Written Exam in Probability Theory, 7.5 ECTS credits
Thursday, 29th of February 2024, 14:00 – 19:00

Examination: On-campus Exam

You are asked to answer below stated questions and motivate your solutions. Grades are assigned as follows: **A** (91+), **B** (75-90), **C** (66-74), **D** (58-65), **E** (50-57), **Fx** (30-49), and **F** (0-29)

You are **allowed** to use calculator. Other supplementary material is attached to your exam questions.

The teacher reserves the right to further examine the students on the answers provided.

1. (12 points) Let the distribution of *Y* conditional on $X = x$ be $N(x, x^2)$ $[Y \mid X = x] \sim N(x, x^2)$ and the marginal distribution of *X* be $U(0,2)$. Calculate $E[Y]$, $Var(Y)$ and $Cov(X, Y)$

2. (10 points) Let $f(x,y) = \begin{cases} x + y, & 0 < x < 1, \quad 0 < y < 1 \\ 0, & otherwise \end{cases}$
   a) Find the probability density function of $X+Y$
   b) Find the median and the mean of the above distribution

3. (12 points) *A* and *B* are hiking and agree to meet at a certain spot between 18:00 and 20:00 on a particular day. Let us suppose they arrive at the meeting place independently and randomly and that both make it during agreed time interval. Find the distribution of the length of time that *A* waits for *B*. (If B arrives before A, define A's waiting time as zero).

4. (12 points) Let $f(x,y) = \begin{cases} c * xy(1 - x^2), & 0 < x < 1, \ 0 < y < 1 \\ 0, & otherwise \end{cases}$

   When you have found the right constant "c" for the above function to be a bivariate pdf, find a probability density function of $X*Y$.

5. (12 points) Let the joint pdf of (X,Y) be $f(x,y)=1,\ 0<y<1,\ y<x<y+1$
   a) Find pdf's of $2X$ and Y explicitly and calculate their means and variances
   b) Further, find $Corr(2X,Y)$

6. (12 points) One tosses two dice: the outcomes are the numbers from 1 to 6. Let *X* be the minimum and *Y* the max of the numbers that one sees on these two dice. Find the joint distribution of (X,Y) and calculate $E[X]$, $E[Y]$, $Var(X)$, $Var(Y)$, and $Corr(X,Y)$. Discuss the value and the sign of the correlation and interpret it in your own words.

7. (10 points)

a) Check that the below stated function is a pdf and find the moment generating function corresponding to $f(x) = \frac{1}{2\beta}\exp\{\frac{-|x-a|}{\beta}\}, -\infty < x, \alpha < \infty, \beta > 0$. Make sure to provide detailed explanations.

b) Derive a moment generating function of random variable *X:= Gamma(α, β)* as defined in appendix. Use derived mgf to calculate *Var[X]*

8. (10 points) Let $-\infty < \mu_X < \infty, -\infty < \mu_Y < \infty, 0 < \sigma_X, 0 < \sigma_Y$, and $-1 < \rho < 1$ be five real numbers. The bivariate normal pdf with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2 and correlation ρ is the bivariate pdf given by

$$f(x,y) = \left(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right)^{-1} \times \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)\right)$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$.

Although the formula for the bivariate normal pdf looks formidable, this bivariate distribution is one of the most frequently used. (In fact, the derivation of the formula need not be formidable at all. See Exercise 4.46.)

The many nice properties of this distribution include these:

a. The marginal distribution of X is $n(\mu_X, \sigma_X^2)$.
b. The marginal distribution of Y is $n(\mu_Y, \sigma_Y^2)$.
c. The correlation between X and Y is $\rho_{XY} = \rho$.

Your task is to prove/derive the properties a)-c) given the formula of the bivariate pdf above. Note that it is not expected that student gives complete derivation for the full points in this part. One can miss few intermediate steps but explain them well.

9. (10 points) Let us assume that the sequence of random variables X_n converges in distribution to a constant *c*. Show that it also converges in probability to the same constant *c*. In other words, convergence in probability and convergence in distribution are equivalent in this particular case. (Hint: start with writing the limiting distribution explicitly as a cdf).

Good Luck