Written re-exam in Probability Theory, 7.5 ECTS credits
Wednesday, 4th of December 2024, 14:00 – 19:00
Time: FIVE hours
Examination Venue: TBD

You are asked to answer below stated questions as well as motivate your solutions. Grades are assigned as follows: **A** (91+), **B** (75-90), **C** (66-74), **D** (58-65), **E** (50-57), **Fx** (30-49), and **F** (0-29)

You are **allowed** to use a calculator and a ruler. Two formula attachments are provided together with the exam.

The teacher reserves the right to further examine the students on the answers provided.

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1. (10 points) Let X and Y be *i.i.d* normal random variables with pdf $N(1,4)$, and $U:=X+Y;\ V:=X-Y$
   a) Calculate the joint pdf: $f_{U,V}(u,v)$
   b) Calculate the $Corr(U,V)$

2. (12 points)

a) Let random variable X have a pdf $f(x) = \begin{cases} \frac{x+2}{3}, & 1 < x < 4 \\ 0, & otherwise \end{cases}$

Find a monotone function $u(x)$ such that the random variable $Y=u(X)$ has a *uniform(0,1)* distribution.

b) Check that below stated function is a pdf and find the moment generating function corresponding to $f(x) = \frac{1}{2\beta}\exp\{\frac{-|x-a|}{\beta}\}, -\infty < x, \alpha < \infty, \beta > 0$. Make sure to provide detailed explanations.

3. (12 points) Let the distribution of Y conditional on $X = x$ be $N(x, x^2)$ $[Y \mid X = x] \sim N(x, x^2)$ and the marginal distribution of X be $U(0,4)$. Find $E[Y]$, $Var(Y)$ and $Cov(X, Y)$.

4. (12 points) One tosses two dice: the outcome is two numbers from 1 to 6. Let X be sum of the two numbers and Y is minimum of the two numbers. Find joint distribution of (X,Y) and calculate $E[X]$, $E[Y]$, $Var(X)$, $Var(Y)$, and $Corr(X,Y)$. Discuss the value and the sign of the correlation and interpret it in your own words.

5. (10 points) Let $f(x,y) = \begin{cases} x+y, & 0<x<1, \quad 0<y<1 \\ 0, & otherwise \end{cases}$

   a) Find the probability density function of the r.v. $(-X-Y)$
   b) Calculate $P(X + Y \leq 0.5)$. Further, find the median and the mean of the r.v. $(-X-Y)$

6. (12 points) Let the joint pdf of *(X,Y)* be *f(x,y)=1, 0<y<1, y<x<y+1*. Find
   a) marginals of 3*X and Y and their first two moments;
   b) *Corr(3*X,Y)*

7. (15 points)) Let *n=4* and

$$f(x_1, x_2, x_3, x_4) = \begin{cases} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2), & 0 < x_i < 1, \; i = 1,2,3,4 \\ 0, & otherwise \end{cases}$$

Show that the above function is a joint pdf. Calculate
   a) $P(X_1 < \frac{1}{2}, X_2 < \frac{3}{4}, X_4 > \frac{1}{2})$
   b) Marginal $f(x_1, x_2)$ and $E([X_1 * X_2])$
   c) Find $f\left(x_3, x_4 \mid x_1 = \frac{1}{3}, x_2 = \frac{2}{3}\right)$ and $P\left(X_3 > \frac{3}{4}, X_4 > \frac{1}{2} \mid X_1 = \frac{1}{3}, X_2 = \frac{2}{3}\right)$

8. (10 points) (Weak Law of Large Numbers) Let $X_1, X_2, \ldots$ be iid random variables with mean μ and finite variance σ^2. Prove, that sample mean of the sample converges in probability: $\overline{X_n} \overset{p}{\Rightarrow} \mu$.

9. (10 points) Let $X_1, \ldots, X_n$ be *i.i.d* $N(\mu, \sigma^2)$ distributed random sample. Let us further assume that you have already proven that $\bar{X}$ and S are independent. Derive distribution of $T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$, where $\bar{X}, S$ denote sample mean and sample variance of the above sample.

Good Luck