and $a \le b \le c$. Using Hill's result (17.57), the limiting value of the single-stage index is:

$$\lim_{r\to+\infty} P^r(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^r(p^0,p^1,q^0,q^1) = \sqrt{\min_i\left(\frac{p_i^1}{p_i^0}\right)\max_i\left(\frac{p_i^1}{p_i^0}\right)} = \sqrt{ac} \qquad (17.67)$$

Now aggregate commodities 1 and 2 into a sub-aggregate and commodities 3 and 4 into another sub-aggregate. Using Hill's result (17.57) again, it is found that the limiting price index for the first sub-aggregate is $[ab]^{1/2}$ and the limiting price index for the second sub-aggregate is $[bc]^{1/2}$. Now apply the second stage of aggregation and use Hill's result once again to conclude that the limiting value of the two-stage aggregation using P^r as the index number formula is $[ab^2c]^{1/4}$. Thus the limiting value as r tends to plus or minus infinity of the single-stage aggregate over the two-stage aggregate is $[ac]^{1/2}/[ab^2c]^{1/4} = [ac/b^2]^{1/4}$. Now b can take on any value between a and c, and so the ratio of the single-stage limiting P^r to its two-stage counterpart can take on any value between $[a/c]^{1/4}$ and $[c/a]^{1/4}$. Since c/a is greater than 1 and a/c is less than 1, it can be seen that the ratio of the single-stage to the two-stage index can be arbitrarily far from 1 as r becomes large in magnitude with an appropriate choice of the numbers a, b and c.

17.60 The results in the previous paragraph show that some caution is required in assuming that *all* superlative indices will be approximately consistent in aggregation. However, for the three most commonly used superlative indices (the Fisher ideal P_F, the Törnqvist–Theil P_T and the Walsh P_W), the available empirical evidence indicates that these indices satisfy the consistency in aggregation property to a sufficiently high degree of approximation that users will not be unduly troubled by any inconsistencies.[42]

The Lloyd–Moulton index number formula

17.61 The index number formula that will be discussed in this section on the single household economic approach to index number theory is a potentially very useful one for statistical agencies that are faced with the problem of producing a CPI in a timely manner.[43]

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[41] For an empirical comparison of the four indices, see Diewert (1978, pp. 894-895). For the Canadian consumer data considered there, the chained two-stage Fisher in 1971 was 2.3228 and the corresponding chained two-stage Törnqvist was 2.3230, the same values as for the corresponding single-stage indices.

[42] See Chapter 19 for some additional evidence on this topic.

[43] Walter Lane pointed out that the timeliness issue for statistical agencies should have been stressed earlier in Chapters 15 and 16, since normally, statistical agencies do not have current period information on quantities or values available. Hence CPI programs are forced to use the Laspeyres, Lowe or Young formulae rather than the theoretically preferred superlative indices since the former indexes do not require current period quantity or

(continued)

The Lloyd–Moulton formula that will be discussed in this section makes use of the same information that is required in order to implement a Laspeyres index except for one additional piece of information.

17.62 In this section, the same assumptions about the consumer are made that were made in paragraphs 17.18 to 17.26 above. In particular, it is assumed that the consumer's utility function $f(q)$ is linearly homogeneous[44] and the corresponding unit cost function is $c(p)$. It is supposed that the unit cost function has the following functional form:

$$c(p) \equiv \alpha_0 \left(\sum_{i=1}^{n} \alpha_i p_i^{1-\sigma} \right)^{1/(1-\sigma)} \text{ if } \sigma \neq 1 \text{ or } \ln c(p) \equiv \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln p_i \text{ if } \sigma = 1 \qquad (17.68)$$

where the α_i and σ are non-negative parameters with $\sum_{i=1}^{n} \alpha_i = 1$. The unit cost function defined by equation (17.68) corresponds to a constant elasticity of substitution (CES) aggregator function, which was introduced into the economics literature by Arrow, Chenery, Minhas and Solow (1961).[45] The parameter σ is the elasticity of substitution; when $\sigma = 0$, the unit cost function defined by equation (17.68) becomes linear in prices and hence corresponds to a fixed coefficients aggregator function which exhibits 0 substitutability between all commodities. When $\sigma = 1$,the corresponding aggregator or utility function is a Cobb–Douglas function. When σ approaches $+\infty$, the corresponding aggregator function f approaches a linear aggregator function which exhibits infinite substitutability between each pair of inputs. The CES unit cost function defined by equation (17.68) is not a fully flexible functional form (unless the number of commodities n being aggregated is 2), but it is considerably more flexible than the zero substitutability aggregator function (this is the special case of equation (17.68) where σ is set equal to zero) that is exact for the Laspeyres and Paasche price indices.

17.63 Under the assumption of cost minimizing behaviour in period 0, Shephard's Lemma (17.15), tells us that the observed first period consumption of commodity i, q_i^0, will be equal to $u^0\, \partial c(p^0)/\partial p_i$, where $\partial c(p^0)/\partial p_i$ is the first-order partial derivative of the unit cost function with respect to the ith commodity price evaluated at the period 0 prices and $u^0 = f(q^0)$ is the aggregate (unobservable) level of period 0 utility. Using the CES functional form defined by equation (17.68) and assuming that $\sigma \neq 1$, the following equations are obtained:

$$\begin{aligned} q_i^0 &= u^0 \alpha_0 \left\{ \sum_{k=1}^{n} \alpha_k \left(p_k^0\right)^r \right\}^{(1/r)-1} \alpha_i \left(p_i^0\right)^{r-1} \text{ for } r \equiv 1-\sigma \neq 0 \text{ and } i = 1,2,\ldots,n \\ &= \frac{u^0 c(p^0) \alpha_i (p_i^0)^{r-1}}{\sum_{k=1}^{n} \alpha_k (p_k^0)^r} \end{aligned} \qquad (17.69)$$

These equations can be rewritten as:

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value information. Nevertheless, superlative indices can be calculated (at least approximately) on a delayed basis or they can be used as target indices.

[44] Thus homothetic preferences are assumed in this section.

[45] In the mathematics literature, this aggregator function or utility function is known as a mean of order r an, where in this context, $r=1-\sigma$; see Hardy, Littlewood and Polyá (1934, pp. 12-13).

$$\frac{p_i^0 q_i^0}{u^0 c(p^0)} = \frac{\alpha_i (p_i^0)^r}{\sum_{k=1}^{n} \alpha_k (p_k^0)^r} \text{ for } i = 1,2,\ldots,n \tag{17.70}$$

where $r \equiv 1 - \sigma$. Now consider the following Lloyd (1975) Moulton (1996a) index number formula:

$$P_{LM}(p^0, p^1, q^0, q^1) \equiv \left\{ \sum_{i=1}^{n} s_i^0 \left(\frac{p_i^1}{p_i^0} \right)^{1-\sigma} \right\}^{1/(1-\sigma)} \quad \text{for } \sigma \neq 1 \tag{17.71}$$

where s_i^0 is the period 0 expenditure share of commodity i, as usual:

$$\begin{aligned} s_i^0 &\equiv \frac{p_i^0 q_i^0}{\sum_{k=1}^{n} p_k^0 q_k^0} \text{ for } i = 1,2,\ldots,n \\ &= \frac{p_i^0 q_i^0}{u^0 c(p^0)} \text{ using the assumption of cost minimizing behaviour} \qquad (17.72) \\ &= \frac{\alpha_i (p_i^0)^r}{\sum_{k=1}^{n} \alpha_k (p_k^0)^r} \text{ using equation (17.70)} \end{aligned}$$

If equation (17.72) is substituted into equation (17.71), it is found that:

$$\begin{aligned} P_{LM}(p^0, p^1, q^0, q^1) &= \left\{ \sum_{i=1}^{n} s_i^0 \left(\frac{p_i^1}{p_i^0} \right)^r \right\}^{1/r} \\ &= \left\{ \sum_{i=1}^{n} \frac{\alpha_i (p_i^0)^r}{\sum_{k=1}^{n} \alpha_k (p_k^0)^r} \left(\frac{p_i^1}{p_i^0} \right)^r \right\}^{1/r} \\ &= \left\{ \frac{\sum_{i=1}^{n} \alpha_i (p_i^1)^r}{\sum_{k=1}^{n} \alpha_k (p_k^0)^r} \right\}^{1/r} \\ &= \frac{\alpha_0 \left\{ \sum_{i=1}^{n} \alpha_i (p_i^1)^r \right\}^{1/r}}{\alpha_0 \left\{ \sum_{k=1}^{n} \alpha_k (p_k^0)^r \right\}^{1/r}} \\ &= \frac{c(p^1)}{c(p^0)} \text{ using } r \equiv 1 - \sigma \text{ and definition (17.68)} \qquad (17.73) \end{aligned}$$

17.64 Equation (17.73) shows that the Lloyd–Moulton index number formula P_{LM} is exact for CES preferences. Lloyd (1975) and Moulton (1996a) independently derived this result, but it was Moulton who appreciated the significance of the formula (17.71) for statistical agency purposes. Note that in order to evaluate formula (17.71) numerically, it is necessary to have information on:

- base period expenditure shares s_i^0;

- the price relatives p_i^1/p_i^0 between the base period and the current period; and
- an estimate of the elasticity of substitution between the commodities in the aggregate, σ.

The first two pieces of information are the standard information sets that statistical agencies use to evaluate the Laspeyres price index P_L (note that P_{LM} reduces to P_L if $\sigma = 0$). Hence, if the statistical agency is able to estimate the elasticity of substitution σ based on past experience,[46] then the Lloyd–Moulton price index can be evaluated using essentially the same information set that is used in order to evaluate the traditional Laspeyres index. Moreover, the resulting CPI will be free of substitution bias to a reasonable degree of approximation.[47] Of course, the practical problem with implementing this methodology is that estimates of the elasticity of substitution parameter σ are bound to be somewhat uncertain, and hence the resulting Lloyd Moulton index may be subject to charges that it is not objective or reproducible. The statistical agency will have to balance the benefits of reducing substitution bias with these possible costs.

Annual preferences and monthly prices

17.65 Recall the definition of the Lowe index, $P_{Lo}(p^0,p^1,q)$, defined by equation (15.15) in Chapter 15. In paragraphs 15.33 to 15.64 of Chapter 15, it is noted that this formula is frequently used by statistical agencies as a target index for a CPI. It is also noted that, while the price vectors p^0 (the base period price vector) and p^1 (the current period price vector) are monthly or quarterly price vectors, the quantity vector $q \equiv (q_1,q_2,\ldots,q_n)$ which appears in this basket-type formula is usually taken to be an annual quantity vector that refers to a base year, b say, that is prior to the base period for the prices, month 0. Thus, typically, a statistical agency will produce a CPI at a monthly frequency that has the form $P_{Lo}(p^0,p^t,q^b)$, where p^0 is the price vector pertaining to the base period month for prices, month 0, p^t is the price vector pertaining to the current period month for prices, month t say, and q^b is a reference basket quantity vector that refers to the base year b, which is equal to or prior to month 0.[48] The question to be addressed in the present section is: Can this index be related to one based on the economic approach to index number theory?

The Lowe index as an approximation to a true cost of living index

17.66 Assume that the consumer has preferences defined over consumption vectors $q \equiv [q_1,\ldots,q_n]$ that can be represented by the continuous increasing utility function $f(q)$. Thus if $f(q^1) > f(q^0)$, then the consumer prefers the consumption vector q^1 to q^0. Let q^b be the annual

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46 For the first application of this methodology (in the context of the CPI), see Shapiro and Wilcox (1997a, pp. 121-123). They calculated superlative Törnqvist indices for the United States for the years 1986–95 and then calculated the Lloyd Moulton CES index for the same period, using various values of σ. They then chose the value of σ (which was 0.7), which caused the CES index to most closely approximate the Törnqvist index. Essentially the same methodology was used by Alterman, Diewert and Feenstra (1999) in their study of United States import and export price indices. For alternative methods for estimating σ, see Balk (2000b).

47 What is a "reasonable" degree of approximation depends on the context. Assuming that consumers have CES preferences is not a reasonable assumption in the context of estimating elasticities of demand: at least a second-order approximation to the consumer's preferences is required in this context. In the context of approximating changes in a consumer's expenditures on the n commodities under consideration, however, it is usually adequate to assume a CES approximation.

48 As noted in Chapter 15, month 0 is called the price reference period and year b is called the weight reference period.

consumption vector for the consumer in the base year b. Define the base year utility level u^b as the utility level that corresponds to $f(q)$ evaluated at q^b:

$$u^b \equiv f(q^b) \tag{17.74}$$

17.67 For any vector of positive commodity prices $p \equiv [p_1,\ldots,p_n]$ and for any feasible utility level u, the consumer's cost function, $C(u, p)$, can be defined in the usual way as the minimum expenditure required to achieve the utility level u when facing the prices p:

$$C(u, p) \equiv \min_q \left\{ \sum_{i=1}^{n} p_i q_i : f(q_1,...,q_n) = u \right\}. \tag{17.75}$$

Let $p^b \equiv [p_1{}^b,\ldots,p_n{}^b]$ be the vector of annual prices that the consumer faced in the base year b. Assume that the observed base year consumption vector $q^b \equiv [q_1{}^b,\ldots,q_n{}^b]$ solves the following base year cost minimization problem:

$$C(u^b, p^b) \equiv \min_q \left\{ \sum_{i=1}^{n} p_i^b q_i : f(q_1,...,q_n) = u^b \right\} = \sum_{i=1}^{n} p_i^b q_i^b \tag{17.76}$$

The cost function will be used below in order to define the consumer's cost of living price index.

17.68 Let p^0 and p^t be the monthly price vectors that the consumer faces in months 0 and t. Then the Konüs true cost of living index, $P_K(p^0, p^t, q^b)$, between months 0 and t, using the base year utility level $u^b = f(q^b)$ as the reference standard of living, is defined as the following ratio of minimum monthly costs of achieving the utility level u^b:

$$P_K(p^0, p^t, q^b) \equiv \frac{C(f(q^b), p^t)}{C(f(q^b), p^0)} \tag{17.77}$$

17.69 Using the definition of the monthly cost minimization problem that corresponds to the cost $C(f(q^b), p^t)$, it can be seen that the following inequality holds:

$$\begin{aligned} C(f(q^b), p^t) &\equiv \min_q \left\{ \sum_{i=1}^{n} p_i^t q_i : f(q_1,...,q_n) = f(q_1^b,...,q_n^b) \right\} \\ &\leq \sum_{i=1}^{n} p_i^t q_i^b \end{aligned} \tag{17.78}$$

since the base year quantity vector q^b is feasible for the cost minimization problem. Similarly, using the definition of the monthly cost minimization problem that corresponds to the month 0 cost $C(f(q^b), p^0)$, it can be seen that the following inequality holds:

$$\begin{aligned} C(f(q^b), p^0) &\equiv \min_q \left\{ \sum_{i=1}^{n} p_i^0 q_i : f(q_1,...,q_n) = f(q_1^b,...,q_n^b) \right\} \\ &\leq \sum_{i=1}^{n} p_i^0 q_i^b \end{aligned} \tag{17.79}$$

since the base year quantity vector q^b is feasible for the cost minimization problem.

17.70 It will prove useful to rewrite the two inequalities (17.78) and (17.79) as equalities. This can be done if non-negative substitution bias terms, e^t and e^0, are subtracted from the right-hand sides of these two inequalities. Thus the inequalities (17.78) and (17.79) can be rewritten as follows:

$$C(u^b, p^t) = \sum_{i=1}^{n} p_i^t q_i^b - e^t \tag{17.80}$$

$$C(u^b, p^0) = \sum_{i=1}^{n} p_i^0 q_i^b - e^0 \qquad (17.81)$$

17.71 Using equations (17.80) and (17.81), and the definition (15.15) in Chapter 15 of the Lowe index, the following approximate equality for the Lowe index results:

$$P_{Lo}(p^0, p^t, q^b) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} = \frac{\{C(u^b, p^t) + e^t\}}{\{C(u^b, p^0) + e^0\}} \approx \frac{C(u^b, p^t)}{C(u^b, p^0)} = P_K(p^0, p^t, q^b) \qquad (17.82)$$

Thus if the non-negative substitution bias terms e^0 and e^t are small, then the Lowe index between months 0 and t, $P_{Lo}(p^0, p^t, q^b)$, will be an adequate approximation to the true cost of living index between months 0 and t, $P_K(p^0, p^t, q^b)$.

17.72 A bit of algebraic manipulation shows that the Lowe index will be exactly equal to its cost of living counterpart if the substitution bias terms satisfy the following relationship:[49]

$$\frac{e^t}{e^0} = \frac{C(u^b, p^t)}{C(u^b, p^0)} = P_K(p^0, p^t, q^b) \qquad (17.83)$$

Equations (17.82) and (17.83) can be interpreted as follows: if the rate of growth in the amount of substitution bias between months 0 and t is equal to the rate of growth in the minimum cost of achieving the base year utility level u^b between months 0 and t, then the observable Lowe index, $P_{Lo}(p^0, p^t, q^b)$, will be exactly equal to its true cost of living index counterpart, $P_K(p^0,p^t,q^b)$.[50]

17.73 It is difficult to know whether condition (17.83) will hold or whether the substitution bias terms e^0 and e^t will be small. Thus, first-order and second-order Taylor series approximations to these substitution bias terms are developed in paragraphs 17.74 to 17.83.

A first-order approximation to the bias of the Lowe index

17.74 The true cost of living index between months 0 and t, using the base year utility level u^b as the reference utility level, is the ratio of two unobservable costs, $C(u^b, p^t)/C(u^b,p^0)$. However, both of these hypothetical costs can be approximated by first-order Taylor series approximations that can be evaluated using observable information on prices and base year quantities. The first-order Taylor series approximation to $C(u^b, p^t)$ around the annual base year price vector p^b is given by the following approximate equation:[51]

[49] This assumes that e^0 is greater than zero. If e^0 is equal to zero, then to have equality of P_K and P_{Lo}, it must be the case that e^t is also equal to zero.

[50] It can be seen that, when month t is set equal to month 0, $e^t = e^0$ and $C(u^b,p^t) = C(u^b,p^0)$, and thus equation (17.83) is satisfied and $P_{Lo} = P_K$. This is not surprising since both indices are equal to unity when $t = 0$.

[51] This type of Taylor series approximation was used in Schultze and Mackie (2002, p. 91) in the cost of living index context, but it essentially dates back to Hicks (1941-42, p. 134) in the consumer surplus context. See also Diewert (1992b, p. 568) and Hausman (2002, p. 8).

$$
\begin{aligned}
C(u^b, p^t) &\approx C(u^b, p^b) + \sum_{i=1}^{n} \left[\frac{\partial C(u^b, p^b)}{\partial p_i}\right]\left[p_i^t - p_i^b\right] \\
&= C(u^b, p^b) + \sum_{i=1}^{n} q_i^b \left[p_i^t - p_i^b\right] \\
&\qquad \text{using assumption (17.76) and Shephard's Lemma (17.12)} \qquad (17.84) \\
&= \sum_{i=1}^{n} p_i^b q_i^b + \sum_{i=1}^{n} q_i^b \left[p_i^t - p_i^b\right] \qquad \text{using (17.76)} \\
&= \sum_{i=1}^{n} p_i^t q_i^b
\end{aligned}
$$

Similarly, the first-order Taylor series approximation to $C(u^b,p^0)$ around the annual base year price vector p^b is given by the following approximate equation:

$$
\begin{aligned}
C(u^b, p^0) &\approx C(u^b, p^b) + \sum_{i=1}^{n} \left[\frac{\partial C(u^b, p^b)}{\partial p_i}\right]\left[p_i^0 - p_i^b\right] \\
&= C(u^b, p^b) + \sum_{i=1}^{n} q_i^b \left[p_i^0 - p_i^b\right] \\
&= \sum_{i=1}^{n} p_i^b q_i^b + \sum_{i=1}^{n} q_i^b \left[p_i^0 - p_i^b\right] \\
&= \sum_{i=1}^{n} p_i^0 q_i^b
\end{aligned}
\qquad (17.85)
$$

17.75 Comparing approximate equation (17.84) with equation (17.80), and comparing approximate equation (17.85) with equation (17.81), it can be seen that, to the accuracy of the first-order approximations used in (17.84) and (17.85), the substitution bias terms e^t and e^0 will be zero. Using these results to reinterpret the approximate equation (17.82), it can be seen that if the month 0 and month t price vectors, p^0 and p^t, are not too different from the base year vector of prices p^b, then the Lowe index $P_{Lo}(p^0, p^t, q^b)$ will approximate the true cost of living index $P_K(p^0, p^t, q^b)$ to the accuracy of a first-order approximation. This result is quite useful, since it indicates that if the monthly price vectors p^0 and p^t are just randomly fluctuating around the base year prices p^b (with modest variances), then the Lowe index will serve as an adequate approximation to a theoretical cost of living index. However, if there are systematic long-term trends in prices and month t is fairly distant from month 0 (or the end of year b is quite distant from month 0), then the first-order approximations given by approximate equations (17.84) and (17.85) may no longer be adequate and the Lowe index may have a considerable bias relative to its cost of living counterpart. The hypothesis of long-run trends in prices will be explored in paragraphs 17.76 to 17.83.

A second-order approximation to the substitution bias of the Lowe index

17.76 A second-order Taylor series approximation to $C(u^b, p^t)$ around the base year price vector p^b is given by the following approximate equation:

$$C(u^b, p^t) \approx C(u^b, p^b) + \sum_{i=1}^{n}\left[\frac{\partial C(u^b, p^b)}{\partial p_i}\right]\left[p_i^t - p_i^b\right]$$
$$+\left(\frac{1}{2}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j}\right]\left[p_i^t - p_i^b\right]\left[p_j^t - p_j^b\right] \tag{17.86}$$
$$= \sum_{i=1}^{n} p_i^b q_i^b + \left(\frac{1}{2}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j}\right] \times\left[p_i^t - p_i^b\right]\left[p_j^t - p_j^b\right]$$

where the last equality follows using approximate equation (17.84).[52] Similarly, a second-order Taylor series approximation to $C(u^b, p^0)$ around the base year price vector p^b is given by the following approximate equation:

$$C(u^b, p^0) \approx C(u^b, p^b) + \sum_{i=1}^{n}\left[\frac{\partial C(u^b, p^b)}{\partial p_i}\right]\left[p_i^0 - p_i^b\right]$$
$$+\left(\frac{1}{2}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j}\right]\left[p_i^0 - p_i^b\right]\left[p_j^0 - p_j^b\right] \tag{17.87}$$
$$= \sum_{i=1}^{n} p_i^0 q_i^b + \left(\frac{1}{2}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j}\right]\left[p_i^0 - p_i^b\right]\left[p_j^0 - p_j^b\right]$$

where the last equality follows using the approximate equation (17.85).

17.77 Comparing approximate equation (17.86) with equation (17.80), and approximate equation (17.87) with equation (17.81), it can be seen that, to the accuracy of a second-order approximation, the month 0 and month t substitution bias terms, e^0 and e^t, will be equal to the following expressions involving the second-order partial derivatives of the consumer's cost function $\partial^2 C(u^b,p^b)/\partial p_i \partial p_j$ evaluated at the base year standard of living u^b and at the base year prices p^b:

$$e^0 \approx -\left(\frac{1}{2}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j}\right]\left[p_i^0 - p_i^b\right]\left[p_j^0 - p_j^b\right] \tag{17.88}$$

$$e^t \approx -\left(\frac{1}{2}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j}\right]\left[p_i^t - p_i^b\right]\left[p_j^t - p_j^b\right] \tag{17.89}$$

Since the consumer's cost function $C(u, p)$ is a concave function in the components of the price vector p,[53] it is known[54] that the n by n (symmetric) matrix of second-order partial

[52] This type of second-order approximation is attributable to Hicks (1941-42, pp. 133-134) (1946, p. 331). See also Diewert (1992b, p. 568), Hausman (2002, p. 18) and Schultze and Mackie (2002, p. 91). For alternative approaches to modelling substitution bias, see Diewert (1998a; 2002c, pp. 598-603) and Hausman (2002).

[53] See Diewert (1993b, pp. 109-110).

[54] See Diewert (1993b, p. 149).

derivatives $[\partial^2 C(u^b, p^b)/\partial p_i \partial p_j]$ is negative semi-definite.[55] Hence, for arbitrary price vectors p^b, p^0 and p^t, the right-hand sides of approximations (17.88) and (17.89) will be non-negative. Thus, to the accuracy of a second-order approximation, the substitution bias terms e^0 and e^t will be non-negative.

17.78 Now assume that there are long-run systematic trends in prices. Assume that the last month of the base year for quantities occurs M months prior to month 0, the base month for prices, and assume that prices trend linearly with time, starting with the last month of the base year for quantities. Thus, assume the existence of constants α_j for $j = 1,\ldots,n$ such that the price of commodity j in month t is given by:

$$p_j^t = p_j^b + \alpha_j (M+t) \text{ for } j = 1,\ldots,n \text{ and } t = 0,1,\ldots T \qquad (17.90)$$

Substituting equation (17.90) into approximations (17.88) and (17.89) leads to the following second-order approximations to the two substitution bias terms, e^0 and e^t:[56]

$$e^0 \approx \gamma M^2 \qquad (17.91)$$

$$e^t \approx \gamma (M+t)^2 \qquad (17.92)$$

where γ is defined as follows:

$$\gamma \equiv -\left(\frac{1}{2}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j}\right]\alpha_i \alpha_j \geq 0 \qquad (17.93)$$

17.79 It should be noted that the parameter γ will be zero under two sets of conditions:[57]

- All the second-order partial derivatives of the consumer's cost function $\partial^2 C(u^b,p^b)/\partial p_i \partial p_j$ are equal to zero.
- Each commodity price change parameter α_j is proportional to the corresponding commodity j base year price p_j^b.[58]

The first condition is empirically unlikely since it implies that the consumer will not substitute away from commodities of which the relative price has increased. The second condition is also empirically unlikely, since it implies that the structure of relative prices

[55] A symmetric n by n matrix A with ijth element equal to a_{ij} is negative semi-definite if, and only if for every vector $z \equiv [z_1,\ldots,z_n]$, it is the case that $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_i z_j \leq 0$.

[56] Note that the period 0 approximate bias defined by the right-hand side of approximation (17.91) is fixed, while the period t approximate bias defined by the right-hand side of (17.92) increases quadratically with time t. Hence, the period t approximate bias term will eventually overwhelm the period 0 approximate bias in this linear time trends case, if t is allowed to become large enough.

[57] A more general condition that ensures the positivity of γ is that the vector $[\alpha_1,\ldots,\alpha_n]$ is not an eigenvector of the matrix of second-order partial derivatives $\partial^2 C(u,p)/\partial p_i \partial p_j$ that corresponds to a zero eigenvalue.

[58] It is known that $C(u,p)$ is linearly homogeneous in the components of the price vector p; see Diewert (1993b, p. 109) for example. Hence, using Euler's Theorem on homogeneous functions, it can be shown that p^b is an eigenvector of the matrix of second-order partial derivatives $\partial^2 C(u,p)/\partial p_i \partial p_j$ that corresponds to a zero eigenvalue and thus $\sum_{i=1}^{n}\sum_{j=1}^{n} [\partial^2 C(u,p)/\partial p_i \partial p_j]\, p_i^b p_j^b = 0$; see Diewert (1993b, p. 149) for a detailed proof of this result.

remains unchanged over time. Thus, in what follows, it will be assumed that γ is a positive number.

17.80 In order to simplify the notation in what follows, define the denominator and numerator of the month t Lowe index, $P_{Lo}(p^0, p^t, q^b)$, as a and b respectively; i.e., define:

$$a \equiv \sum_{i=1}^{n} p_i^0 q_i^b \tag{17.94}$$

$$b \equiv \sum_{i=1}^{n} p_i^t q_i^b \tag{17.95}$$

Using equation (17.90) to eliminate the month 0 prices p_i^0 from equation (17.94) and the month t prices p_i^t from equation (17.95) leads to the following expressions for a and b:

$$a = \sum_{i=1}^{n} p_i^b q_i^b + \sum_{i=1}^{n} \alpha_i q_i^b M \tag{17.96}$$

$$b = \sum_{i=1}^{n} p_i^b q_i^b + \sum_{i=1}^{n} \alpha_i q_i^b (M + t) \tag{17.97}$$

It is assumed that a and b[59] are positive and that

$$\sum_{i=1}^{n} \alpha_i q_i^b \geq 0 \tag{17.98}$$

Assumption (17.98) rules out a general deflation in prices.

17.81 Define the bias in the month t Lowe index, B^t, as the difference between the true cost of living index $P_K(p^0, p^t, q^b)$ defined by equation (17.77) and the corresponding Lowe index $P_{Lo}(p^0, p^t, q^b)$:

[59] It is also assumed that $a - \gamma M^2$ is positive.

$$
\begin{aligned}
B^t &\equiv P_K(p^0,p^t,q^b) - P_{Lo}(p^0,p^t,q^b) \\
&= \left\{\frac{C(u^b,p^t)}{C(u^b,p^0)}\right\} - \left(\frac{b}{a}\right) && \text{using (17.94) and (17.95)} \\
&= \left\{\frac{[b-e^t]}{[a-e^0]}\right\} - \left(\frac{b}{a}\right) && \text{using (17.80) and (17.81)} \\
&\approx \left\{\frac{[b-\gamma(M+t)^2]}{a-\gamma M^2}\right\} - \left(\frac{b}{a}\right) && \text{using (17.91) and (17.92)} \\
&= \gamma \frac{\{(b-a)M^2 - 2aMt - at^2\}}{\{a[a-\gamma M^2]\}} && \text{simplifying terms} \\
&= \gamma \frac{\left\{\left[\sum_{i=1}^{n}\alpha_i q_i^b t\right]M^2 - 2\left[\sum_{i=1}^{n} p_i^b q_i^b + \sum_{i=1}^{n}\alpha_i q_i^b M\right]Mt - at^2\right\}}{\{a[a-\gamma M^2]\}} && \text{using (17.96) and (17.97)} \\
&= -\gamma \frac{\left\{\left[\sum_{i=1}^{n}\alpha_i q_i^b t\right]M^2 + 2\left[\sum_{i=1}^{n} p_i^b q_i^b\right]Mt + at^2\right\}}{\{a[a-\gamma M^2]\}} && (17.99) \\
&< 0 && \text{using (17.98).}
\end{aligned}
$$

Thus, for $t \geq 1$, the Lowe index will have an upward bias (to the accuracy of a second-order Taylor series approximation) compared to the corresponding true cost of living index $P_K(p^0,p^t,q^b)$, since the approximate bias defined by the last expression in equation (17.99) is the sum of one non-positive and two negative terms. Moreover, this approximate bias will grow quadratically in time t.[60]

17.82 In order to give the reader some idea of the magnitude of the approximate bias B^t defined by the last line of equation (17.99), a simple special case will be considered at this point. Suppose there are only two commodities and that, at the base year, all prices and quantities are equal to 1. Thus, $p_i^b = q_i^b = 1$ for $i = 1,2$ and $\sum_{i=1}^{n} p_i^b q_i^b = 2$. Assume that $M = 24$ so that the base year data on quantities take two years to process before the Lowe index can be implemented. Assume that the monthly rate of growth in price for commodity 1 is $\alpha_1 = 0.002$ so that after one year, the price of commodity 1 rises 0.024 or 2.4 per cent. Assume that commodity 2 falls in price each month with $\alpha_2 = -0.002$ so that the price of commodity 2 falls 2.4 per cent in the first year after the base year for quantities. Thus the relative price of the two commodities is steadily diverging by about 5 per cent per year. Finally, assume that $\partial^2 C(u^b,p^b)/\partial p_1 \partial p_1 = \partial^2 C(u^b,p^b)/\partial p_2 \partial p_2 = -1$ and $\partial^2 C(u^b,p^b)/\partial p_1 \partial p_2 = \partial^2 C(u^b,p^b)/\partial p_2 \partial p_1 = 1$. These assumptions imply that the own elasticity of demand for each commodity is -1 at the base year consumer equilibrium. Making all of these assumptions means that:

$$
2 = \sum_{i=1}^{n} p_i^b q_i^b = a = b \quad \sum_{i=1}^{n} \alpha_i q_i^b = 0 \quad M = 24;\, \gamma = 0.000008 \qquad (17.100)
$$

[60] If M is large relative to t, then it can be seen that the first two terms in the last equation of (17.99) can dominate the last term, which is the quadratic in t term.

Substituting the parameter values defined in equation (17.100) into equation (17.99) leads to the following formula for the approximate amount that the Lowe index will exceed the corresponding true cost of living index at month t:

$$-B^t = 0.000008 \frac{\left(96t + 2t^2\right)}{2\left(2 - 0.004608\right)} \qquad (17.101)$$

Evaluating equation (17.101) at t = 12, t = 24, t = 36, t = 48 and t = 60 leads to the following estimates for $-B^t$: 0.0029 (the approximate bias in the Lowe index at the end of the first year of operation for the index); 0.0069 (the bias after two years); 0.0121 (the bias after three years); 0.0185 (the bias after four years); 0.0260 (the bias after five years). Thus, at the end of the first year of the operation, the Lowe index will only be above the corresponding true cost of living index by approximately a third of a percentage point but, by the end of the fifth year of operation, it will exceed the corresponding cost of living index by about 2.6 percentage points, which is no longer a negligible amount.[61]

17.83 The numerical results in the previous paragraph are only indicative of the approximate magnitude of the difference between a cost of living index and the corresponding Lowe index. The important point to note is that, to the accuracy of a second-order approximation, the Lowe index will generally exceed its cost of living counterpart. The results also indicate, however, that this difference can be reduced to a negligible amount if:

- the lag in obtaining the base year quantity weights is minimized; and
- the base year is changed as frequently as possible.

It should also be noted that the numerical results depend on the assumption that long-run trends in prices exist, which may not be true,[62] and on elasticity assumptions that may not be justified.[63] Statistical agencies should prepare their own carefully constructed estimates of the differences between a Lowe index and a cost of living index in the light of their own particular circumstances.

The problem of seasonal commodities

17.84 The assumption that the consumer has annual preferences over commodities purchased in the base year for the quantity weights, and that these annual preferences can be used in the context of making monthly purchases of the same commodities, was a key one in relating the economic approach to index number theory to the Lowe index. This assumption that annual preferences can be used in a monthly context is, however, somewhat questionable because of the seasonal nature of some commodity purchases. The problem is that it is very likely that consumers' preference functions systematically change as the season of the year changes. National customs and weather changes cause households to purchase certain goods and services during some months and not at all for other months. For example, Christmas trees are purchased only in December and ski jackets are not usually purchased during

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[61] Note that the relatively large magnitude of M compared to t leads to a bias that grows approximately linearly with t rather than quadratically.

[62] For mathematical convenience, the trends in prices were assumed to be linear, rather than the more natural assumption of geometric trends in prices.

[63] Another key assumption that was used to derive the numerical results is the magnitude of the divergent trends in prices. If the price divergence vector is doubled to $\alpha_1 = 0.004$ and $\alpha_2 = -0.004$, then the parameter γ quadruples and the approximate bias will also quadruple.

summer months. Thus, the assumption that annual preferences are applicable during each month of the year is only acceptable as a very rough approximation to economic reality.

17.85 The economic approach to index number theory can be adapted to deal with seasonal preferences. The simplest economic approach is to assume that the consumer has annual preferences over commodities classified not only by their characteristics but also by the month of purchase.[64] Thus, instead of assuming that the consumer's annual utility function is $f(q)$ where q is an n-dimensional vector, assume that the consumer's annual utility function is $F[f^1(q^1), f^2(q^2),\ldots,f^{12}(q^{12})]$ where q^1 is an n dimensional vector of commodity purchases made in January, q^2 is an n dimensional vector of commodity purchases made in February, …, and q^{12} is an n dimensional vector of commodity purchases made in December.[65] The sub-utility functions $f^1, f^2, \ldots ,f^{12}$ represent the consumer's preferences when making purchases in January, February, …. , and December, respectively. These monthly sub-utilities can then be aggregated using the macro-utility function F in order to define overall annual utility. It can be seen that these assumptions on preferences can be used to justify two types of cost of living index:

- an annual cost of living index that compares the prices in all months of a current year with the corresponding monthly prices in a base year;[66] and
- 12 monthly cost of living indices where the index for month m compares the prices of month m in the current year with the prices of month m in the base year for $m = 1,2,\ldots,12$.[67]

17.86 The annual Mudgett-Stone indices compare costs in a current calendar year with the corresponding costs in a base year. However, any month could be chosen as the year-ending month of the current year, and the prices and quantities of this new non-calendar year could be compared to the prices and quantities of the base year, where the January prices of the non-calendar year are matched to the January prices of the base year, the February prices of the non-calendar year are matched to the February prices of the base year, and so on. If further assumptions are made on the macro-utility function F, then this framework can be used in order to justify a third type of cost of living index: a moving year annual index.[68] This index compares the cost over the past 12 months of achieving the annual utility achieved in the base year with the base year cost, where the January costs in the current moving year are matched to January costs in the base year, the February costs in the current moving year are matched to February costs in the base year, and so on. These moving year indices can be

[64] This assumption and the resulting annual indices were first proposed by Mudgett (1955, p. 97) and Stone (1956, pp. 74-75).

[65] If some commodities are not available in certain months m, then those commodities can be dropped from the corresponding monthly quantity vectors q^m.

[66] For further details on how to implement this framework, see Mudgett (1955, p. 97), Stone (1956, pp. 74-75) and Diewert (1998b, pp. 459-460).

[67] For further details on how to implement this framework, see Diewert (1999a, pp. 50-51).

[68] See Diewert (1999a, pp. 56-61) for the details of this economic approach.

calculated for each month of the current year and the resulting series can be interpreted as (uncentred) seasonally adjusted (annual) price indices.[69]

17.87 It should be noted that none of the three types of indices described in the previous two paragraphs is suitable for describing the movements of prices going from one month to the following month; i.e., they are not suitable for describing short-run movements in inflation. This is obvious for the first two types of index. To see the problem with the moving year indices, consider a special case where the bundle of commodities purchased in each month is entirely specific to each month. Then it is obvious that, even though all the above three types of index are well defined, none of them can describe anything useful about month-to-month changes in prices, since it is impossible to compare like with like, going from one month to the next, under the hypotheses of this special case. It is impossible to compare the incomparable.

17.88 Fortunately, it is not the case that household purchases in each month are entirely specific to the month of purchase. Thus month-to-month price comparisons can be made if the commodity space is restricted to commodities that are purchased in each month of the year. This observation leads to a fourth type of cost of living index, a month-to-month index, defined over commodities that are available in every month of the year.[70] This model can be used to justify the economic approach described in paragraphs 17.66 to 17.83. Commodities that are purchased only in certain months of the year, however, must be dropped from the scope of the index. Unfortunately, it is likely that consumers have varying monthly preferences over the commodities that are always available and, if this is the case, the month-to-month cost of living index (and the corresponding Lowe index) defined over always-available commodities will generally be subject to seasonal fluctuations. This will limit the usefulness of the index as a short-run indicator of general inflation since it will be difficult to distinguish a seasonal movement in the index from a systematic general movement in prices.[71] Note also that if the scope of the index is restricted to always-available commodities, then the resulting month-to-month index will not be comprehensive, whereas the moving year indices will be comprehensive in the sense of using all the available price information.

17.89 The above considerations lead to the conclusion that it may be useful for statistical agencies to produce at least two consumer price indices:

- a moving year index which is comprehensive and seasonally adjusted, but which is not necessarily useful for indicating month-to-month changes in general inflation; and
- a month-to-month index which is restricted to non-seasonal commodities (and hence is not comprehensive), but which is useful for indicating short-run movements in general inflation.

[69] See Diewert (1999a, pp. 67-68) for an empirical example of this approach applied to quantity indices. An empirical example of this moving year approach to price indices is presented in Chapter 22.

[70] See Diewert (1999a, pp. 51-56) for the assumptions on preferences that are required in order to justify this economic approach.

[71] One problem with using annual weights in the context of seasonal movements in prices and quantities is that a change in price when a commodity is out of season can be greatly magnified by the use of annual weights. Baldwin (1990, p. 251) noted this problem with an annual weights price index: “But a price index is adversely affected if any seasonal good has the same basket share for all months of the year; the good will have an inappropriately small basket share in its in season months, an inappropriately large share in its off season months.” Seasonality problems are considered again from a more pragmatic point of view in Chapter 22.

The problem of a zero price increasing to a positive price

17.90 In a recent paper, Haschka (2003) raised the problem of what to do when a price which was previously zero is increased to a positive level. He gave two examples for Austria, where parking and hospital fees were raised from zero to a positive level. In this situation, it turns out that basket-type indices have an advantage over indices that are weighted geometric averages of price relatives, since basket-type indices are well defined even if some prices are zero.

17.91 The problem can be considered in the context of evaluating the Laspeyres and Paasche indices. Suppose as usual that the prices p_i^t and quantities q_i^t of the first n commodities are positive for periods 0 and 1, but that the price of commodity n+1 in period 0 is zero but is positive in period 1. In both periods, the consumption of commodity n+1 is positive. Thus the assumptions on the prices and quantities of commodity n+1 in the two periods under consideration can be summarized as follows:

$$p_{n+1}^0 = 0 \quad p_{n+1}^1 > 0 \quad q_{n+1}^0 > 0 \quad q_{n+1}^1 > 0 \qquad (17.102)$$

Typically, the increase in price of commodity n+1 from its initial zero level will cause consumption to fall so that $q_{n+1}^1 < q_{n+1}^0$, but this inequality is not required for the analysis below.

17.92 Let the Laspeyres index between periods 0 and 1, restricted to the first n commodities, be denoted as P_L^n and let the Laspeyres index, defined over all n+1 commodities, be defined as P_L^{n+1}. Also let $v_i^0 \equiv p_i^0 q_i^0$ denote the value of expenditures on commodity i in period 0. Then by the definition of the Laspeyres index defined over all n+1 commodities:

$$P_L^{n+1} \equiv \frac{\sum_{i=1}^{n+1} p_i^1 q_i^0}{\sum_{i=1}^{n+1} p_i^0 q_i^0} \qquad (17.103)$$

$$= P_L^n + \frac{p_{n+1}^1 q_{n+1}^0}{\sum_{i=1}^{n} v_i^0}$$

where $p_{n+1}^0 = 0$ was used in order to derive the second equation above. Thus the complete Laspeyres index P_L^{n+1} defined over all n+1 commodities is equal to the incomplete Laspeyres index P_L^n (which can be written in traditional price relative and base period expenditure share form), plus the mixed or hybrid expenditure $p_{n+1}^1 q_{n+1}^0$ divided by the base period expenditure on the first n commodities, $\sum_{i=1}^{n} v_i^0$. Thus the complete Laspeyres index can be calculated using the usual information available to the price statistician plus two additional pieces of information: the new non-zero price for commodity n+1 in period 1, p_{n+1}^1, and an estimate of consumption of commodity n+1 in period 0 (when it was free), q_{n+1}^0. Since it is often governments that change the previously zero price to a positive price, the decision to do this is usually announced in advance, which will give the price statistician an opportunity to form an estimate for the base period demand, q_{n+1}^0.

17.93 Let the Paasche index between periods 0 and 1, restricted to the first n commodities, be denoted as P_P^n and let the Paasche index, defined over all n+1 commodities, be defined as P_P^{n+1}. Also let $v_i^1 \equiv p_i^1 q_i^1$ denote the value of expenditures on commodity i in period 1. Then, by the definition of the Paasche index defined over all n+1 commodities:

$$P_P^{n+1} \equiv \frac{\sum_{i=1}^{n+1} p_i^1 q_i^1}{\sum_{i=1}^{n+1} p_i^0 q_i^1}$$

$$= P_P^n + \frac{v_{n+1}^1}{\sum_{i=1}^{n} p_i^0 q_i^1} \qquad (17.104)$$

$$= P_P^n + \frac{v_{n+1}^1}{\sum_{i=1}^{n} v_i^1 / (p_i^1 / p_i^0)}$$

where $p_{n+1}{}^0 = 0$ was used in order to derive the second equation above. Thus the complete Paasche index $P_P{}^{n+1}$ defined over all n+1 commodities is equal to the incomplete Paasche index $P_P{}^n$ (which can be written in traditional price relative and current period expenditure share form), plus the current period expenditure on commodity n+1, $v_{n+1}{}^1$, divided by a sum of current period expenditures on the first n commodities, $v_i{}^1$, divided by the ith price relative for the first n commodities, $p_i{}^1/p_i{}^0$. Thus the complete Paasche index can be calculated using the usual information available to the price statistician plus information on current period expenditures.

17.94 Once the complete Laspeyres and Paasche indices have been calculated using equations (17.103) and (17.104), then the complete Fisher index can be calculated as the square root of the product of these two indices:

$$P_F^{n+1} = [P_L^{n+1} P_P^{n+1}]^{1/2} \qquad (17.105)$$

It should be noted that the complete Fisher index defined by equation (17.105) satisfies the same exact index number results as were demonstrated in paragraphs 17.27 to 17.32 above; i.e., the Fisher index remains a superlative index even if prices are zero in one period but positive in the other. Thus the Fisher price index remains a suitable target index even in the face of zero prices.