PROBABILITY MODELS FOR ECONOMIC DECISIONS

by Roger B. Myerson (Duxbury, 2005)

Chapter 8: Risk Sharing and Finance

In our study of decision analysis, we initially assumed (starting in Chapter 2) that the basic criterion for defining optimal decisions under uncertainty is maximization of the decision-maker's expected payoff. This assumption seemed questionable, however, when we observed that individual decision-makers are often risk averse. Then utility theory (in Chapter 3) taught us to use expected utility values, rather than expected monetary values, as our general criterion for optimal decision-making. We have focused on the theory of utility functions with constant risk tolerance, as a practical framework for analyzing the effect of risk aversion on people's decisions. But utility theory was still about decision-making by individuals. In this chapter we make the transition from individual decision-making to decision-making in partnerships and corporations.

A large business enterprise generally has a chief executive officer, but it typically has many owners (partners or stockholders), each of whom may have a different risk tolerance. How should such partnerships or corporations make decisions under uncertainty? Should they use the utility function of the chief executive officer, or of the owners? If the owners, how do we resolve their differences when they have different utility functions?

We begin this chapter with a general analysis of optimal risk sharing among individuals who have constant risk tolerance. We find that, in an optimal allocation of risks, more risk tolerant people should hold more risks, in proportion to their risk tolerance. We show that a partnership with such optimal risk sharing should evaluate investment opportunities by applying a risk tolerance that is the sum of the partners' individual risk tolerances.

We then consider the effect of incentive constraints that may prevent managers from achieving such optimal risk sharing with outside investors. The trade-off between risk sharing and incentives is analyzed in simple principal-agent problems. This analysis teaches us that senior managers of big businesses should be expected to bear a personally significant share of the corporate risks. Such managers may then want corporate decision-making to be guided by their own personal utility functions applied to their personal shares of the corporate risks. But a very different approach is needed if we want to analyze corporate decision-making on behalf of the stockholders.

When a publicly held corporation makes decisions on behalf of its stockholders, it should generally assume that its stock is held by investors as a part of a well-diversified portfolio, and that these stockholders want the corporation to maximize the value of its shares in the stock market. So to understand optimal corporate decision-making for the stockholders, we need a theory of how the prices of financial assets are determined in the stock market. We develop here a model of financial asset pricing, using the assumption that the stock market includes many investors who have constant risk tolerance. This model is somewhat different from the well-known capital asset pricing model (CAPM), but it yields similar and closely related results. Both these asset-pricing models teach us that the magnitude of a corporation's risks alone may be less important than the relationship between these risks and the greater aggregate risks of the whole stock market.

Any system of asset pricing that does not create arbitrage opportunities must be consistent with a generalized expected-value criterion that applies some modified probabilities which may be determined in the stock market. These general results of arbitrage pricing theory are introduced at the end of this chapter, and are shown to be include our asset-pricing model as a special case.

8.1. Optimal risk sharing in a partnership of individuals with constant risk tolerance

To introduce the basic ideas of optimal risk sharing, let us begin with an example of two individuals (numbered 1 and 2) who are considering a real-estate development project. Suppose that they have an option to buy a tract of land for $125,000, after which they would then need to spend an additional $40,000 on improvements (including an allocation for the cost of their own time in supervising the project) before they could sell the land in subdivided lots. The total revenue that they could then earn from selling these lots would be uncertain, but has an expected value of $200,000 and a standard deviation of $25,000. For simplicity, let us assume here that the time to complete this real estate project is small enough that we can ignore the interest costs of borrowing money to cover the expenses before the revenues come in. So the net returns from this real estate project next year will have expected value

$$\mu = 200{,}000 - (125{,}000 + 40{,}000) = \$35{,}000$$

and standard deviation

$$\sigma = \$25{,}000.$$

Suppose that each of these two individuals evaluates risky incomes using a utility function with constant risk tolerance, where individual 1 has risk tolerance

$$r_1 = \$20{,}000,$$

and individual 2 has risk tolerance

$$r_2 = \$30{,}000.$$

They must decide whether to undertake this real estate project, and if so, how to divide the returns among themselves. Let us assume that the uncertainty about profits from this project can be described by a Normal distribution.

In Section 4.7 of Chapter 4 we saw that, when an individual with constant risk tolerance r has a gamble that will pay a random amount of money drawn from a Normal probability distribution with mean μ and standard deviation σ, his certainty equivalent for the gamble is

$$CE = \mu - (0.5/r)*\sigma\verb|^|2$$

So if individual 1 were to undertake this project himself, his certainty equivalent would be

$$\mu - (0.5/r_1)*(\sigma\verb|^|2) = 35000 - (0.5/20000)*(25000\verb|^|2)$$

$$= 35000 - 15625 = \$19{,}375.$$

That is, the option to buy this land and undertake this project would be worth $19,375 to individual 1, if he had to undertake all the risks of the project alone.

If individual 2 were to undertake this project by herself, then its value to her would be

$$\mu - (0.5/r_2)*(\sigma\verb|^|2) = 25000 - (0.5/30000)*(25000\verb|^|2)$$

$$= 35000 - 10417 = \$24{,}583$$

So if individual 1 had the option to buy this land, then individual 2 would be willing to pay up to $24,583 to buy the option from him, and individual 1 would be glad to sell the option for any price above $19,375. Of course it is not surprising that this risky project should be more valuable to the individual who has greater risk tolerance.

But even though individual 2 is strictly more risk tolerant than individual 1, the project could be even more valuable to these individuals if they undertake the project as partners, with individual 1 taking a positive share of the project's risks. For example, if they each took 50% of the net profits from the project, then each individual would anticipate a payment drawn from a Normal distribution with mean 0.50*35000 = $17,500 and standard deviation 0.50*25000 = $12,500. For his 50% share, individual 1 would have certainty equivalent

CE(1) = 17500 − (0.5/20000)*(12500^2) = 17500 − 3906 = $13,594,

For her 50% share, individual 2 would have certainty equivalent

CE(2) = 17500 − (0.5/30000)*(12500^2) = 17500 − 2604 = $14,896.

So the total certainty-equivalent value of the project to the two individuals when they share it equally is

CE(1) + CE(2) = 13594 + 14896 = $28,490

Thus, the project is worth more to them when it is shared equally than when the more risk tolerant individual 2 owns it completely.

Such risk sharing is beneficial because each individual j's risk premium $(0.5/r_j)$*σ^2 is proportional to the square of the standard deviation (the variance) of his or her income. So halving individual 2's share from 100% to 50% would halve the standard deviation of her monetary returns from $25,000 to $12,500, which in turn would reduce her risk premium to a quarter of its former value from $10,417 to $2604. This decrease in individual 2's risk premium from giving up 50% of the project (10417−2604 = 7813) is much greater than the increase in individual 1's risk premium when he takes on 50% of the project (3906−0).

	A	B	C	D	E	F	G	H
1	Suppose profits will be drawn from a Normal distribution							
2	Mean	35000						
3	Stdev	25000						
4								
5	Profits can be shared by individuals 1 and 2							
6	Individ	RiskTol	%Share	Mean	Stdev	CE	RiskPremium	
7	1	20000	0.4	14000	10000	11500	2500	
8	2	30000	0.6	21000	15000	17250	3750	
9								
10		Sum(RTs)	CE(total,sumRTs)			Sum(CEs)	Sum(RPs)	
11		50000	28750			28750	6250	
12	SOLVER(1): Maximize F11 by changing C7.							
13	FORMULAS							
14	C8. =1-C7				B33. =NORMINV(RAND(),B2,B3)			
15	D7. =C7*$B$2				D27. =1-C27		D28. =-C28	
16	E7. =C7*$B$3				C34. =C$28+C$27*$B34			
17	F7. =D7-(0.5/B7)*E7^2				D34. =D$28+D$27*$B34			
18	G7. =D7-F7				C34:D34 copied to C34:D534			
19	D7:G7 copied to D8:G8				C31. =CE(C34:C534,C30)			
20	F11. =SUM(F7:F8)				D31. =CE(D34:D534,D30)			
21	G11. =SUM(G7:G8)				F31. =SUM(C31:D31)			
22	B11. =SUM(B7:B8)				F30. =SUM(C30:D30)			
23	C11. =B2-(0.5/B11)*B3^2				F33. =CE(B34:B534,F30)			
24					F28. =C28+D31			
25	SOLVER(2): Maximize F31 by changing C27:C28.							
26		Individual	1	2				
27		Sharing rate	0.4	0.6		C28 value for CE2=0		
28		Fixed payment	17000	-17000		17475		
29								
30		RiskTols	20000	30000		50000	Sum(RiskTols)	
31		CE	28650	475		29125	Sum(CEs)	
32	(sim'd)	Total$	Net incomes					
33	SimTable	72184.7	Pay 1	Pay 2		29125	CE(total$,sumRTs)	
34	0	59273	40709	18564				
35	0.002	396	17158	-16763				
36	0.004	34138	30655	3483				
37	0.006	48928	36571	12357				
38	0.008	5586	19235	-13648				

Figure 8.1. Sharing a Normal gamble.

The spreadsheet in Figure 8.1 is set up to analyze the effect on the individuals' certainty of other ways of sharing the risks of this project. When we enter individual 1's share of the risks into cell C7, then the expected value and standard deviation of 1's income are calclulated in cells D7 and E7 by the formulas `=C7*$B$2` and `=C7*$B$3`, where cells B2 and B3 contain the mean 35000 and standard deviation 25000 of the project's total profits. Then individual 1's certainty equivalent is calculated in cell F7 by the formula `=D7–(0.5/B7)*E7^2`, where B7 contains individual 1's risk tolerance 20000. Individual 2's share is calculated by `=1-C7` in cell C8, and copying D7:F7 to D8:F8 yields individual 2's certainty equivalent for her share in cell F8. Cell F11 computes the sum of the individuals' certainty equivalents by the formula `=SUM(F7:F8)`.

Now we can use Solver in this spreadsheet to maximize the sum of the computed certainty equivalents in cell F11 by changing individual 1's percentage share of the project in cell C7. The result is that Solver returns the value 0.4 in cell C7, as shown in Figure 8.1. When individual 1 takes a 40% share, his expected monetary value is 0.40*35000 = $14,000 and his standard deviation is 0.40*25000 = $10,000, and so his certainty equivalent is

CE(1) = 14000 – (0.5/20000)*(10000^2) = 14000 – 2500 = $11,500

When individual 2 takes a 60% share, her expected monetary value is 0.60*35000 = $21,000 and her standard deviation is 0.60*25000 = $15,000, and so her certainty equivalent is

CE(2) = 21000 – (0.5/30000)*(15000^2) = 21000 – 3750 = $17,250

When they plan to share the risks in this way, their total certainty equivalent of the project is

CE(1) + CE(2) = 11500 + 17250 = $28,750

This total $28,750 is the maximal sum of certainty equivalents that the partners can achieve by sharing the profits of this project.

In this optimal sharing rule, the ratio of 2's share to 1's share is 0.6/0.4 = 1.5. Notice that the ratio of 2's risk tolerance to 1's risk tolerance is exactly the same 30000/20000 = 1.5. This result is not a coincidence, as the following general fact asserts.

<u>Fact 1.</u> Suppose that a group of individuals have formed a partnership to share the risky profits from some joint venture or gamble, and each individual j in this group has a constant risk tolerance that we may denote by r_j. Let R denote the sum of all the partners' risk tolerances

($R = \sum_j r_j$). Then these individuals can maximize the sum of their certainty equivalents by sharing the risky profits among themselves in proportion to their risk tolerances, with each individual j taking the fractional share r_j/R of the risky profits.

For this example, Fact 1 yields the same optimal shares that Solver returned in Figure 8.1. The sum of the partners' risk tolerances here is

$$R = r_1 + r_2 = 20000+30000 = \$50,000.$$

So the optimal share for individual 1 is 20000/50000 = 0.4, the same share that Solver generated in cell C7.

For any such partnership, we may define the <u>total risk tolerance</u> of a partnership to be the sum of the risk tolerances of the individual partners. For this example, we have seen that the partnership's total risk tolerance is R = $50,000. Now, if we considered the partnership as a corporate person with constant risk tolerance equal to this total R, then a Normal lottery with mean $35,000 and standard deviation $25,000 would have certainty equivalent

$$\mu - (0.5/R)*\sigma\text{^}2 = 35000 - (0.5/50000)*(25000\text{^}2) = \$28,750$$

for this partnership, as is calculated in cell C11 of Figure 8.1. Notice that this corporate certainty equivalent is exactly the same as maximized sum of the partners' individual certainty equivalents in cell F11 under the optimal sharing rule. The following general fact asserts that this result is also not a coincidence.

<u>Fact 2.</u> Consider a group of individuals who have formed a partnership to share the risky profits from some joint venture or gamble, where each individual has constant risk tolerance, as assumed in Fact 1. Let R denote the sum of all the partners' individual risk tolerances ($R = \sum_j r_j$). Then the maximal sum of the partners' certainty equivalents that can be achieved by optimal risk sharing (as described in Fact 1) is equal to the certainty equivalent of the whole gamble to an individual who has a constant risk tolerance equal to the sum of these partners' risk tolerances. Thus, to maximize the sum of their certainty equivalents, the partnership should evaluate gambles according to its total risk tolerance, whenever the partners have a choice about which gambles to undertake.

Facts 1 and 2 here do not require the gamble to be Normal. In illustrating these two facts, we have used the special formula for certainty equivalents of Normal gambles, but the same results can also be obtained with simulation analysis, as shown in the lower portion of Figure 8.1 (row 25 and below). A simulation table here holds a sample of 501 independent simulations of the Normally distributed of profits for this project. We consider sharing rules where each partner gets a fractional share of the profits as listed in cell C27 or D27 plus a fixed payment listed in cell C28 or D28 (for 1 or 2 respectively). A negative payment in D27 represents a payment from individual 2 to individual 1, as when she must buy into a project that was initially owned by individual 1. The fractional shares in C27:D27 must sum to 1, because the partners must share 100% of the profits, and the fixed payments in C28:D28 must sum to 0, because any fixed payment to one partner must come from the other. These constraints are represented in this spreadsheet by the formulas `=1-C27` in cell D27 and `=-C28` in cell D28. Under the sharing rule in C27:D28, the net incomes from the project's simulated profits for individuals 1 and 2 are listed below in cells C34:C534 and D34:D534, and the corresponding certainty equivalents are computed in cells C31 and D31 with the formulas

`=CE(C34:C534,C30)` and `=CE(D34:D534,D30)`

(where C30 and D30 contain the individuals' risk tolerances). The sum of the individuals' certainty equivalents is computed in cell F31.

If we ask Solver to maximize the sum of the individuals' certainty equivalents in cell F31 of Figure 8.1 by changing the sharing-rule parameters in cells C27:C28, then Solver will report that individual 1 should keep a 40% share of the profits (as shown in cell C27) and individual 2 should take the remaining 60% (D27), as Fact 1 predicts. Solver will leave 1's fixed payment in C28 at any arbitrary value, because changing it would not affect the sum of the certainty equivalents in G31. (In making Figure 8.1, I arbitrarily entered 17000 into C28 before running Solver, and Solver left it unchanged.)

Cell F33 in Figure 8.1 calculates the certainty equivalent of the total profits from this project, based on the simulation data in B34:B534, by the formula

`=CE(B34:B534,F30)`

where F30 contains the sum of the partners' risk tolerances R = $50,000. The value in cell F33 ($29,125) is exactly the same as the maximized sum of the partners's individual certainty equivalents in cell F31, as Fact 2 predicts. Cells F33 and F31, being estimates from simulation data that only approximates the given Normal distribution, are slightly different from the values in cells F11 and C11, which use the exact formula for certainty equivalents of Normal gambles. But these simulation estimates also confirm Facts 1 and 2, because these facts do not depend on Normality.

Fact 2 can give us some sense of why businesses are typically more risk tolerant than individuals, because the risks of a business may be shared among many investors. When shares of a company are owned by 50 people whose average risk tolerance is $20,000, then Fact 2 asserts that the company itself should evaluate risks with a risk tolerance of $1,000,000. Fact 1 tells us that, among these 50 people, the ones with greater risk tolerance should have a greater share of the company.

The above discussion assumes that partners should want to maximize the sum of their certainty equivalents. This is a good assumption, but it needs some defense. After all, any single partner may care only about his own certainty equivalent of what he gets from the partnership. Why should anyone care about maximizing this sum of all certainty equivalents? The answer is given by the following fact.

Fact 3. Consider a risk-sharing partnership where all partners have constant risk tolerance. If the partners were planning to share risks according to a sharing rule that does not maximize the sum of the partners' certainty equivalents, then any partner j could propose another sharing rule rule that would increase j's own certainty equivalent and would not decrease the certainty equivalents of any other partners.

To understand Fact 3, notice first that adding any fixed payment from one partner to another partner would not change the sum of the partners' certainty equivalents. A net payment of x dollars from partner 2 to partner 1 (when there is no uncertainty about this amount x) would decrease 2's certainty equivalent by x and would increase 1's certainty equivalent by x, because each partner is assumed to have constant risk tolerance. Thus the net payment of x dollars would leave the sum of their certainty equivalents unchanged.

Now, suppose that the partners were originally planning to use some sharing rule does not maximize the sum of the partners' certainty equivalents. Then consider any other sharing rule that is optimal, in the sense of maximizing the sum of the partners' certainty equivalents. Changing to this "optimal" sharing rule would increase some partners' certainty equivalents, but it might also decrease other partners' certainty equivalents. But let us now modify this optimal rule by adding some net payments that will cancel out these changes for all partners except one, say partner j. Any partner whose certainty equivalent would decrease should receive an additional payment equal to the amount of his decrease, to be paid by this partner j. Any other partner whose certainty equivalent would increase should make an additional payment equal to the amount of his increase, paying it to partner j. So when these payments have been added into the optimal sharing rule, everybody other than partner j is getting exactly the same overall certainty equivalent as under the original plan. But adding these fixed payments does not change the sum of the partners' certainty equivalents. So our modified optimal plan (with the additional payments) still maximizes the sum of the partners' certainty equivalents, and so it must generate a strictly greater sum of certainty equivalents than the original plan. Thus, with everybody else's certainty equivalent unchanged, partner j must be enjoying a strictly greater certainty equivalent under this new plan. This proves Fact 3.

Fact 3 tells us that it is always optimal for partners to maximize the sum of their certainty equivalents. To apply Fact 3, consider our sharing example from the perspective of individual 1, in a situation where the option to buy and develop the land was originally his alone, and so he has the option to undertake the project without any participation from individual 2. Individual 2, of course, has the alternative of not participating in the project, in which case she would get \$0. Any sharing rule that gives 2 a certainty equivalent more than \$0 would be better for her than nonparticipation, and so could be accepted by her. The best possible sharing rule for individual 1 would be one that maximizes 1's certainty equivalent subject to the constraint that 2's certainty equivalent should not be less than \$0. Fact 3 tells us that this can be achieved by sharing in the optimal proportions, to maximize the sum of the individuals' certainty equivalents, with an additional payment from individual 2 to individual 1 that reduces 2's certainty equivalent to \$0 (or to some value slightly greater than \$0). By Fact 1, the optimal share for individual 2 is 60% of this project, because 30000/(20000+30000) = 0.6, and we have seen that a 60% share with no additional payment would have certainty equivalent $17,250 to individual 2 (see cell F8 in Figure 8.1). So the best possible sharing rule for individual 1 would be to sell individual 2 a 60% share of this project for an initial payment of $17,250 (or slightly less than this), which just exhausts 2's perceived gains from participating in the partnership. After selling 60% of the investment to individual 2 for this maximal price, individual 1 would have $17,250 in cash plus a risky investment that is worth $11,500 to him (his certainty equivalent for a 40% share). Thus, selling 60% to individual 2 for $17,250 would make individual 1's overall certainty equivalent from the project 17250 + 11500 = $28,750. This is the most he could possibly hope for in any sharing rule, because it allocates to him all the maximal sum of certainty equivalents that the two partners can get from this project.

Of course, individual 2 would prefer to pay less than $17,250 for a 60% share, and she might try to negotiate for a lower price in this situation. Recall that $19,375 was 1's certainty equivalent for undertaking the project himself, and so 1 would not accept any certainty equivalent less than $19,375 when his alternative is owning 100% of the project himself. Because 1's certainty equivalent for 40% of the project is $11,500, he needs an additional payment of 19375−11500 = $7875 to raise his certainty equivalent to this level. So the best possible sharing rule for individual 2 here would be for her to buy 60% of the project (her optimal share) for just a bit above $7875, which is the lowest price that individual 1 would be willing to accept.

But regardless of who initially owns the project, the partners can can agree that they should maximize the sum of their certainty equivalents by sharing the risky returns in proportion to their risk tolerances. How this maximal value is divided among them is a bargaining problem. If one of them initially owns more than his or her optimal share of the project, there will exist a range of transfer prices at which the individuals could both gain by changing to their optimal shares. In this situation, the price that individual 2 may actually pay to buy 60% of the project must be a question of bargaining between the two individuals, and without a theory of bargaining we can only say here that it should be somewhere between $7875 and $17,250.

Facts 1, 2, and 3 here require the assumption that all partners have constant risk tolerance, but they do not require any assumption about the probability distribution from which the partnership's profits will be drawn. Normality here was only used to compute exact certainty equivalents in the top 11 rows of Figure 8.1.

	A	B	C	D	E	F	G	H	I	J
1	Sharing profits drawn from a Gen-Lognormal distribution									
2		with quartiles:		50	80	120	($1000s)			
3	Total $profits (sim'd)									
4		53								
5				Partners						
6			1	2	3		Sums			
7	Sharing rates		0.3	0.3	0.4		1	Sum(Rates)		
8	Fixed payment		0	0	0		0	Sum(Payments)		
9										
10		RiskTols	30	30	40		100	Sum(RiskTols)		
11		CE	24.959	24.959	33.279		83.198	Sum(CEs)		
12		Total$								
13	SimTabl	53.341	Partners' incomes				83.198	CE(total$s,sumRTs)		
14	0	66.709	20.013	20.013	26.683					
15	0.002	70.729	21.219	21.219	28.291					
16	0.004	19.302	5.7906	5.7906	7.7208	FORMULAS				
17	0.006	97.585	29.276	29.276	39.034	B4. =GENLINV(RAND(),D2,E2,F2)				
18	0.008	68.36	20.508	20.508	27.344	B13. =B4				
19	0.01	141.21	42.363	42.363	56.484	C7. =1-SUM(D7:E7)				
20	0.012	86.192	25.858	25.858	34.477	C8. =-SUM(D8:E8)				
21	0.014	84.812	25.444	25.444	33.925	C14. =C$8+C$7*$B14				
22	0.016	105.66	31.697	31.697	42.262	C14 copied to C14:E514.				
23	0.018	137.95	41.385	41.385	55.18	C11. =CE(C14:C514,C10)				
24	0.02	136.17	40.851	40.851	54.468	C11 copied to C11:E11.				
25	0.022	87.173	26.152	26.152	34.869	G7. =SUM(C7:E7)				
26	0.024	52.619	15.786	15.786	21.048	G7 copied to G8,G10:G11.				
27	0.026	155.76	46.729	46.729	62.305	G13. =CE(B14:B514,G10)				
28	0.028	55.537	16.661	16.661	22.215	SOLVER: maximize G11				
29	0.03	193.94	58.183	58.183	77.577	by changing D7:E8				

Figure 8.2. Optimal risk sharing among three partners with constant risk tolerance.

Figure 8.2 shows an example of optimal linear sharing among three partners where the partnership's profits are generated by a Generalized-Lognormal distribution that is not Normal. A linear sharing rule for partners 2 and 3 is parameterized in cells D7:E8 here, with 1's share being determined in cells C7 and C8 so as to keep the sum of shares equal to 100% and the sum of the fixed payments equal to $0. When Solver is asked to adjust these sharing rules so as to maximize the sum of the partners' certainty equivalents in cell G11, then Solvers' optimal solution should give the partners fractional shares in C7:E7 that are proportional to their respective risk tolerances in cells C10:E10, as predicted by Fact 1. Then as predicted by Fact 2, the maximized sum of the three partners' certainty equivalents in cell G11 is equal to the partnerships' certainty equivalent for the total risky profit which is computed in cell G13, using a risk tolerance for the partnership that is the sum of the partners' individual risk tolerances.

8.2 Optimality of linear rules in the larger class of nonlinear sharing rules

In Figures 8.1 and 8.2, when we asked Solver to find an optimal sharing rule, we implicitly assumed that the two partners would share the profits linearly. Here a partner's share is linear if each extra dollar of profit would increase the partner's income by the same amount. But this linearity assumption is not necessary. Even when we allow that a partner's income may be a nonlinear function of the total profit earned, the linear sharing rule that we described in Fact 1 is still optimal for maximizing the sum of the certainty equivalents among partners who all have constant risk tolerance. (For an illustration of a nonlinear sharing rule, see the inset graph in Figure 8.7 below.)

I want to show you that nonlinear sharing rules cannot do better for the partners in this example, but it is more complicated to evaluate nonlinear sharing rules. Even with profits coming from a Normal random variable, the individuals' incomes will not be Normal with nonlinear sharing rules, and so we cannot use the simple quadratic formula to compute exact certainty equivalents. So we must use a simulation model. Furthermore, seaching among nonlinear sharing rules is much harder, because there are so many nonlinear rules. But Figure 8.3 shows a spreadsheet in which we can evaluate a large set of nonlinear sharing rules and show the optimality of the (40%, 60%) linear sharing rule in this set.

Data from 501 simulations of the partnership's total profit are contained in cells B22:B522 of Figure 8.3. (Note: Rows 25 to 519 have been hidden in Figure 8.3, using the menu command Data:Group.) The smallest simulated profit (−40750) is shown in cell B5, and a value slightly above the largest simulated profit (104222) is shown in cell B15. Cells B6:B14 have been filled with an increasing sequence of values that were chosen (somewhat arbitrarily) between these smallest and largest profits (0, 10000,... , 80000).

	A	B	C	D	E	F	G	H	I
1	Sharing profits drawn from a Normal distribution								
2		with Mean	35000		and Stdev	25000			
3									
4		Total	PayTo1	Slope1	Intercept1				
5		-40750	700	0.4	17000				
6		0	17000	0.4	17000				
7		10000	21000	0.4	17000				
8		20000	25000	0.4	17000				
9		30000	29000	0.4	17000				
10		40000	33000	0.4	17000				
11		50000	37000	0.4	17000				
12		60000	41000	0.4	17000				
13		70000	45000	0.4	17000				
14		80000	49000	0.4	17000				
15		104222	58689						
16					SOLVER: Max F19 by changing C5:C15				
17			Partner1	Partner2					
18		RiskTols	20000	30000		50000	Sum(RiskTols)		
19		CE	28746	618.46		29364	Sum(CEs)		
20	(sim'd)	Total$							
21	SimTabl	24223	PayTo1	PayTo2		CE(total$s,sumRTs)			
22	0	68354	44342	24012		29364			
23	0.002	18508	24403	-5895					
24	0.004	11732	21693	-9961					
520	0.996	34555	30822	3733					
521	0.998	24301	26720	-2420					
522	1	46912	35765	11147					
523									
524	FORMULAS								
525	B21. =NORMINV(RAND(),C2,F2)								
526	B5. =MIN(B22:B522)								
527	B15. =MAX(B22:B522)+1								
528	D5. =(C6-C5)/(B6-B5)				D5 copied to D5:D14				
529	E5. =C5-D5*B5				E5 copied to E5:E14				
530	C22. =VLOOKUP(B22,$B$5:$E$14,4)+B22*VLOOKUP(B22,$B$5:$E$14,3)								
531	D22. =B22-C22								
532	C22:D22 copied to C22:C522								
533	C19. =CE(C22:C522,C18)				D19. =CE(D22:D522,D18)				
534	F18. =SUM(C18:D18)				F19. =SUM(C19:D19)				
535	F22. =CE(B22:B522,F18)								

Figure 8.3. A spreadsheet to evaluate nonlinear sharing rules.

We will consider continuous sharing rules such that 1's income depends linearly on profit inside the interval between each pair of adjacent values in cells B5:B15 of Figure 8.3, but a different linear function may be used in each of these intervals. That is, 1's income could be specified by one linear formula for profits in the interval from −40750 to 0, by another linear formula for profits in the interval from 0 to 10000, and so on. For continuity, we require that the linear formulas on the intervals below and above 0 must specify the same income when profit is $0, with a similar requirement at each of the other interval-separators in B6:B14. Such rules are called piecewise linear. By using more small intervals, we could closely approximate any nonlinear sharing rule by such piecewise linear rules. (For a picture of a piecewise linear function for a similar example, see the inset graph in Figure 8.7.)

Such a piecewise-linear sharing rule can be specified in Figure 8.3 by listing the income that individual 1 would get for each of the profit value listed in cells B5:B15. In this spreadsheet, these incomes for 1 are listed in cells C5:C15. That is, each cell in C5:C15 specifies the income that individual 1 would get, under this profit-sharing rule, if the profit were equal to the corresponding value in B5:B15. For profits in the interval between any adjacent pair of values in B5:B15, we will determine 1's income is by linear interpolation, that is, by applying the linear function that matches the specified income for 1 at each of the two endpoints of the interval. So for any profit x in the interval between B5 and B6, 1's income is supposed to be a linear function of x that has the form A*x+B, where the slope A is computed in cell D5 by the formula

```
=(C6-C5)/(B6-B5)
```

and the intercept B is computed in cell E5 by the formula

```
=C5-D5*B5
```

Cells D5:E5 have been copied down the range D5:E14, to show the corresponding slope and intercepts for the linear sharing rule that is applied in the interval between each value in B5:B14 and the next value below it.

Now to apply this piecewise-linear sharing rule to the simulated profits in cell B22 of Figure 8.3, cell C22 contains the formula

```
=VLOOKUP(B22,$B$5:$E$14,4)+B22*VLOOKUP(B22,$B$5:$E$14,3)
```

The first VLOOKUP in this formula finds the lowest row in the range B5:E15 where the B-cell's value is not greater than B22, and then returns the intercept listed in the E-column (the 4th column of B5:E14) in that row. The second VLOOKUP in this formula returns the corresponding slope in the 3rd column of the B5:E14 table, which is then multiplied by B22 and added to the intercept. Cell D22 computes the corresponding income for individual 2 from the B22 profit value, by the formula

```
=B22-C22
```

Then copying C22:D22 down to C22:D522, we get the incomes for the two partners when the piecewise-linear sharing rule from B5:E14 is applied to the simulated profits in B22:B522. The corresponding certainty equivalents for individuals 1 and 2 are computed in cells C19 and D19, using the CE function, and the sum of these certainty equivalents is computed in cell F19.

Now we can ask Solver to maximize cell F19 by changing cells C5:C15 in Figure 8.3, which specify 1's income values in this piecewise-linear function. Having so many cells to adjust makes this a hard problem for Solver, and it may need to work through forty trial solutions which could take an hour of computing time on an older computer from the 1990s, but which can be done in less than a minute on newer machines in 2002. The result that Solver returns, as shown in Figure 8.3, has the same linear sharing rule applied in all intervals, and the slope of this rule in cells D5:D14 always gives individual 1 his optimal share of risky profits as specified by Fact 1. (If you happen to specify values in C5:C15 that depend on the B5:B15 profits according to a linear formula that has the optimal slope 0.4, then Solver will quickly report that these initial values constitute an optimal solution and will leave them unchanged. The intercept that Solver returns in cells E5:E15 may be any number, and will depend on the initial values that you specified in cells C5:C15 before applying Solver.)

Cell F22 in Figure 8.3 applies the CE function to estimate the value of the total profits of the project (as sampled in B22:B522) to the partnership, when the partnership is treated as a corporate person with a contant risk tolerance equal to the sum of the partners' individual risk tolerances ($50,000, computed in cell F18). Fact 2 tells us that the optimal sum of certainty equivalents in cell F19, after it has been maximized by Solver, must be equal to this value in cell F22, and this equality can be seen in Figure 8.3.

Finally, let us consider in Figure 8.4 a discrete example where two partners with constant risk tolerances of $20,000 and $30,000 have to share a gamble that will pay an amount of money drawn from the following discrete distribution:

Partnership's total profit	Probability
$0	0.2
$25,000	0.3
$50,000	0.4
$75,000	0.1

These profit values and probabilities are shown in cells A6:A9 and B6:B9 of Figure 8.4. Cells C6:C9 are used to specify how much income individual 1 should get from the partnership for each possible amount of profit. The corresponding net incomes for individual 2 are computed in cells D6:D9, by entering the formula `=A6-C6` in cell D6, and then copying D6 to D6:D9. The resulting certainty equivalent for individual 1 is computed in cell C12 by the formula

```
=CEPR(C6:C9,$B$6:$B$9,C2)
```

where cell C2 contains 1's risk tolerance $20,000. (Recall from Section 3.1 in Chapter 3 that CEPR(*values*, *probabilities*, *risktolerance*) returns the certainty equivalent of a discrete gamble where the given values have the given probabilities, for an individual with the given constant risk tolerance.) Copying C12 to D12 yields individual 2's certainty equivalent of her income from this discrete gamble. The CEPR function is also applied in cell A14 to compute the certainty-equivalent value of the whole gamble to an individual whose risk tolerance is the sum of these partners' risk tolerances.

In Figure 8.4, Solver has been asked to maximize 1's certainty equivalent in cell C12 by changing the sharing-rule parameters in cells C6:C9, subject to the constraint that 2's certainty equivalent in cell D12 must satisfy D12>=0. So cells C6:D9 here show the best possible sharing rule for individual 1, subject to the constraint that individual 2 should be willing to stay in the partnership, when 2's alternative is to leave the partnership and get nothing ($0) from this gamble. In this optimal sharing rule, individual 1 gets a fixed payment of $17,857 from individual 2 (1's income in C6 when the gamble pays $0), and then individual 1 gets $0.40 of each dollar that is earned from the gamble (computed in cells E6:E8). Individual 2 gets the remaining $0.60 of each dollar earned from the gamble, and this 60% share is just worth the fixed payment of $17,857 to her. Thus, the partners' optimal shares are linear in profits and are proportional to their risk tolerances, as predicted by Fact 1. Also, as predicted by Fact 2, the optimal sum of the partners' certainty equivalents (in cell D14) is equal to the certainty equivalent of the whole gamble to an individual with the total risk tolerance of the partners (in cell A14).

	A	B	C	D	E	F	G
1			Partner 1	Partner 2			
2	Risk Tolerance		20000	30000			
3							
4	POSSIBLE OUTCOMES						
5	Total $	Proby	PayTo1	PayTo2	Rate1		
6	0	0.2	17857	-17857	0.4		
7	25000	0.3	27858	-2858	0.4		
8	50000	0.4	37857	12143	0.4		
9	75000	0.1	47859	27141			
10							
11	Sum(RTs)		CE(1)	CE(2)			
12	50000		29763	0			
13	CE(total$,sumRTs)			Sum of CEs			
14	29763			29763			
15							
16	SOLVER 1 (no moral hazard):						
17	Max C12 by changing C6:C9 subject to D12>=0.						
18							
19							
20	FORMULAS						
21	D6. =A6-C6			D6 copied to D6:D9			
22	E6. =(C7-C6)/(A7-A6)			E6 copied to E6:E8			
23	C12. =CEPR(C6:C9,$B$6:$B$9,C2)						
24	D12. =CEPR(D6:D9,$B$6:$B$9,D2)						
25	D14. =SUM(C12:D12)						
26	A12. =SUM(C2:D2)			A14. =CEPR(A6:A9,B6:B9,A12)			

Figure 8.4. Optimal risk sharing in a discrete example.

8.3. Risk sharing subject to moral-hazard incentive constraints

In real life, people do not always share every risk in proportion to their individual risk tolerances. One basic reason is that people who are well insured against risks sometimes do not work hard enough to avoid them. This problem is called moral hazard in the insurance industry. To avoid such moral hazard problems, workers and managers in an enterprise are often forced to bear more of the enterprise's risks than they would bear under an ideal risk-sharing system.

To introduce the ideas of moral hazard, let us reconsider in Figure 8.5 an extension of the discrete example from Figure 8.4 in the previous section. In this example, individuals 1 and 2, who have constant risk tolerances $20,000 and $30,000 respectively, are sharing a gamble that will pay a total dollar amount to be drawn from the probability distribution shown in cells A5:B9. Cells C5:D9 in Figure 8.4 show the sharing rule that maximizes 1's certainty equivalent, subject to the constraint that 2's certainty equivalent should be at least $0. This sharing rule is equivalent to 1 selling a 60% share of the gamble to 2 for $17,857.

But suppose now that the distribution of returns listed in cells A5:B9 can be achieved only if individual 1 attends to some managerial duties which individual 2 cannot directly observe. Suppose that, if individual 1 neglected these duties, then there would be no chance of the partnership earning $75,000, and the profit would instead be either $0 or $25,000 or $50,000, each with probability 1/3, as shown in cells G5:H9 of Figure 8.5. So 1's neglect of his duties would reduce the partnership's expected profit from $35,000 to $25,000 (computable here by SUMPRODUCT(A6:A9,B6:B9) and SUMPRODUCT(G6:G9,H6:H9) respectively). But suppose that this neglect of his duties would enable individual 1 to take up another private project that would be worth $6000 to him.

Under the sharing rule that was shown in Figure 8.4, if individual 1 neglected his duties, then his income would be either $17,857 or $27,857 or $37,857, each with probability 1/3, and this gamble would have a certainty equivalent of $26,224 to him (given his constant risk tolerance of $20,000). So when the additional $6000 that he could earn privately is taken into account, neglecting his duties to the partnership would enable individual 1 to get an overall certainty-equivalent value of $32,224, which is better than the certainty-equivalent value of $29,763 that he would get by properly fulfilling his duties to the partnership (shown in cell C12 of Figure 8.4). Thus, under the sharing rule that is shown in Figure 8.4, individual 1 would prefer to neglect his duties. But if 1 neglects his duties then individual 2 should not be willing to pay $17,857 for a 60% share of the profits!

	A	B	C	D	E	F	G	H	I
1			Partner 1	Partner 2			MORAL-HAZARD INCENTIVES		
2	Risk Tolerance		20000	30000			1's private$ if negligent		
3							6000		
4	POSSIBLE OUTCOMES						Probys if 1 is negligent		
5	Total $	Proby	PayTo1	PayTo2	Rate1		Total $	Proby	
6	0	0.2	6823	-6823	0.95		0	0.33333	
7	25000	0.3	30449	-5449	0.54		25000	0.33333	
8	50000	0.4	43956	6044	0.67		50000	0.33333	
9	75000	0.1	60739	14261			75000	0	
10									
11	Sum(RTs)		CE(1)	CE(2)			If 1 neglects his duties		
12	50000		27184	0			his share has CE		
13	CE(total$,sumRTs)			Sum of CEs			21184		
14	29763			27184			CE(1) including private $		
15							27184		
16	SOLVER 1 (no moral hazard):								
17	Max C12 by changing C6:C9 subject to D12>=0.								
18	SOLVER 2 (moral hazard):								
19	Max C12 by changing C6:C9 subject to D12>=0, C12>=G15.								
20	FORMULAS								
21	D6. =A6-C6			D6 copied to D6:D9					
22	E6. =(C7-C6)/(A7-A6)			E6 copied to E6:E8					
23	C12. =CEPR(C6:C9,$B$6:$B$9,C2)								
24	D12. =CEPR(D6:D9,$B$6:$B$9,D2)								
25	D14. =SUM(C12:D12)								
26	A12. =SUM(C2:D2)			A14. =CEPR(A6:A9,B6:B9,A12)					
27	G13. =CEPR(C6:C9,H6:H9,C2)								
28	G15. =G13+G3								

Figure 8.5. Optimal risk sharing with moral hazard, in a discrete example.

To find a sharing rule that avoids this difficulty, we must add a constraint that individual 1 should not prefer to neglect his duties. Such a constraint may be called a <u>moral-hazard incentive constraint</u>, because it says that the risk sharing should not insure 1 so well that he does not want to exert appropriate efforts to avoid bad outcomes. This moral-hazard incentive constraint can be expressed in Figure 8.5 by the inequality C12>=G15, where C12 is 1's certainty equivalent for his share of the partnership when he fulfills his duties (applying the probabilities in B6:B9), and G15 is 1's certainty equivalent for his private income (in G3) plus his share of the partnership when he neglects his duties (applying the probabilities in H6:H9).

Figure 8.5 shows the results when Solver is asked to maximize 1's certainty equivalent in cell C12 by changing the sharing-rule parameters in cells C6:C9, subject to the constraints D12>=0 and C12>=G15. The first constraints here says that individual 2 should not prefer to quit the partnership, and the second constraint says that individual 1 should not prefer to neglect his duties. Under the optimal sharing rule in cells C6:D9 of Figure 8.5, if profit is $0 then individual 1 gets a payment of $6823 from individual 2, as shown in cell C6 of Figure 8.5. This payment is smaller than the corresponding payment in cell C6 of Figure 8.4, but now individual 1 keeps more than half of the partnership's risky profits. As shown in cells E6:E8 of Figure 8.5, individual 1 gets $0.95 from each dollar of the partnership's profit between $0 and $25,000, $0.54 from each dollar of profit between $25,000 and $50,000, and $0.67 from each dollar of profit between $50,000 and $75,000. So individual 1 here holds a larger share of the risks than the ideal share (40%) that Fact 1 would predict, because the Facts in the preceding section assumed that there were no moral-hazard incentive constraints.

Now let us consider a moral-hazard incentive problem in a more realistic example where the profit that an investment may earn is a continuous random variable that has infinitely many possible values. Suppose that a large group of investors are hiring an agent to manage some investment for them. The investors will not be able to directly monitor the manager to see whether he is working or shirking, but they will observe the profit that the investment earns under his management. If the manager works diligently, then this profit will be a Generalized Lognormal random variable with quartile points $230,000, $280,000, and $340,000. But if the manager shirks his responsibilities, then the profit will instead be a Generalized Lognormal random variable with quartile points $190,000, $230,000, and $280,000. Shirking his responsibilities would allow the manager to attend to some personal affairs which would generate private rewards worth $10,000 to him. The manager has constant risk tolerance $20,000. The investors who are hiring this manager have total risk tolerance $480,000. (You may think of these investors as group of 24 partners, each of whom also has constant risk tolerance $20,000.) The manager's alternative employment opportunities would pay him $50,000 during the period when he is being asked to manage this investment, so his certainty equivalent when he agrees to manage this investment cannot be lower than $50,000. Furthermore, no matter how badly the investment turns out, the investors cannot subject the manager to a penalty worse than a $0 wage.

(This last minimum-wage condition may deserve some explanation. It says that the investors cannot ask their agent to put money in an escrow account that would he would forfeit in the event of a particularly bad investment performance. Such punitive conditions may unacceptable because the agent might simply not have the money to put in such an account. Or they may be unacceptable because, if the investors requested such punitive conditions in bad events, then the agent would become suspicious that the investors might actually know something bad about this investment project which would increase the probability of the events where they would profit at his expense.)

Among the feasible compensation plans that would give the manager an incentive to work diligently on managing this investment, let us try to find the plan that would yield the highest sum of certainty equivalents for these investors. This is a difficult optimization problem. So we begin by considering by considering a simpler class of compensation plans in Figure 8.6, and then we can go on to consider a more general class of compensation plans later in Figure 8.7.

The given parameters of the optimal compensation problem are listed in the range A1:C15 of Figure 8.6. The profit quartiles if the manager works diligently are in B3:B5, the profit quartiles if the manager shirks are in C3:C5, the risk tolerances of the manager and the investors are in B9 and C9, the manager's certainty equivalent under his best alternative employment option is in B11, the manager's private gain from shirking is in B13, and the required minimum wage is in B15. (All monetary values in Figure 8.6 are in $1000s.)

Cells B20:B520 contain 501 simulated values of the profit returned by this investment when the manager works diligently, and cells C20:C520 contain the corresponding simulated profit values when the manager shirks. This simulation data was generated by recalculations of the random variables in cells B19 and C19, where we have assumed that the working and shirking profits are maximally correlated by having the same RAND (in cell A18) drive both of these random variables. (Because working and shirking are alternatives that cannot both happen at once, it would not have been wrong to simulate the working and shirking profits by independent random variables. But using correlated random variables here improves the expected accuracy of our estimates from any limited number of simulations, because it reduces the probability that the no-shirking constraint may be distorted by a false contrast between unusually high simulated profits from one alternative and unusually low simulated profits from the other alternative.)

In Figure 8.6, we consider compensation plans where the manager is paid a linear function of the total profit that is returned by the investment, except that the manager can never be paid less than the required minimum base. The slope and intercept of this linear function are entered into cells E12 and E13, and the required minimum base ($0) has been specified in cell B15. (We will ask Solver to choose E12 and E13 optimally, so we may start by putting any arbitrary values in E12 and E13.) Then for our simulated profit data, the manager's corresponding wages when he works diligently can be computed in cells E20:E520 by entering the formula

```
=MAX($E$13+$E$12*B20, $B$15)
```

into cell E20, and copying E20 to E20:E520. The manager's simulated wages from shirking are similarly calculated in by entering the formula

```
=MAX($E$13+$E$12*C20, $B$15)
```

into cell F20, and copying F20 to F20:F520. The remaining profits that will be paid to the investors, in the case where the manager works diligently, are computed by entering the formula

```
=B20-E20
```

into cell G20, and copying G20 to G20:G520. Then the manager's certainty equivalent from working can be estimated in cell E17 by the formula

```
=CE(E20:E520,$B$15)
```

Then the manager's certainty equivalent from shirking (including the private rewards worth listed in cell B13), can be estimated in cell F17 by the formula

```
=CE(F20:F520,$B$15)+B13
```

By Fact 2 (which can be applied to the investors because they have no moral-hazard incentive constraints among themselves), the sum of the investors' certainty equivalents can be computed in cell G17 by the formula

```
=CE(G20:G520,C9)
```

	A	B	C	D	E	F	G	H	I	J	K
1	Profit quartiles										
2		Work	Shirk								
3	Q1	230	190								
4	Q2	280	230								
5	Q3	340	280								
6		(in $1000s)									
7	Risk tolerances										
8		Manager	Investors		SOLVER: Max G17 by changing E12:E13						
9		20	480		subject to E17>=B11, E17>=F17.						
10	Mgr's alternative CE										
11		50			Mgr's compensation plan						
12	Mgr's extra $ if shirks				0.2658	Slope					
13		10			-16.71	Intercept					
14	Mgr's required minimum				(Minimum is paid when profit <					62.843	)
15		0			Mgr's CE		Investors' CE				
16					Work	Shirk	with work				
17	(rand)	Sim'd profit			50	50	226.05				
18	0.5636	Work	Shirk		Mgr's income		Investors' income				
19	SimTabl	293.28	240.89		Work	Shirk	with work				
20	0	423.17	352.41		95.79	76.981	327.38				
21	0.002	324.7	267.06		69.615	54.291	255.09				
22	0.004	312.85	257.13		66.464	51.649	246.39				
518	0.996	282.43	231.99		58.377	44.966	224.06				
519	0.998	227.04	187.68		43.65	33.188	183.39				
520	1	363.59	300.19		79.951	63.098	283.64				
521											
522	FORMULAS										
523	J14. =(B15-E13)/E12										
524	A18. =RAND()										
525	B19. =GENLINV($A$18,B3,B4,B5)										
526	C19. =GENLINV($A$18,C3,C4,C5)										
527	E20. =MAX($E$13+$E$12*B20,$B$15)										
528	F20. =MAX($E$13+$E$12*C20,$B$15)										
529	G20. =B20-E20										
530	E20:G20 copied to E20:G520										
531	E17. =CE(E20:E520,$B$9)										
532	F17. =CE(F20:F520,$B$9)+B13										
533	G17. =CE(G20:G520,C9)										
534	SOLVER: Max G17 by changing E12:E13										
535	subject to E17>=B11, E17>=F17.										

Figure 8.6. Optimal linear incentive plan for an agent with moral hazard.

In Figure 8.6, the investors' optimal linear compensation plan has been found by asking Solver to maximize the investors' total certainty equivalent in cell G17 by changing the compensation parameters in cells E12:E13, subject to the constraints E17>=B11 and E17>=F17.

The first constraint here says that the manager should not prefer his alternative employment opportunities to working for these investors. The second constraint says that the manager should not prefer shirking to working. The results in E12:E13 tell us that the manager should be paid 26.58% of the profits, minus a constant deduction of $16,710, except that he would get the $0 minimum wage if this linear formula yielded an amount less than $0. Cell J14 computes that this minimum wage would apply if profit were less than $62,843, and so in effect the manager is getting a 26.58% share of all profits above $62,843. When the manager works under this optimal linear compensation plan, the investors' total certainty equivalent is $226,050.

Fact 1 does not apply here because of the moral hazard problem. To see why, notice first that Fact 1 would suggest that the manager's share of profits be 20000/(20000+480000) = 0.04, because the manager's risk tolerance is $20,000 and the investors' total risk tolerance is $480,000. With this simulation data, the manager's certainty equivalent from working here could be made equal to the competitive $50,000 certainty equivalent by offering the manager 4% of the profits plus a fixed payment of $38,662, and the resulting total certainty equivalent for the investors would then be increased to $233,450. That is, if we entered 0.04 and 38.662 into cells E12 and E13 of Figure 8.6, then we would get the values 50 in cell E17 and 233.45 in cell G17. But we would also get the value 58.101 in cell F17, which would mean that the manager could increase his own certainty equivalent from $50,000 to $58,101 by shirking. If the manager shirked, then the investors would get only the shirking profits (simulated in C20:C520) minus the corresponding payments to the manager (in F20:F520), which would actually yield a certaintly equivalent of less than $190,000 for the investors. Thus, because of the moral-hazard problem, Solver has recommended instead that the manager should be paid 26.58% of all profits above $62,843, to make sure that he has an incentive to work diligently.

This optimal linear incentive plan is relatively insensitive to different assumptions about the investors' total risk tolerance. If the investors were risk neutral, then the investors' total certainty-equivalent formula in cell G17 in Figure 8.6 would be changed to

```
=AVERAGE(G20:G520)
```

If Solver were asked change E12:E13 so as to maximize this new formula in cell G17, with the same constraints as before, the optimal linear incentive plan would actually remain the same.

Note on Solver: Numerical difficulties may cause Solver to stall at a false solution when different changing cells take values that have very different magnitudes. Such difficulties could occur in an example like Figure 8.6 when the optimal intercept in cell E13 is many times larger than the slope in cell E12. If you find that Solver seems unable to change such cells in a sensible way, then you should try Solver's automatic-scaling option, which is accessed by clicking the Options button in the Solver Parameters dialogue box, and then selecting the "Use Automatic Scaling" option.

8.4 Piecewise-linear sharing rules with moral hazard

Figure 8.7 reconsiders the same problem as Figure 8.6, but it allows nonlinear compensation plans that are piecewise linear on the intervals between adjacent profits in cells E3:E13. The basic parameters of the problem, as listed in the range A1:C15, are the same in Figure 8.7 as in Figure 8.6. But in the range E1:H13 in Figure 8.7, we can construct a nonlinear sharing rule similarly to Figure 8.3.

Cell E3 in Figure 8.7 contains the lowest profit level that is observed under working or shirking in our simulation data in B20:C520, and cell E13 contains a value above the highest profit level in this simulation data. Cells E4:E12 contain a sequence of selected profit values that break the interval between E3 and E13 into ten similar-sized subintervals. A compensation plan can be specified in cells F3:F13 by listing here the manager's wage for each of the corresponding profit values in E3:E13. (We will ask Solver to choose F3:F13 optimally, so we could start by entering any arbitrary initial values in F3:F13.) For profits between any adjacent pair in E3:E13, the manager's wage is computed by a linear interpolation between the corresponding pair of wages in F3:F13. So for profits between E3 and E4, the manager's wage is to be determined by a linear function with the slope and intercept computed in cells G3 and H3 respectively by the formulas

`=(F4-F3)/(E4-E3)` and `=F3-G3*E3`

Then copying G3 and H3 to G3:G12 and H3:H12 yields the slopes and intercepts that are applied in each of the ten subintervals. Thus, cells E3:H12 form a table which lists the low ends of our ten subintervals in the first column (E), and which lists the slope and intercepts of the piecewise-linear wage function for each subinterval in the third and fourth columns (G and H).

So the manager's wage for the simulated profit in cell B20 can be computed in cell E20 of Figure 8.7 by the formula

```
=VLOOKUP(B20,$E$3:$H$12,4)+B20*VLOOKUP(B20,$E$3:$H$12,3)
```

Copying cell E20 to E20:F520, we get the manager's simulated wages from working in cells E20:E250, and we get his simulated wages from shirking in cells F20:F520. Entering the formula

```
=B20-E20
```

in cell G20, and copying G20 to G20:G520, we get in cells G20:G520 the profits that are paid to the investors when the manager works diligently. Then the certainty equivalents that the manager would get from working and from shirking are computed in cells E17 and F17 respectively by the formulas

`=CE(E20:E520,$B$9)` and `=CE(F20:F520,$B$9)+$B$13`

The investors' total certainty equivalent is computed in cell G17 by the formula

```
=CE(G20:G520,C9)
```

In Figure 8.7, the investors' optimal piecewise-linear compensation plan has been found by asking Solver to maximize the investors' total certainty equivalent in cell G17 by changing the compensation parameters in cells F3:F13, subject to the constraints F3>=B15, G3:G12>=0, G3:G12<=1, E17>=F17, and E17>=B11. The constraint G3:G12>=0 asserts that, in each subinterval, the slope of the wage function must not be negative. That is, the manager's wage must never be decreased by an increase in the investment's total profits, because otherwise the manager could have an incentive to incur unnecessary costs simply to reduce profits where this reduction would increase his wage. The constraint G3:G12<=1 says that the manager must never gain more than $1 for an additional $1 of profit, because otherwise the manager could have an incentive to artificially inflate profits by covertly adding some of his own money. The constraint F3>=B15 guarantees that the manager's wage is never below the required minimum wage of $0 in cell B15, because F3 is the manager's wage at the lowest profit, and the nonnegative slopes in G3:G12 imply that the wages cannot be lower at higher profits. The constraint E17>=B11 says that the manager must not prefer his alternative employment opportunities (which pay the amount $50,000 shown in B11) over working to manage this investment. The constraint E17>=F17 says that the manager must not prefer shirking to working here.

With so many variables to adjust and so many constraints to satisfy, this Solver problem can take a long time on older computers, but newer computers in 2002 can do all this in a few minutes. The optimal plan, as shown in Figure 8.7, pays the minimum wage at the lowest possible profit level in our simulation data, but then increases the manager's wage with a share of profits that starts above 50% and declines gradually as profit increases. For profits above $300,000, this Solver solution gives the manager a share of incremental profits that varies somewhat erratically but that averages around 4% over the interval from $300,000 to $685,000. (The wrinkles in the compensation curve for profits above $300,000 could be caused by small random clusters in our simulated profit data, which the Solver may exploit by wrinkling the compensation curve to pay less at the shirking-profit clusters and more at the working-profit clusters. Such effects would tend to disappear if we used a larger simulation sample or wider subintervals.) The investors' total certainty equivalent with this optimal piecewise-linear incentive plan is $229,990, shown in cell G17, which is about $4000 more than the certainty equivalent that they got from their best linear incentive plan in Figure 8.6. (Note: Figure 8.7 actually uses the same simulation data in B20:C520 as Figure 8.6, so this increase is really due to the change of allowing some nonlinearity in the manager's compensation plan.)

	A	B	C	D	E	F	G	H	I	J
1	Profit quartiles				Mgr's compensation plan					
2		Work	Shirk		Profit	PayMgr	Slope	Intercept		
3	Q1	230	190		114.58	0	0.5675	-65.02		
4	Q2	280	230		150	20.102	0.5384	-60.66		
5	Q3	340	280		200	47.021	0.1138	24.253		
6		(in $1000s)			250	52.714	0.1763	8.6494		
7	Risk tolerances				300	61.526	0.0153	56.931		
8		Manager	Investors		350	62.292	0.0437	47		
9		20	480		400	64.477	0.1045	22.662		
10	Mgr's alternative CE				450	69.704	0	69.704		
11		50			500	69.704	0.0678	35.787		
12	Mgr's extra $ if shirks				550	73.095	0.0345	54.131		
13		10			685.35	77.762				
14	Mgr's required minimum									
15		0			Mgr's CE		Investors' CE			
16					Work	Shirk	with work			
17	(rand)	Sim'd profit			50	50	229.99			
18	0.7439	Work	Shirk		Mgr's income		Investors' income			
19	SimTabl	338.15	278.43		Work	Shirk	with work			
20	0	423.17	352.41		66.898	62.398	356.27			
21	0.002	324.7	267.06		61.905	55.721	262.8			
22	0.004	312.85	257.13		61.723	53.969	251.13			
518	0.996	282.43	231.99		58.43	50.663	224			
519	0.998	227.04	187.68		50.099	40.39	176.94			
520	1	363.59	300.19		62.886	61.529	300.7			
521										
522										
523										
524										
525	FORMULAS									
526	A18. =RAND()									
527	B19. =GENLINV($A$18,B3,B4,B5)									
528	C19. =GENLINV($A$18,C3,C4,C5)									
529	E3. =MIN(B20:C520)									
530	E13. =MAX(B20:C520)+1									
531	G3. =(F4-F3)/(E4-E3)									
532	H3. =F3-G3*E3									
533	G3:H3 copied to G3:H12									
534	E20. =VLOOKUP(B20,$E$3:$H$12,4)+B20*VLOOKUP(B20,$E$3:$H$12,3)									
535	F20. =VLOOKUP(C20,$E$3:$H$12,4)+C20*VLOOKUP(C20,$E$3:$H$12,3)									
536	G20. =B20-E20				E20:G20 copied to E20:G520					
537	E17. =CE(E20:E520,$B$9)				F17. =CE(F20:F520,$B$9)+B13					
538	G17. =CE(G20:G520,C9)									
539	SOLVER: Max G17 by changing F3:F13 subject to									
540	F3>=B15, G3:G12>=0, G3:G12<=1, E17>=F17, E17>=B11.									

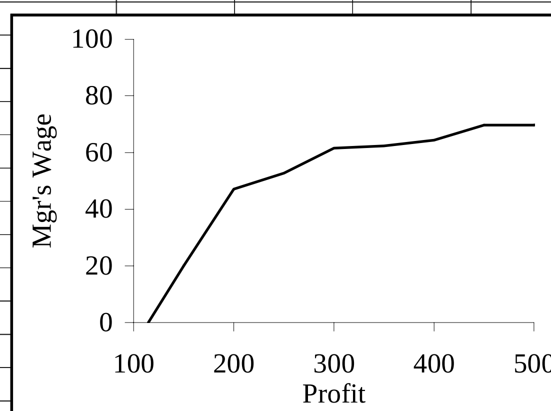

Figure 8.7. Optimal piece-wise linear incentive plan for an agent with moral hazard.

If the investors were risk neutral, or if there were so many investors that the sum of their risk tolerances was too large to measure, then the investors' total certainty equivalent would be their expected share of profits, which could be estimated in cell G17 by the formula

```
=AVERAGE(G20:G520)
```

If we ask Solver to maximize this measure of the investors' expected profit by changing the same variables with the same constraints as we used in Figure 8.7 (using the same simulation data as used in Figures 8.6 and 8.7), then the optimal compensation plan (in $1000s) would become:

Profit	Pay Manager	Slope	Intercept
114.58	0	0.474	-54.323
150	16.795	0.711	-89.821
200	52.334	0.04	44.396
250	54.319	0.133	20.963
300	60.990	0	60.990
350	60.990	0	60.990
400	60.990	0.046	42.446
450	63.308	0	63.308
500	63.308	0	63.308
550	63.308	0.003	61.931
685.35	63.646		

The main difference between this compensation plan and the one in Figure 8.7 is that the manager here gets almost no wage increases for profit increases above $300,000. So the positive wage increases for profit increases above $300,000 that we see in cells G7:G12 of Figure 8.7 are due mainly to the advantages of adding the manager as another partner in risk sharing. Most of the required incentives for working instead of shirking are generated in Figure 8.7 by the big pay increases that the manager can get from increasing profits up to $300,000.

The optimality of the nonlinear incentive plan in Figure 8.7 may depend, however, on our implicit assumption that the manager must decide once whether to work or shirk, and only thereafter can he learn how much profit will be earned. But in a real world that is more complex than our simple model, the profits from such an investment project might actually be earned over a period of weeks or months, and the manager might be able to decide about working or shirking every day, knowing how much profit has already been earned so far. If our optimal incentive plan from Figure 8.7 were applied in such a world, then the manager might work diligently only until the accumulated profits reach \$300,000, and thereafter he might start shirking, which would significantly reduce the investors' chances of earning any higher profits above \$300,000. To avoid this dynamic moral-hazard problem, we may recommend that the investors should restrict their attention to linear compensation plans, as assumed in Figure 8.6, because the incentives to continue working under linear compensation plans would be less affected by good news in the early weeks of the project.

8.5. Corporate decision-making and asset pricing in the stock market

The preceding two sections provide the basis for understanding optimal decision-making under uncertainty in big publicly held corporations. Actually, we should have two theories of decision analysis in corporations, depending on whether the decision analysis is supposed to be for the benefit of senior managers who control the corporation, or for the benefit of the stockholders who legally own the corporation. The latter assumption is the main focus of this section, but let us first devote one paragraph to the former assumption.

Under a well-designed compensation plan, as we saw in Sections 8.3 and 8.4, senior managers may anticipate that their personal rewards will depend substantially on the profits that they generate for the corporation, because they should have strong incentives to increase these profits. But then corporate risks will generate substantial personal risks for senior managers. So a senior manager may want to evaluate corporate risks using a risk-averse corporate utility function similar to those we have studied for individual decision-making. If F(**X**) denotes the senior manager's income when **X** is the corporation's total profit (under the manager's compensation plan), and if U denotes the manager's personal utility function for income, then the manager should personally prefer that corporate decisions under uncertainty should maximize E(U(F(**X**))). In effect, the manager would want the corporation to evaluate risky profits by the expected value of a utility function V such that V(x) = U(F(x)) for any amount of corporate profits x. But as agents of the corporation, senior managers are not supposed to define the corporation's interests so blatantly in terms of their own, and so decision analysts rarely use such an approach to assessing a corporate utility function. Nevetheless, the sensitivity of a corporation to risks cannot be well understood unless we recognize that corporate risks typically generate substantial personal risks for the risk-averse senior managers who are responsible for making corporate decisions. In this context, decision analysts have found that senior managers may be comfortable with an expected-utility analysis of corporate risks, using risk tolerances for corporate profits that are sometimes about 1/6 of the total value of the corporation.

The perspective of the stockholders on corporation decision-making is often very different. Individual stockholders may be very risk averse, but a single corporation's risks may have only a very small impact on the overall net worth of a typical well-diversified individual stockholder. From the perspective of such well-diversified stockholders, the goal of corporate decision-making should be to adopt strategies that, when generally understood by investors, will increase the market value of the corporations's stock. To apply this criterion, we need some theory to predict how prices in the stock market are determined. So we will discuss here the elements of such a theory, which is based on a simplified idealized version of the stock market, but which can offer real practical insights into the pricing of capital assets.

The most important fact about the stock market is that the future prices of stocks and other financial assets are unknown quantities. So any individuals' beliefs about a collection of asset prices at some future date should be described by a joint probability distribution. In our financial theory, we assume that all investors' beliefs about future prices at some future date (say, a year from now) can be described by some joint probability distribution. That is, we assume that everybody shares the same beliefs which can be measured by a joint probability distribution for these unknown quantities. The end result of our asset-pricing theory will be to show how to compute what an asset should be worth now in the stock market, given such a joint probability distribution of future asset prices.

Let us assume here that investors in the stock market have constant risk tolerance, so that we can apply what we have already learned about optimal risk-sharing among such individuals. The market portfolio is the collection of all the risky assets that are available to be bought and sold in the stock market (or in other financial markets). All the risks in this market portfolio must be shared in some way among the investors, and we may think of the entire stock market as a risk-sharing partnership that includes all investors. So Fact 1 tells us that, when the stock market achieves an optimal allocation of risks, each investor should have a share in the whole market portfolio of risky assets that is proportional to his or her risk tolerance. If any investor owned more or less of some asset than his proportional share then, by Fact 3, investors could make mutually beneficial trades of money and risky assets until an optimal allocation of risks is achieved. So in a market equilibrium, the prices of all financial assets should be such that an investor with constant risk tolerance is willing to hold the same share of every risky asset in the market portfolio, and the only difference among investors should be that their shares differ in proportion to their risk tolerances. Thus, all investors could simply buy shares in well-diversified mutual funds that hold all the stocks in the market portfolio. But every corporation's stock is a part of this overall market portfolio. So the price of a stock today must be such that, when an investor has bought his optimal share of the market portfolio, then he does not want to buy or sell any additional amounts of this stock now.

To understand all these complex ideas of asset pricing, let us apply them to a specific example, as illustrated in Figures 8.8 and 8.9.

	A	B	C	D	E	F	G	H	I	J
1	M = $Returns next year to $1 now in the Market Portfolio									
2	X,Y = $Values (per share) of Assets 1 and 2 next year									
3		M	X	Y	SOLVER: max F13 by changing F11.					
4	LogMean	0.09	20	20	Q1					
5	LogStdev	0.25	25	25	Q2					
6			32	32	Q3	What if prices now are:				
7		Correl(M,X)						for X		for Y
8		0.8						25		25
9	1+interest		RiskTol		Invest now			...invest more?		
10	1.06		10000			$ in M		$ in X	or	$ in Y
11	Y is independent of M and X					9703		1		1
12	corands	0.9703	0.9772			CE		CE change		
13		M	X	Y		332.713		-0.0377		0.04125
14	SimTable	1.753	54.96	23.66		Income				
15	0	1.024	17.74	23.53		-348.09		-348.45		-348.21
16	0.002	1.091	30.98	41.16		299.817		299.996		300.403
17	0.004	1.097	20.80	48.35		362.924		362.696		363.798
18	0.006	1.189	23.84	18.74		1252		1251.9		1251.69
19	0.008	1.419	32.84	24.72		3483.57		3483.82		3483.5
511	0.992	1.508	49.79	19.99		4348.74		4349.67		4348.48
512	0.994	0.689	19.60	36.04		-3597.1		-3597.4		-3596.7
513	0.996	1.005	29.54	25.08		-537.17		-537.05		-537.22
514	0.998	0.912	18.92	19.57		-1434.4		-1434.7		-1434.7
515	1	1.110	32.08	22.38		486.686		486.909		486.521
516										
517	FORMULAS									
518	B12:C12. {=CORAND(B8)}									
519	B14. =EXP(NORMINV(B12,B4,B5))									
520	C14. =GENLINV(C12,C4,C5,C6)									
521	D14. =GENLINV(RAND(),D4,D5,D6)									
522	F15. =$F$11*(B15-$A$10)									
523	F15 copied to F15:F515									
524	F13. =CE(F15:F515,C10)									
525	H15. =F15+$H$11*(C15/$H$8-$A$10)									
526	J15. =F15+$J$11*(D15/$J$8-$A$10)									
527	H15:J15 copied to H15:J515									
528	H13. =CE(H15:H515,$C$10)-$F$13									
529	J13. =CE(J15:J515,$C$10)-$F$13									

Figure 8.8. Comparing investments that differ in correlation with the market portfolio.

In this example, we consider probability information about values next year of the market portfolio and a couple of selected assets in it. We assume that a dollar invested today in the market portfolio (that is, in a well-diversified mutual fund that holds this market portfolio on behalf of its investors) will be worth an amount next year that has a Lognormal probability distribution with log-mean 0.09 and log-standard-deviation 0.25. That is, the logarithmic growth rate of an investment in the market portfolio has expected value 0.09 and standard deviation 0.25. We also assume that investors can borrow or lend at a risk-free interest rate such that $1 today returns $1.06 next year, as indicated in cell A10 of Figure 8.8.

Cells B15:B515 of Figure 8.8 contain 501 simulated values of the returns next year per dollar now in the market portfolio, generated by a random variable in cell B14 that has this Lognormal distribution. (Notice that rows 20 to 510 are hidden in Figure 8.8.) Now consider an individual who has constant risk tolerance $10,000, as indicated in cell C10, suppose that cell F11 denotes the amount of money that this individual chooses to invest now in the market portfolio. Let us account the results of this individual's investments in the stock market in terms of gains or losses next year relative to the wealth that he would have next year if he depositing all his wealth now into a risk-free bank account that pays $1.06 next year for each $1 invested now. Relative to this safe strategy, each dollar invested in the market portfolio costs him $1.06 next year (regardless of whether he takes the dollar from his bank account or he borrows the money, given our simplifying assumption that he can both borrow and lend at the same risk-free rate). So if B15 were the market portfolio's growth ratio over the next year, then his net gain next year from investing F11 dollars now in the market portfolio can be computed in cell F15 by the formula

```
=$F$11*(B15-$A$10)
```

Copying cell F15 to F15:F515 yields the investor's net gains next year from investing F11 dollars now in the market portfolio, for all of our simulated market outcomes. The certainty equivalent of these gains is computed in cell F13 by the formula

```
=CE(F15:F515,C10)
```

Now we can ask Solver to maximize the investor's certainty equivalent in cell F13 by changing his investment in cell F11, and the resulting optimal solution is as shown in Figure 8.8.

That is, his optimal strategy (with this simulation data) is to invest $9703 in the market portfolio now, which will yield a net gamble that is as good as the certainty equivalent gain of $332.71 (relative to what he could get next year by putting all his wealth now in the bank). If changed the risk tolerance and ran Solver again, then the optimal investment in cell F11 would change in proportion to the risk tolerance. For example, if we doubled the risk tolerance to $20,000 into cell C10 and ran Solver again, then Solver would maximize the certainty equivalent F13 by doubling the investment amount to $19,406 in cell F11.

The optimal investment quantity for each investor in these calculations depends on the risk-free interest rate that banks are assumed to pay (in cell A10). If this interest rate were lower, then the net gains from putting money in the stock market would be greater, and the optimal investment in the risky market portfolio would increase for any individual investor. In a market equilibrium, the risk-free interest rate has been determined by the condition that the sum of all individuals' investments in shares of the market portfolio must equal the whole portfolio of stocks and other financial assets available in the market.

So in a market equilibrium, all financial assets must be priced now in such a way that, when individual investors solve the optimal investment problem that we have just described, they buy different shares of the general market portfolio, and no one wants to buy or sell any further shares of any financial asset. This no-further-trade condition can be used to characterize the prices must be for financial assets in a market equilibrium.

Two other financial assets are considered in Figure 8.8. You should think of these as being just two among the very large number of stocks in the stock market that make up the market portfolio. Let **X** denote the price per share of the first stock next year, and let **Y** denote the price per share of the second stock next year. To be specific, we suppose here that these two stock prices next year have the same marginal probability distribution which is a Generalized Lognormal distribution with quartile points $20, $25, and $32. But suppose that the first stock's future price **X** has a correlation 0.8 with the growth ratio of the market portfolio, while the second stock's price **Y** is independent of the market portfolio. These two random variables are simulated in cells C14 and D14 of Figure 8.8, where the array formula `{=CORAND(0.8)}` in cells B12:C12 provides the specified correlation between the simulated **X** in cell C14 and the simulated growth of the market portfolio (**M**) in cell B14. The simulated **Y** in cell D14 here depends independently on its own RAND.

Column H analyzes the effects of making another small investment now in the first stock, in addition to the individual's F11 optimal investment in the market portfolio. The effect of such an additional investment would depend, of course, on the price that the investor has to pay now for a share of this stock. Our goal in this exercise is to compute what this price should be (based on the probability information about **X**), but let us begin here by entering in cell H8 some arbitrary guess about what this current stock price might be. Next, let us specify in cell H11 any small amount of money that the investor might consider adding now to his investment in this stock. Figure 8.8 considers the possibility of investing an additional $1 (H11) to this stock when its price per share now is $25 (H8). For the first outcome of our simulation model, stored in row 15 of the simulation table, this additional investment would change the investor's net gains next year from the amount in cell F15 to

```
=F15+$H$11*(C15/$H$8-$A$10)
```

which has been entered into cell H15. To understand this formula, notice that each of the H11 (1) additional dollars that he invests now buys 1/H8 (1/25) shares of this stock at the current price, which will yield next year C15/H8 dollars in this simulated market outcome, but investing the dollar now also reduces the individual's bank account by A10 (1.06) dollars next year. Then copying the formula from H15 to cells H15:H515, we get the net gains next year with this additional investment for all the outcomes in our simulation table. So the change of the investor's certainty equivalent next year that would be generated by this additional investment in the first stock is computed in cell H15 by the formula

```
=CE(H15:H515,$C$10)-$F$13
```

A similar calculation is done for the second stock in column J of Figure 8.8. Here cell J8 contains our guess about the current price per share of this stock, and cell J11 contains an amount of money that we are thinking of investing in this stock now, as an addition to the optimal investment in the market portfolio. Then, by copying cells H13:H515 to J13:J515, we similarly compute in cell J13 the change of the investor's certainty equivalent next year that would be generated by this additional investment in the second stock. (These additional investments are being considered here as alternative possibilities, and so the simulated gains in J15:J515 include the J11 additional investment in the second stock but do not include the H11 additional investment in the first stock.)

In Figure 8.8, the computed changes of certainty equivalent in cells H13 and J13 are −0.0377 and 0.04125 respectively. This means that, if the prices of these stocks were both $25 now (as assumed in cells H8 and J8), then investing an additional dollar in the first stock would decrease the investor's certainty equivalent next year by almost 4 cents, but investing an additional dollar in the second stock would increase the investor's certainty equivalant next year by a bit more than 4 cents. But remember, in a market equilibrium, every investor should hold an optimal share in the whole market portfolio that is proportional to his risk tolerance, and should not want to add any other disproportionate investment in particular stocks. So if the positive value of cell J13 indicates that this investor could do strictly better by adding another $1 (J11) in the second stock here when its current price is as specified in cell J8 ($25) , then the assumed price in cell J8 must be too low. So in an equilibrium of the stock market now, the price per share of the second stock must be strictly more than $25, because investors like this one would want to buy more of this stock for $25 now.

The negative value of cell H13 in Figure 8.8 indicates that this investor could not improve his portfolio by adding another $1 (H11) in the first stock. But what about selling some of the first stock? Remember, every stock is a small part of the market portfolio, and so his optimal investment in the market portfolio must implicitly include some small amount of money in this stock, which (in principle) he could sell separately now. Selling the first stock can be represented in this spreadsheet by entering a negative number in cell H11. For example, if we entered the value −1 into cell H11, to represent the investor selling $1's worth of the first stock now, then the calculations in cell H13 would show him increasing his certainty equivalent next year by $0.0377. (His increase of CE from selling $1's worth would be almost the same size as his decrease of CE from buying $1's worth.) This result indicates that, in an equilibrium of the stock market now, the price per share of the first stock must be strictly less than $25, because investors like this one would want to sell their shares of this stock for $25 now.

So the current price per share of the first stock should be less than $25 and the current price per share of the second stock should be more than $25, even though investors anticipate that the prices per share of these stocks next year have the same marginal probability distribribution. The only difference between these stocks is that the first (**X**) has a positive correlation with the market portfolio, while the second (**Y**) is independent and so has zero correlation with the market portfolio. This observation illustrates an important general fact: A higher correlation with the market portfolio tends to decrease the current value of the asset (when the marginal probability distribution of its value at some future date is held fixed).

Intuitively, the second stock is worth more than the first stock now because the second stock can offer better insurance against the risks in the market portfolio that the investors are holding. The investors, being risk averse, would get more extra utility from an extra dollar of income next year in an event where their overall portfolio has done badly than in an event when their overall portfolio has done well. But the first stock, being highly correlated with the market portfolio, tends to pay its highest returns in the same events where the market portfolio does well. Both of these stocks have the same probability of selling for more than $32 next year (0.25), but there is a much smaller probability of the first stock being so profitable when the overall market portfolio does badly, because of its positive correlation with the market portfolio.

Now let me tell you how you could use the spreadsheet in Figure 8.8 to compute the current price of the first stock. Remember, cell F11 contains the optimal investment in the market portfolio that Solver generated to maximize the certainty equivalent in cell F13. Keeping this value in cell F11, you could now ask Solver to change the additional investment in cell H11 to maximize the additional certainty equivalent that it generates in cell H13. If Solver returns a positive investment in cell H11, then the price for the first stock that you assumed in cell H8 is too low. If Solver returns a negative investment in cell H11 (indicating that investors would want to sell this stock at this price), then the price for the first stock that you assumed in cell H8 is too high. When the right current price for the first stock is entered into cell H8, Solver should report that the additional certainty equivalent in H13 is maximized by setting the additional investment in H11 equal to $0. For the simulation data in this spreadsheet, the correct current price of the first stock turns out to be $24.11. (Results may differ slightly with other simulation data.)

Similarly, you can find the current equilibrium price for the second stock in Figure 8.8 by adjusting the price in cell J8 so that, when Solver is asked to maximize J13 by changing J11, Solver will reports an optimal additional investment of $0 in cell J11. For the simulation data in this spreadsheet, the correct current price of the second stock turns out to be $25.97. The difference between this $25.97 for the second stock and the price $24.11 for the first stock is caused by the difference between these stocks' correlations with the market portfolio.

Running Solver so many times may be awkward, so Figure 8.9 shows a way to directly calculate the current price of a stock from its distribution of future returns. To simplify the spreadsheet, we only consider the first stock (the one that will be worth **X** next year), but the simulation model and data in columns B and C of Figure 8.9 are the same as in Figure 8.8. The interest rate in A10 and the investor's risk tolerance in C10 are also the same in both Figures. Column E of Figure 8.9 repeats the analysis of the individual's optimal investment in the market portfolio that we already saw in column F of Figure 8.8. That is, cell E11 contains the individual's monetary investment now in the market portfolio, cells E15:E515 contain his resulting monetary gains next year from this investment (compared to the alternative of putting all his wealth in bank accounts) with the simulated market outcomes, and cell E13 computes his certainty equivalent for this gamble with the formula

```
=CE(E15:E515,C10)
```

The investment amount shown in E11 ($9703) is the result of asking Solver to maximize E13 by changing E11.

In Column F in Figure 8.9, you can find a review of the calculations by which the CE function computes the certainty equivalent in cell E13. First, the utility values of the monetary incomes in cells E15:E515 are computed, by entering the formula

```
=UTIL(E15,$C$10)
```

into cell F15, and then copying F15 to F15:F515. Then these utility values are averaged, and the average utility is converted back to the equivalent monetary value by UINV by the formula

```
=UINV(AVERAGE(F15:F515),C10)
```

in cell F13. The result in cell F13 is the same as the certainty equivalent value that is computed by the CE function in cell E13.

	A	B	C	D	E	F	G	H	I	J	K
1	M = $Returns next year to $1 now in the Market Portfolio										
2	X = $Value (per share) of Asset next year										
3		M	X			SOLVER(1): max E13 by changing E11.					
4	LogMean	0.09	20	Q1		SOLVER(2): max I13 by changing I11.					
5	LogStdev	0.25	25	Q2			(Should find I11=0 optimal!)				
6			32	Q3							
7		Correl(M,X)				Value of Asset now = E'(X)/(1+i)					
8		0.8							24.11		
9	1+interest		RiskTol		Invest now				Invest more?		
10	1.06		10000		$ in M				$ in X		
11					9703				0		
12	corands	0.5563	0.4176		CE	UINV(EU)			CE change		
13		M	X		332.71	332.71			0		
14	SimTable	1.134	23.27		Income	Util	Weights		Income		
15	0	1.024	17.74		-348.1	-1.035	0.0021		-348.1		
16	0.002	1.091	30.98		299.82	-0.97	0.0020		299.82		
17	0.004	1.097	20.80		362.92	-0.964	0.0020		362.92		
18	0.006	1.189	23.84		1252	-0.882	0.0018		1252		
19	0.008	1.419	32.84		3483.6	-0.706	0.0015		3483.6		
511	0.992	1.508	49.79		4348.7	-0.647	0.0013		4348.7		
512	0.994	0.689	19.60		-3597	-1.433	0.0030		-3597		
513	0.996	1.005	29.54		-537.2	-1.055	0.0022		-537.2		
514	0.998	0.912	18.92		-1434	-1.154	0.0024		-1434		
515	1	1.110	32.08		486.69	-0.952	0.0020		486.69		
516											
517	FORMULAS										
518	B12:C12. {=CORAND(B8)}								E'(X)		
519	B14. =EXP(NORMINV(B12,B4,B5))								25.56		
520	C14. =GENLINV(C12,C4,C5,C6)										
521	E15. =$E$11*(B15-$A$10)								E(X)		
522	E15 copied down								27.66		
523	E13. =CE(E15:E515,C10)										
524	F15. =UTIL(E15,$C$10)										
525	F15 copied down										
526	F13. =UINV(AVERAGE(F15:F515),C10)										
527	G15. =F15/SUM($F$15:$F$515)										
528	G15 copied down										
529	I8. =SUMPRODUCT(C15:C515,G15:G515)/A10										
530	I15. =E15+$I$11*(C15/$I$8-$A$10)										
531	I15 copied down										
532	I13. =CE(I15:I515,C10)-E13										
533	I519. =SUMPRODUCT(C15:C515,G15:G515)										
534	I522. =AVERAGE(C15:C515)										

Figure 8.9. Computing an asset's value in a market with constant risk-tolerant investors.

But now let us do something else with these utility values in F15:F515. Recall from Chapter 4 that the constant risk-tolerance utility functions are defined by the following mathematical formula

UTIL(x, r) = −EXP(−x/r),

for any monetary amount x and any positive risk tolerance r. This exponential formula has the special property that, as the amount of money x is varied with a fixed risk tolerance, the slope of the utility function is proportional to the absolute value of the utility function itself. If you look back at the graph in Figure 3.1 in Chapter 3, you can see that the slope of a constant risk-tolerance utility curve is proportionally steeper in regions where the utility value is farther below zero. Thus, with these special exponential utility functions, the investor's utility benefit from a small increase in income should be greater in the bad outcomes where his utility is farther below zero, in direct proportion to the distance of his utility below zero. So investor's relative sensitivity to income changes in the various simulated outcomes here may be measured by the weights shown in G15:G515 of Figure 8.9, which are computed entering by the formula

```
=F15/SUM($F$15:$F$515)
```

into cell G15 and then copying G15 to G15:G515. The utility values in F15:F515 are all negative numbers, but then their sum is also negative, and so these weights in G15:G515 are all positive numbers. (A negative number divided by a negative number yields a positive number.) Furthermore, the sum of these weights in G15:G515 is equal to 1. In effect, these weights look like a probability distribution over the various outcomes in our simulation table, where bad outcomes are given more probability than the good outcomes, in proportion to the slopes of his utility curve at these different outcomes. For example, row 511 in Figure 8.9 gets a small weight 0.0013 in cell G11, because this row represents a good outcome where the market portfolio goes up and the investor gains $4349 (E11). But row 512 gets a relatively large weight 0.0030 in cell G12, because this row represents a bad outcome where the market portfolio goes down and the investor loses $3597, which would make him much more sensitive to additional income than he would be in the good outcome.

Until now, we have always analyzed simulation data as if every outcome in every row of our simulation table represented an equally likely outcome from our probability model. But now let us instead pretend that the outcomes in the various rows have instead the probabilities shown in G15:G515, which we may call the market-adjusted probabilities. Let us use E' to denote the expected value of a random variable when we apply these mysterious market-adjusted probabilities. With these market-adjusted probabilities, the expected price-per-share of our selected stock next year would be computed by the formula

$$E'(\mathbf{X}) = \text{SUMPRODUCT(C15:C515, G15:G515)},$$

because cells C15:C515 contain the simulation data for this stock's future value **X**. With the simulation data, this market-adjusted expected value is E'(**X**) = $25.56. For comparison, if we treated all rows as equally likely and conventionally estimated the expected value E(**X**) by AVERAGE(C15:C515), then we would get E(**X**) = $27.66 with this simulation data. The market-adjusted expected value E'(**X**) is smaller than E(**X**) here because **X** is positively correlated with the market portfolio **M**, and so the high values of **X** tend to come in rows where the market portfolio does well and thus where the market-adjusted probability is relatively small.

The price **X** is a future value of this stock, and so it is denominated in future dollars, a year from today. But such future dollars can be exchanged for dollars today by borrowing or lending at a bank, at the ratio defined by the bank's interest rate, here 1+i = 1.06 as shown in cell A10 in Figure 8.9. So to convert future monetary values into current monetary values here, we should divide by A10. In particular, the current monetary equivalent of the market-adjusted expected value of the price per share of this stock next year is

$$E'(\mathbf{X})/(1+i) = \text{SUMPRODUCT(C15:C515, G15:G515)}/\text{A10}$$

This important formula

```
=SUMPRODUCT(C15:C515,G15:G515)/A10
```

has been entered into cell I8, and it is the answer to our question of what should be the equilibrium price of this stock today.

When today's price per share of this stock is as computed in cell I9 of Figure 8.9, the investor will not want to buy or sell any additional amounts of this stock now, once he has made his optimal (E11) investment in the market portfolio. To demonstrate this result, cell I13 in Figure 8.9 computes the net change in this investor's certainty equivalent next year if he were to invest in this stock today the additional amount of money that is entered in cell I11, given the current price per share in cell I9. If we asked Solver to maximize this certainty equivalent in I13 by changing the investment amount in I11 (holding fixed the previously optimized market-portfolio investment in E11), then Solver cannot do better than by letting the investment amount in I11 be $0. So the formula in cell I9 gives us the correct equilibrium price today for this stock that will have the price **X** next year, because individual investors will not want to change their optimal investment in the market portfolio by buying or selling this stock at this price today.

The price that we have computed in cell I9 of Figure 8.9 does not depend on the particular risk tolerance of $10,000 that we assumed in cell C10.   For example, suppose we doubled the risk tolerance by entering $20,000 into cell C10, and we then asked Solver again to maximize the new certainty equivalent in cell E13 by changing cell E11. Then the new optimal market-portfolio investment in cell E11 would double, and all the net incomes in cells E15:E515 would double, relative to the amounts shown in Figure 8.9. But then the utility values in cells F15:F515 would remain the same as shown in Figure 8.9, because UTIL(x,t) = −EXP(−x/r) depends only on the ratio of the income x to the risk tolerance r, and so doubling them both would leave the utility unchanged. So the market-adjusted probabilities in cells G15:G515 (which depend only on F15:F515) and the computed stock price in cell I9 (which depends only on G15:G515 and the simulation data and the interest rate) would also be unchanged. Thus, the market-adjusted probabilities and the computed stock price in Figure 8.9 would not be affected by a change of the risk tolerance in cell C10, once Solver has changed the optimal investment in E11 to maximize the new certainty equivalent in E13. Our results here depend only on the assumption that investors are risk averse and have some constant risk tolerance.

This financial asset-pricing model has important applications to corporate decision-making. To develop these applications, we just need to extend our interpretation of the random variable **X** in cell C14.

Suppose that a big publicly-held corporation is considering a major new investment project which would cost 25 $million now. For simplicity, suppose that all the monetary returns from this project would be realized one year from now, and that these returns next year would have a Generalized Lognormal probability distribution with quartile points 20, 25, and 32 $million. Suppose also that these returns would have a correlation 0.8 with the growth ratio of the market portfolio next year. Thus, the returns from this project next year would have the same marginal probability distribution and the same relationship with the market portfolio as the random variable **X** that we considered in Figures 8.8 and 8.9. So let us now reinterpret **X** as this unknown quantity that the project will earn next year.

Should this project be recommended now as being to the benefit of the corporation's stockholders? This is a decision under uncertainty, because the payoff **X** next year is an unknown quantity that could be much greater or much less than the cost of the project. But any current stockholder in this corporation could sell his shares tomorrow, and so all stockholders can benefit from decisions that will increase the current market value of the corporations's stock. So we should ask whether a decision to undertake this project would tend to increase or decrease the value of this corporation in the stock market now (once the decision becomes widely known to investors).

According to our analysis in Figure 8.9, a stock coupon (or a lottery ticket) that will be worth **X** dollars next year should have a value of $24.11 in the stock market today. When this corporation invests in this project that will pay **X** million dollars next year, it is essentially buying a million such coupons. So the corporations investment in this project should be worth 24.11 $million in the stock market today. That is, the prospect of earning **X** $million next year from this project should add 24.11 $million to the total value of the corporation's stock. But undertaking this project requires an expense of 25 $million today. This expense would make the corporation increase its debt (or decrease its liquid assets) by 25 $million, which should have the effect of subtracting 25 $million now from the total market value of the corporation. So according to our analysis, a decision to undertake this project should change the current total value of this corporation by 24.11−25 = −0.89 $million, as soon as the decision becomes understood by investors in the stock market. Thus, the project should not be recommended on behalf of the stockholders, because it would tend now to decrease the value of their shares in the corporation.

This conclusion depends critically on our assumption that the project's future earnings have correlation 0.8 with the future performance of the market portfolio. If the project's future earning were considered instead to be independent of the future performance of the market portfolio, then the project's future earnings would have a current value that is greater than 25 $million, because it would be like the gamble that pays **Y** in Figure 8.8. In general, the value of a project or risky investment to investors in the stock market cannot be determined without analyzing how the project's risks are related to the broader macroeconomic risks that investors bear in the market portfolio. To do such analysis, we must use a model of the joint probability distribution of the investment's returns **X** and the market portfolio **M**, as in cells B14:C14 of Figure 8.9.

(Financial analysts often estimate asset values using the capital asset pricing model, or CAPM, which is similar to the constant risk-tolerance model that we have developed here. CAPM is based on an assumption that investors care only about the mean and standard deviation of their portfolio's value, and so it yields an asset-pricing formula which depends only on the asset's expected value and covariance of returns with the market portfolio. When future asset returns are drawn from a Multivariate Normal distribution, the CAPM model is equivalent to the model that we have developed here. Even for the nonNormal data in Figure 8.9, a CAPM analysis would yield a stock price differing by only $0.05 from the value that we computed in cell I9. But CAPM can yield nonsensical results for some extremely skewed nonNormal distributions that are avoided by the method that we have developed here.)

*8.6. Fundamental ideas of arbitrage pricing theory

In arbitrage pricing theory, we assume that there is some list of possible states of the world such that exactly one of these states will occur and the future values of financial assets will be determined by the state that occurs. We would say that an arbitrage opportunity existed if there were some portfolio of loans and assets that an individual could acquire with zero initial investment of his own but which would guarantee him a positive future return in all states. That is, an arbitrage opportunity would be an opportunity to get something for nothing without any risks. Arbitrage pricing theory characterizes financial asset prices when such arbitrage opportunities do not exist.

Let us begin by considering a simple two-state example in which State 1 is a state of the world with high oil prices and State 2 is a state of the world with low oil prices. Suppose that the value of some automotive company's stock will be $80 per share next year if State 1 occurs, but it will be $140 if State 2 occurs. Similarly, suppose that the future value of some oil company's stock will be $140 per share next year if State 1 occurs, but it will be $80 if State 2 occurs. Suppose also that investors can borrow and lend at a 10% annual interest rate, so that 1.1 is the annual return ratio for risk-free bonds. If these two stocks are currently selling for $90 per share, then an arbitrage opportunity exists. For each $180 that we borrow, we could buy one share of each of these two stocks now, and then next year (after selling the shares and repaying the debt) we would could take a sure profit of (80+140)−180*1.10 = $22 in all possible states. By borrowing more money we could make as much money as we like, with no risk to ourselves.

If the current prices for these two stocks are each $100 per share, however, then such arbitrage opportunities would not exist. At this price, if we assessed a probability 0.5 for each of the two states, then the expected returns next year per dollar invested now would be (.5*80+.5*140)/100 = $1.10, which exactly equals the cost next year of borrowing a dollar now. If every financial asset that we can buy or sell has an expected return of 10% then, no matter how we mix investments and debts, the expected value of our portfolio next year will be 10% more than its value this year. In particular, if we start with no initial investment of our own funds, then the expected total value of our portfolio must be $0 (=0*1.10) next year, and so there cannot exist any arbitrage strategy that offers positive returns in all states with zero net initial investment.

In general, if there exists some way of assigning probabilities to the various possible states such that every financial asset offers the same expected return ratio as risk-free bonds, then arbitrage opportunities cannot exist. With each investment offering the same expected return ratio as the risk-free bonds, no portfolio can offer a higher (or lower) expected return ratio than this risk-free return ratio on the net initial investment. But an arbitrage strategy which guarantees a positive return with $0 initial investment would be offering an infinite expected return ratio. So there cannot exist any arbitrage strategy for generating positive returns in all states with zero net initial investment.

Conversely, if arbitrage opportunities do not exist in a financial market then, there must exist some way of assigning probabilities to the possible states such that, when we compute expected returns using this probability distribution, the expected return ratio of every financial asset is equal to the risk-free return ratio. This important fact is the main result of arbitrage pricing theory.

To illustrate this result, consider the example shown in Figure 8.10, which describes a simple imaginary financial market in which three stocks are traded, and the return ratios for these stocks over the next year will depend on which of four possible states of the world occurs. The table in cells B5:E7 lists the return ratio for each stock in each possible state of the world. The return ratio for risk-free bonds is 1.10, which is listed in cell A1.

Given this financial data, Figure 8.10 shows how to use Solver to find an arbitrage opportunity if one exists. Cells G5:G7 represent the money to be invested in each of the three stocks now, in our investment strategy. If any of these cells becomes negative, it can be interpreted as the amount of money to be raised now by selling the corresponding stock short. The net investment in these stocks is assumed to come out of bonds that pay the risk-free interest rate of 10% per year, and so the formula

```
=SUMPRODUCT(B5:B7,$G$5:$G$7)-SUM($G$5:$G$7)*$A$1
```

in cell B9 represents the net returns from our investments if State 1 occurs. Copying cell B9 to B9:E9 gives us cells representing the net returns from our investments in each of the four possible states. In cell G10, we enter a "goal" of returns that our investment strategy will try to achieve in all states. The shortfall from this goal in each state is computed in cells B11:E11, by entering the formula

```
=$G$10-B9
```

in cell B11, and then copying B11 to B11:E11. Thus, a positive shortfall in cells B11:E11 denotes a failure to achieve the goal in some state. With this spreadsheet formulation, we can now ask Solver to find the highest goal that can be achieved in all states by an investment strategy.

	A	B	C	D	E	F	G	H
1	1.1	Risk-free return ratio ($ next year per $1 invested now)						
2								
3	$Returns next year per $1 invested now							
4		State 1	State 2	State 3	State 4		Invest now	
5	Stock A	0.95	0.90	1.30	1.20		0	
6	Stock B	0.95	1.35	0.85	1.00		0	
7	Stock C	1.30	1.10	1.15	1.00		0	
8								
9	Net $return	0	0	0	0		Goal	
10							0	
11	Shortfall	0	0	0	0	(+ is bad!)		
12								
13	SOLVER (with Options:AssumeLinearModel):							
14	Maximize G10 by changing G5:G7,G10 subject to B11:E11<=0.							
15	Select SensitivityReport when Solver finishes.							
16								
17		State 1	State 2	State 3	State 4			
18	ShadowProby	0.01504	0.43609	0.34586	0.20301			
19		ShadowE($Returns)						
20	Stock A	1.1						
21	Stock B	1.1						
22	Stock C	1.1						
23								
24	FORMULAS FROM RANGE A1:G22							
25	B9. =SUMPRODUCT(B5:B7,$G$5:$G$7)-SUM($G$5:$G$7)*$A$1							
26	B9 copied to B9:E9							
27	B11. =$G$10-B9							
28	Bll copied to B11:E11							
29	B20. =SUMPRODUCT($B$18:$E$18,B5:E5)							
30	B20 copied to B20:B22							
31	Shadow probabilities (or Lagrange Multipliers) in B18:E18 are copied							
32	(with paste-special,transpose) from Solver's SensitivityReport.							
33								
34	Fact: If arbitrage opportunities do not exist, then Solver must							
35	terminate with 0 optimal value in cell G10, and the Shadow Prices							
36	(or Lagrange Multipliers) in the Sensitivity Report will give us a							
37	shadow probability distribution over the states such that all assets							
38	have the same shadow-expected return ratio as risk-free bonds.							

Figure 8.10. A simple example of arbitrage pricing theory.

In the Solver dialogue box, let us tell Solver to maximize the target cell G10 by changing cells G5:G7,G10 subject to the constraints B11:E11<= 0. (It is important to include G10 among the changing cells as shown. Solver will accept multiple ranges separated by commas in the changing-cells box.) To take advantage of the special linear structure of this problem, which will enable Solver to analyze this problem more accurately, let us also go to the Solver "Options" dialogue box and check the "Assume Linear Model" option (then "OK"). Then we can go back to the basic Solver dialogue box and click the "Solve" button. When Solver finishes and announces that it has found a solution, we should also select the "Sensitivity" option in the "Reports" box, which causes Solver to add a Sensitivity-Report page in our workbook.

For this example, Solver will report the value 0 in cell G10 as the maximum return that can be guaranteed with no net investment. (Solver may report a value of G10 slightly different form 0, such as 1.1E−06, which denotes 1.1*10^(−6) = 0.0000011, but this tiny deviation from 0 is just due to roundoff error.) Getting this Solver output tells us that an arbitrage strategy for guaranteeing a positive return in all states with no net investment does not exist. Thus, the main result of arbitrage pricing theory tells us that there must exist some probability distribution that makes each stock have an expected return ratio equal to the return ratio of risk-free bonds. But where can we find this probability distribution?

The answer is in the Sensitivity Report that Solver added to the workbook, when we selected the Sensitivity option when Solver finished the optimization. Solver's Sensitivity Report, as shown in Figure 8.11, includes a lot of numbers, but the only important numbers for our purposes are those listed for the constraints under the heading Shadow Price or Lagrange Multiplier. (Solver uses the term "Shadow Price" when the "Assume Linear Model" option has been selected, and it uses the equivalent term "Lagrange Multiplier" otherwise.) In general, the shadow price of a constraint is a measure of the rate at which the optimal value of the target cell would increase if we started increasing the value on the right-hand side of the constraint. But Solver's shadow prices have a special interpretation here in this problem of searching for arbitrage opportunities. This problem includes one constraint for each of the possible states, and the shadow prices of these constraints are nonnegative numbers that sum to 1. So the shadow prices generated by Solver for this problem can be interpreted as a shadow probability distribution over the set of possible states.

This shadow probability distribution from the sensitivity report has been copied (and pasted-special transposed) to the range B18:E18 in Figure 8.10. Then the formula

```
=SUMPRODUCT($B$18:$E$18,B5:E5)
```

in cell B20 computes the expected return ratio for Stock A under this shadow probability distribution. Copying this formula to cells B20:B22, we find that all three stocks have shadow-expected return ratios equal to the return ratio on risk-free bonds in this market.

Microsoft Excel Sensitivity Report

Changing Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$G$5	Stock A Invest now	0	0	0	0.05227	0.00370
$G$6	Stock B Invest now	0	0	0	0.08846	0.00455
$G$7	Stock C Invest now	0	0	0	0.05610	0.02500
$G$10	Goal	0	0	1	1E+30	1

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$11	Shortfall State 1	0	0.01504	0	1E+30	1E+30
$C$11	Shortfall State 2	0	0.43609	0	1E+30	1E+30
$D$11	Shortfall State 3	0	0.34586	0	1E+30	1E+30
$E$11	Shortfall State 4	0	0.20301	0	1E+30	1E+30

Figure 8.11. Solver Sensitivity Report for the problem in Figure 8.10.

This result is completely general. If you change the returns listed in cells B5:E7 of Figure 8.10, or the risk-free return ratio listed in cell A1, then one of two cases will hold. Case 1 is that the Solver will find an optimal solution with 0 as the best goal that can be guaranteed in cell G10, in which case the shadow prices in the Sensitivity Report will form a shadow probability distribution with which all stocks offer the same expected return ratio as risk-free bonds. Case 2 is that Solver may report "the set cell values do not converge," which means that arbitrage opportunities are possible and returns that exceed any positive goal can be guaranteed. (If you want to see what these arbitrage strategies look like, you can add the constraint G10<=1000, which tells Solver to try to stop when it finds a way to guarantee returns higher than $1000 with no initial investment. To get an example where such arbitrage opportunities exist, you can change the value of cell B5 in Figure 8.10 from 0.95 to 1.19 or higher, leaving all other parameters the same.)

Now look back at our asset-pricing model in Figure 8.9. The current value of the asset, as computed in cell I9, is an expected value of the asset next year divided by the return ratio for risk-free bonds, but where the expected value is computed using the market-adjusted probabilities in cells G15:G515. So when these market-adjusted probabilities are used to compute the asset's expected return ratio (its future values in C15:C515 divided by its current value in I9), the resulting market-adjusted expected return ratio is exactly the risk-free return ratio in cell A10. Thus, the market-adjusted probabilities in cells G15:G515 of Figure 8.9 are the shadow probabilities that make this asset-pricing model a special case of arbitrage pricing theory.

But Figure 8.9 also teaches us more about how such market-adjusted shadow probabilities may differ from traditional probabilities (which are defined by relative frequency, and so would be all be equal to 1/501 in the analysis of 501 simulation outcomes). The difference is that higher market-adjusted probabilities are assigned to the bad outcomes where the market portfolio does poorly, and lower market-adjusted probabilities are assigned to the good outcomes where the market portfolio does well. This over-weighting of bad outcomes occurs because risk-averse investors should be more sensitive to income changes in the states where their overall portfolio has done badly.

8.7. Summary

Individuals with constant risk tolerance can maximize the sum of their certainty equivalents by sharing risks linearly in proportion to their individual risk tolerances. If the members of an investment partnership were not using such an optimal sharing rule, then one partner could propose such an optimal sharing rule, together with some fixed payments among the partners, with the result that he does better and no one else does worse (in terms of their certainty equivalents). When partners share a risky gamble in this optimal way, the sum of their certainty equivalents of their separate shares is the same as what the certainty equivalent of the whole gamble would be to a single individual whose risk tolerance was the sum of the partners' risk tolerances. In this sense, a partnership should evaluate gambles and choose among them just as if it were a single person with a risk tolerance equal to the sum of the partners' individual risk tolerances.

This simple theory of sharing risks in proportion to risk tolerances is based on an assumption that the way that profits are divided among the partners does not affect the probability distribution of total profits earned. But this assumption may fail, and so this simple theory of optimal risk sharing may not apply, when the partnership's profits depend on the efforts of an individual whose work cannot be perfectly monitored by the other partners. His temptation to shirk instead of work is an example of moral hazard problems that can be caused by systems of risk sharing and insurance. In such cases, this individual may have to bear a larger share of the risks, to give him more motivation for diligent efforts on behalf of the partnership. We learned how to compute optimal linear and nonlinear sharing rules that satisfy the moral-hazard incentive constraints for such an agent.

We then moved from the study of partnerships to publicly held corporations with a stock market that includes many investors with constant risk tolerance. Among two financial assets that have the same marginal probability distribution of values next year, the one that has lower correlation with the overall market portfolio should have a higher value now, because it offers better insurance against the risks of the diversified market portfolio that investors hold. We developed a spreadsheet model for computing an asset's current value in the stock market from the joint distribution of the future values of this asset and the market portfolio. This model yields asset values such that, when expected values are computed using some market-adjusted probabilities for the possible outcomes, the expected growth ratio of the every asset's value is equal to the return ratio on risk-free bonds or bank accounts. This market-adjusted probability distribution puts relatively more weight on bad outcomes where the market portfolio does poorly, because risk-averse investors are relatively more sensitive to income changes in such bad outcomes. The property of all assets having equal expected return ratios in terms of some shadow probability distribution is a general characteristic of any asset-pricing system that does not create arbitrage opportunities.

EXERCISES

1. G. Washington owns a real-estate development which will pay returns drawn from a Normal distribution, with expected value $50,000, and standard deviation $12,000. Washington has constant risk tolerance, and his risk-tolerance index is $8000. J. Madison's risk-tolerance index is $5000. (The development does not require any significant unobservable effort from Washington himself.)

(a) What is Washington's certainty equivalent for the real-estate development, if he keeps the development for himself?

(b) Washington is planning to sell Madison a share in the development. To maximize their total certainty-equivalent values of their shares, what share should Washington sell to Madison? What is the minimum price that Washington should accept for selling this share? What is the maximum price that Madison would be willing to pay for this share?

(c) B. Arnold owns a real-estate development up river that will pay returns drawn from a Normal distribution with expected value $64,000 and standard deviation $20,000. Arnold has offered to exchange his up-river development for Washington's development. Washington expects to take Madison on as a partner in his real-estate deals in any case. Should Washington accept Arnold's offer to exchange developments? Explain your answer.

2. Case: The Wilson Estate

Daniel Wilson and Rebecca Wilson Tisler have just inherited their mother's real estate holdings, which consist of scattered commercial properties, some of which have substantial mortgages. Without their mother to manage the properties, they plan to sell the properties and split the returns equally.

A sudden offer by P. J. Cooney to buy all of their mother's properties for $140,000 over the cost of repaying all mortages has caused disagreement among the brother and sister, however. Daniel is eager to accept the offer, but Rebecca feels strongly that they should try to do better by selling the various properties separately over the coming year. You have been asked to help resolve their conflict.

After lengthy discussions with the Daniel and Rebecca, you have found that they essentially agree about their prospective risks if they turn down Cooney's offer. Appraisals of the properties by independent real-estate agents have suggest that the expected returns from selling the properties separately is $200,000 over the cost of repaying all mortages, although their actual realized returns could be substantially above or below that amount. So both Daniel and Rebecca agree that a Normal distribution with this mean of $200,000 and a standard deviation of $75,000 can accurately describe their beliefs about their combined returns from selling the properties separately after rejecting Cooney's offer.

Where the siblings differ is in their risk tolerance, however. Daniel, with children about to enter college, is substantially less risk-tolerant than Rebecca. By asking them about their willingness to take hypothetical risks, you have assessed Daniel's risk-tolerance index to be $20,000, and you have assessed Rebecca's risk-tolerance index to be $45,000. It seems reasonable to assume that they each have constant risk tolerance.

(a) What is Daniel's certainty equivalent for a 50% share of the returns from selling the properties after rejecting Cooney's offer? What is Rebecca's certainty equivalent for a 50% share of the returns from selling the properties after rejecting Cooney's offer?

(b) Consider an alternative plan in which Rebecca pays her brother some amount of money now to buy a larger share of the returns from the properties. To increase the sum of their certainty equivalents as much as possible, what share should Rebecca take? How much money would she have to pay Daniel now to buy this increased share, so that his certainty equivalent for his remaining share plus her payment to him should be equal to $70,000 (the amount that he would get if they sold to Cooney now and divided the money equally)?

(c) If Rebecca paid her brother and increased her share of the returns from selling the properties as you described in part (b), then what would be her certainty equivalent for her share of the returns minus her payment to her brother?

(d) Make a chart showing the (inverse) cumulative distribution of the net returns (including the payment to or from the other sibling) that Daniel and Rebecca will each get under your plan from part (b).

<u>3.</u> If the manager is diligent, then the gross profits that our company will earn from a new project will be a random variable drawn from a Normal distribution with mean $900,000 and standard deviation $200,000. But if the manager shirks then these profits will be reduced by 20%. We cannot observe whether the manager is diligent or shirking, but we will be able to observe the gross profits that are generated. The manager has constant risk tolerance $50,000. Shirking would be worth an additional $40,000 in compensation to the manager. The manager's pay may depend on the gross profit earned, but this pay cannot be less than $80,000 in any case, and the manager would quit now if the certainty equivalent of his compensation plan was less than $200,000. We want to maximize our company's expected net profit, after subtracting the amount that we pay the manager.

(a) Let us consider linear compensation plans, adjusted to the minimum wage $80,000 where necessary. That is, suppose that the wage that we pay the manager will be some fixed constant plus a fixed fraction of the gross profits, or $80,000, whichever is larger. Find the compensation plan of this linear form that maximizes our expected net profit. Under this plan, what is the highest gross profit for which the manager gets only $80,000? What fraction of profits over this

amount do we pay to the manager?
(b) If we consider nonlinear compensation plans, how much can you increase our expected net profit?
(c) Suppose instead that we compensate the manager in such a way that he does not quit but he shirks. What is the highest expected net profit that our company can earn from this project with a shirking manager?
(d) How would your optimal linear compensation plan in part (a) change if shirking reduced gross profits by 25%, but everything else is the same?

4. Consider again the Part C of the Superior Semiconductor case, as described at the beginning of Chapter 4 and analyzed in Section 4.6. Let us assume, for simplicity, that all costs and revenues of this T-regulator product will accrue 2 years from now, but the interest rate on risk-free bonds is essentially zero. The value in 2 years of $1 that is invested now in the well-diversified market portfolio will be a Generalized Lognormal random variable with quartile boundary points $0.80, $1.10, and $1.50. Suppose that, in the T-regulator project, the development costs, the event of successful development, and the number of entering competitors are believed to be independent of the returns to the market portfolio. But the total value of the market for the T-regulator product in this case will be dependent on the same macroeconomic forces that will determine the returns to the market portfolio in the stock market. To be specific, suppose that Superior Semiconductor's business-marketing manager says that, if she were told that the stock market portfolio would decrease to $0.80 in 2 years per dollar invested now, then she would revise her median value of the whole T-regulator market to $85 million.
(a) Assuming that investors in the stock market understood all the facts described above and in Part C of the Superior Semiconductor case, would you recommend that Superior Semiconductor develop the new T-regulator product? How would the total value of Superior Semiconductor's stock change if the company announced that it was developing the T-regulator device?
(b) How would your answer change if the business-marketing manager instead assessed a conditional median of $90 million for value of the T-regulator market given the stock market portfolio decreasing to $0.80 in 2 years?