Stockholms universitet

STATISTISKA INSTITUTIONEN HT-2023

Written Exam in Probability Theory, 7.5 ECTS credits
Thursday, 04th of January 2024, 08:00 – 13:00

Examination: On-campus Exam

You are asked to answer below stated questions and motivate your solutions. Grades are assigned as follows: **A** (91+), **B** (75-90), **C** (66-74), **D** (58-65), **E** (50-57), **Fx** (30-49), and **F** (0-29)

You are **allowed** to use calculator. Other supplementary material is attached to your exam questions.

The teacher reserves the right to further examine the students on the answers provided.

1. (12 points) Let $f(x,y) = \begin{cases} 6xy^2, & 0<x<1, \quad 0<y<1 \\ 0, & otherwise \end{cases}$

   a) Show that $f(x,y)$ defines a proper density function
   b) Calculate $P(X + Y \geq 0.9)$
   c) Calculate $[P(0.5 < X < 1) - P(0 < X < 0.5)]$. When you get the answer, discuss what sign it has and argue if it is reasonable and intuitively expected.

2. (10 points) Let $f(x,y) = \begin{cases} x+y, & 0<x<1, \quad 0<y<1 \\ 0, & otherwise \end{cases}$

   a) Find the probability density function of $X+Y$
   b) Find the median and the mean of the above distribution

3. (12 points) *A* and *B* are hiking and agree to meet at a certain place between 13:00 and 18:00. Let us suppose they arrive at the meeting place independently and randomly and that both make it during agreed time interval. Find the distribution of the length of time that *A* waits for *B*. (If B arrives before A, define A's waiting time as zero).

4. (12 points) Let *X* and *Y* be *i.i.d* normal random variables with pdf $N(1,4)$, and $U:=X+Y$; $V:=X-Y$

   a) Calculate the joint pdf: $f_{U,V}(u,v)$
   b) Calculate the $Corr(U,V)$

5. (12 points) Let the joint pdf of (X,Y) be $f(x,y)=1$, $0<y<1$, $y<x<y+1$
   a) Find pdf's of X and $5Y$ explicitly and calculate their means and variances
   b) Further, find $Corr(X,5Y)$

6. (12 points) One tosses two dice: the outcomes are the numbers from 1 to 6. Let X be the minimum and Y the max of the two. Find the joint distribution of (X,Y) and calculate $E[X]$, $E[Y]$, $Var(X)$, $Var(Y)$, and $Corr(X,Y)$. Discuss the value and the sign of the correlation and interpret it in your own words.

7. (10 points)

a) Check that the below stated function is a pdf and find the moment generating function corresponding to $f(x) = \frac{1}{2\beta}\exp\{\frac{-|x-a|}{\beta}\}, -\infty < x, \alpha < \infty, \beta > 0$. Make sure to provide detailed explanations.

b) Derive a moment generating function of random variable $X := Gamma(\alpha, \beta)$ as defined in appendix. Use derived mgf to calculate $Var[X]$

8. (10 points) Let $X_1, \ldots, X_n$ be *i.i.d* $N(\mu, \sigma^2)$ distributed random sample. Let us further assume that you have already proven that $\bar{X}$ and S are independent. Derive distribution of $T = \frac{\bar{X}-\mu}{S/\sqrt{n}}$, where $\bar{X}, S$ denote sample mean and sample variance of the above sample.

9. (10 points) Let us assume that the sequence of random variables X_n converges in distribution to a constant c. Show that it also converges in probability to the same constant c. In other words, convergence in probability and convergence in distribution are equivalent in this particular case. (Hint: start with writing the limiting distribution explicitly as a cdf).

Good Luck