Written Exam in Probability Theory, 7.5 ECTS credits
Tuesday, 07th of March 2023, 08:00 – 13:00

Examination: On-campus Exam

You are asked to answer below stated questions as well as motivate your solutions. Grades are assigned as follows: **A** (91+), **B** (75-90), **C** (66-74), **D** (58-65), **E** (50-57), **Fx** (30-49), and **F** (0-29)

You are **allowed** to use any calculator. Other supplementary material is attached to your exam questions.

The teacher reserves the right to further examine the students on the answers provided.

1. (12 points) *A* and *B* are hiking and agree to meet at a certain place on a certain day (12 hours). Let us suppose they arrive at the meeting place independently and randomly during these 12 hours. Find the distribution of the length of time that *A* waits for *B*. (If B arrives before A, define A's waiting time as zero)

2. (12 points) Let $f(x,y) = \begin{cases} x+y, & 0<x<1, \quad 0<y<1 \\ 0, & otherwise \end{cases}$
   a) Find the probability density function of $X+Y$
   b) Calculate $P(X+Y>1)$

3. (15 points) Let the distribution of Y conditional on $X=x$ be $N(x, x^2)$ $[Y \mid X=x] \sim N(x,x^2)$ and the marginal distribution of X be $U(0,3)$. Find $E[Y]$, $Var(Y)$ and $Cov(X, Y)$.

4. (15 points) Show that the following function is a joint pdf and calculate

$$f(x_1,x_2,x_3,x_4) = \begin{cases} \frac{3}{4}(x_1^2+x_2^2+x_3^2+x_4^2), & 0<x_i<1, \ i=1,2,3,4 \\ 0, & otherwise \end{cases}$$

   a) $P(X_1<\frac{1}{2}, X_2<\frac{3}{4}, X_4>\frac{1}{2})$
   b) Marginal $f(x_1,x_2)$ and $E([X_1 * X_2])$
   c) Find $f\left(x_3,x_4 \mid x_1=\frac{1}{3}, x_2=\frac{2}{3}\right)$ and $P\left(X_3>\frac{3}{4}, X_4>\frac{1}{2} \mid X_1=\frac{1}{3}, X_2=\frac{2}{3}\right)$

5. (12 points) Let the joint pdf of (X,Y) be $f(x,y)=1,\ 0<y<1,\ y<x<y+1$
   a) Find pdf's of $-2X$ and Y explicitly and calculate their means and variances
   b) Further, find $Corr(-2X,Y)$

6. (10 points) (Weak Law of Large Numbers) Let $X_1, X_2, \ldots$ be iid random variables with mean μ and finite variance σ^2. Prove, that sample mean of the sample converges in probability: $\overline{X_n} \overset{p}{\Rightarrow} \mu$.

7. (12 points)

a) Let random variable X have a pdf $f(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1 \\ 0, & otherwise \end{cases}$

Find a monotone function *u(x)* such that the random variable *Y=u(X)* has a *uniform(0,1)* distribution

b) Derive moment generating function of random variable *X:= Gamma(α, β)* as defined in appendix. Use derived mgf to calculate *E[X]*

8. (12 points) Let $f(x,y) = \begin{cases} 6xy^2, & 0 < x < 1, \quad 0 < y < 1 \\ 0, & otherwise \end{cases}$

a) Show that *f(x,y)* defines a proper density function
b) Calculate $P(X + Y \geq 1.1)$
c) Calculate $[P(0.6 < X < 1) - P(0 < X < 0.6)]$

Good Luck