Designing an Optimal Stock Portfolio

Due: Mon. Apr. 24

We first need some terminology. The *rate of return* of a stock is the percentage increase in its value over some time period. So, if a stock has price P_i at the beginning of a time period and price P_f at the end, then the rate of return for that time period is:

$$\text{Rate of return } = \frac{P_f - P_i}{P_i}$$

Suppose you monitor some stocks' prices every Friday in the newspaper for several weeks. Then you could determine a weekly rate of return for each of the weeks in question.

If you did that for long enough, you could estimate the *average* (weekly) *rate of return*—let's call it ρ_i for stock i. A useful indicator of *risk* is the *variance* of the rates of return—a statistical measurement of how much those weekly rates of return fluctuate from the average ρ_i. Let's call the variances s_i; if you want to know how to compute them given your data from the Friday papers, see any statistics book, but it's not necessary for this project. If $\rho_4 = 0.01$ and $s_4 = 0.005$, then you might think of your expected rate of return from stock 4 as something like 1 % ± 0.5 %.

OK, now to the problem. A basic philosophy of investing is that splitting one's investments into several opportunities is a better strategy than investing in only one opportunity. This diversification strategy tends to protect us from the fluctutations of any given stock. Finding the optimal balance of investments is the focus of this project. The variables we will use are w_i, the fraction of our total wealth we invest in stock i.

Basic statistics determines the overall ρ and s of our portfolio from the individual ρ_i and s_i:

$$\rho = w_1\rho_1 + w_2\rho_2 + \cdots + w_n\rho_n$$
$$s = w_1^2 s_1 + w_2^2 s_2 + \cdots + w_n^2 s_n$$

We need one final item for our model: a measurement of how much risk we are willing to take for a given average rate of return. It is obviously hard to accurately model such a psychological effect, but here's our model. We'll say that our happiness with a portfolio is given by

$$Happiness = \rho - as$$

where a is a constant called a risk-aversion factor. The bigger a is, the less risk we like.

Assume we have three stocks to invest in. How should we divide our wealth to make the best portfolio? To start out, let's assume it's OK for any of the w's to be negative – that may sound silly, but it represents an actual practice in the market called holding a "short position" in a stock. [I'm no expert, but I understand it to mean that you accept payment from someone equal to the current price of the stock, and agree that when that person wants to sell the stock in the future, you will buy it at the current market price (so it's like you're selling instead of buying).]

(1) Find the optimal portfolio assuming first that all the s_i's are greater than zero.

(2) Find the optimal portfolio assuming that $s_1 = 0$ (a totally safe investment, like a CD or a government bond) and the rest of the s_i's are greater than zero.

(3) Using what you've done above, how should we invest if $\rho_1 = 0.05$, $\rho_2 = 0.10$, $\rho_3 = 0.18$, $s_1 = 0.03$, $s_2 = 0.04$, $s_3 = 0.10$? Using Mathematica, or any plotting program, plot w_1, w_2, and w_3 versus a for $1 \le a \le 10$. Comment on what you observe.

(4) For some values of a, you should have found the optimal strategy involved a negative w_i, holding a "short position" as described above. Pick such a value of a, and now suppose that the option of holding a short position is not available to you. Re-solve the optimal portfolio problem under the assumption that all the w_i's must be positive.

(5) Repeat question 3 using the values $\rho_1 = 0.05$, $\rho_2 = 0.05$, $\rho_3 = 0.18$, $s_1 = 0.03$, $s_2 = 0.04$, $s_3 = 0.10$. Note that stock 2 has the same average rate of return as stock 1 but is riskier, a clearly inferior stock. Do you ever end up investing in stock 2? Comment on that.

(6) Finally, repeat question 3 using the values $\rho_1 = 0.05$, $\rho_2 = 0.10$, $\rho_3 = 0.18$, $s_1 = 0$, $s_2 = 0.04$, $s_3 = 0.10$. How does the involvement of a safe investment change the structure of the portfolio?

If you want, you can head to the library and use actual stock data for the ρ_i and s_i instead of my made-up data. You can replace the data in question 3 with your data and answer the same questions—also, if you want to use more than 3 stocks, you could probably do so pretty easily by just reworking your question 1 derivation slightly. Leave the data in question 5 and 6 as it is, though, so you can investigate the issues involved in those questions.