Written re-exam in Probability Theory, 7.5 ECTS credits

Wednesday, 9th of January 2025, 14:00 – 19:00

Time: FIVE hours

Examination Venue: TBD

You are asked to answer below stated questions as well as motivate your solutions. Grades are assigned as follows: **A** (91+), **B** (75-90), **C** (66-74), **D** (58-65), **E** (50-57), **Fx** (30-49), and **F** (0-29)

You are **allowed** to use a calculator and a ruler. Two formula attachments are provided together with the exam.

The teacher reserves the right to further examine the students on the answers provided.

1. (10 points) If $X_1, X_2, \ldots$ are *iid uniform(0,1)* and $X_{(n)} = \max_{1 \le i \le n} X_i$,
   a) (3p) where the sequence $X_{(n)}$ converges in distribution?
   b) (7p) If you set $\epsilon = \frac{t}{n}$, and use definition of convergence in distribution, what is the limiting distribution for the sequence $n*(1-X_{(n)})$?

2. (12 points) Let $-\infty < \mu_X < \infty, -\infty < \mu_Y < \infty, 0 < \sigma_X, 0 < \sigma_Y$, and $-1 < \rho < 1$ be five real numbers. The bivariate normal pdf with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2 and correlation ρ is the bivariate pdf given by

$$f(x,y) = \left(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right)^{-1} \times \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)\right)$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$.

Although the formula for the bivariate normal pdf looks formidable, this bivariate distribution is one of the most frequently used. (In fact, the derivation of the formula need not be formidable at all. See Exercise 4.46.)

The many nice properties of this distribution include these:

a. The marginal distribution of X is $\mathrm{n}(\mu_X, \sigma_X^2)$.

b. The marginal distribution of Y is $\mathrm{n}(\mu_Y, \sigma_Y^2)$.

c. The correlation between X and Y is $\rho_{XY} = \rho$.

Your task is to prove/derive the properties a)-c) given the formula of the bivariate pdf above. Note that it is not expected that student gives complete derivation for the full points in this part. One can miss few intermediate steps but explain them w7ell.

3. (12 points)
   a) Derive a moment generating function of random variable *X:= Gamma(α, β)* as defined in appendix. Use derived mgf to calculate *Var[X]*

b) Let $\bar{X}_1$ and $\bar{X}_2$ be respective means of two independent samples of size n drawn from a population having variance σ^2. Find the value of n such that $P(|\bar{X}_1 - \bar{X}_2| < \sigma) \approx 0.9$.

Please, justify your calculations. How your calculation of the above probability would change if both samples were quadrupled (4 times larger = "*4n*"). Provide calculation and give intuitive explanation for your answer.

4. (10 points) *A* and *B* are hiking and agree to meet at a certain spot between 18:00 and 20:00 on a particular day. Let us suppose they arrive at the meeting place independently and randomly and that both make it during agreed time interval. Find the distribution of the length of time that *A* waits for *B*. (If B arrives before A, define A's waiting time as zero).

5. (12 points) Let $f(x,y) = \begin{cases} c * xy(1-x^2), & 0 < x < 1,\ 0 < y < 1 \\ 0, & otherwise \end{cases}$

When you have found the right constant "c" for the above function to be a bivariate pdf, find a probability density function of *X*Y.*

6. (10 points) Let $f(x,y) = \begin{cases} 6xy^2, & 0 < x < 1, \quad 0 < y < 1 \\ 0, & otherwise \end{cases}$

a) Show that *f(x,y)* defines a proper density function
b) Calculate $P(X + Y \geq 0.9)$
c) Calculate $[P(0.5 < X < 1) - P(0 < X < 0.5)]$. When you get the answer, discuss what sign it has and argue if it is reasonable and intuitively expected.

7. (10 points) One tosses two dice: the outcome is two numbers from 1 to 6. Let *X* be minimum of the two numbers and *Y* is the maximum of the two numbers. Find joint distribution of *(X,Y)* and calculate *E[X], E[Y], Var(X), Var(Y),* and *Corr(X,Y).* Discuss the value and the sign of the correlation and interpret it in your own words.

8. (12 points) Let $f(x,y) = \begin{cases} x + y, & 0 < x < 1, \quad 0 < y < 1 \\ 0, & otherwise \end{cases}$

a) Find the probability density function of the r.v. (*-X-Y)*
b) Calculate $P(X + Y \leq 0.3)$. Further, find the median and the mean of the r.v. (*-X-Y*)

9. (12 points) Let the joint pdf of *(X,Y)* be $f(x,y)=1,\ 0<y<1,\ y<x<y+1$. Find
   a) marginals of 2**X* and 3**Y* and their first two moments;
   b) *Corr(2*X,3*Y)*

Good Luck