Stockholms universitet

STATISTISKA INSTITUTIONEN HT-2023

Written Exam in Probability Theory, 7.5 ECTS credits
Monday, 27th of November 2023, 08:00 – 13:00

Examination: On-campus Exam

You are asked to answer below stated questions and motivate your solutions. Grades are assigned as follows: **A** (91+), **B** (75-90), **C** (66-74), **D** (58-65), **E** (50-57), **Fx** (30-49), and **F** (0-29)

You are **allowed** to use calculator. Other supplementary material is attached to your exam questions.

The teacher reserves the right to further examine the students on the answers provided.

1. (12 points) Let the joint pdf of (X,Y) be $f(x,y)=1,\ 0<y<1,\ y<x<y+1$
   a) Find pdf's of $3X$ and Y explicitly and calculate their means and variances
   b) Further, find $Corr(3X,Y)$

2. (12 points) One tosses two dice: the outcomes are the numbers from 1 to 6. Let X be the minimum and Y the max of the two. Find the joint distribution of (X,Y) and calculate $E[X]$, $E[Y]$, $Var(X)$, $Var(Y)$, and $Cov(X,Y)$. Discuss the value of the correlation and interpret it in your own words.

3. (12 points) Let continuous random vector (X,Y) have a joint pdf: $f(x,y)=e^{-y}$, $0<x<y<\infty$. Write explicitly $f(x,y)$ as one formula that contains both the domain and the functional expression of the pdf. Provide detailed solutions while computing the following:

   a) the marginal $f(x)$
   b) $f(y|\,x)= (Y\mid X=x)$; $E[Y\mid X=x]$; $Var(Y\mid X=x)$
   c) $P(X+Y>1)$

4. (12 points) Let $f(x,y)=\begin{cases} x+y, & 0<x<1,\quad 0<y<1 \\ 0, & otherwise \end{cases}$
   a) Find the probability density function of $X+Y$
   b) Calculate $P(X+Y\leq 1.5)$

5. (12 points) Let X and Y be *i.i.d* standard normal random variables, and $U:=X+Y;\ V:=X-Y$
   a) Calculate the joint pdf: $f_{U,V}(u,v)$
   b) Calculate the $Corr(U,V)$
   c) Are the r.v. U and V independent? Support your statement by probabilistic argument.

6. (10 points)

a) Let r.v. X have a pdf $f(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1 \\ 0, & otherwise \end{cases}$

   Find a monotone function *u(x)* such that the random variable *Y=u(X)* has a *uniform(0,1)* distribution

b) Derive a moment generating function of random variable *X:= Gamma(α, β)* as defined in appendix. Use derived mgf to calculate *Var[X]*

7. (15 points)) Let *n=4* and

$$f(x_1, x_2, x_3, x_4) = \begin{cases} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2), & 0 < x_i < 1,\ i = 1,2,3,4 \\ 0, & otherwise \end{cases}$$

Show that the above function is a joint pdf. Calculate

a) $P(X_1 < \frac{1}{2}, X_2 < \frac{3}{4}, X_4 > \frac{1}{2})$

b) Marginal $f(x_1, x_2)$ and $E([X_1 * X_2])$

c) Find $f\left(x_3, x_4 \mid x_1 = \frac{1}{3}, x_2 = \frac{2}{3}\right)$ and $P\left(X_3 > \frac{3}{4}, X_4 > \frac{1}{2} \mid X_1 = \frac{1}{3}, X_2 = \frac{2}{3}\right)$

8. (12 points) Let $f(x, y) = \begin{cases} c * xy(1 - x^2), & 0 < x < 1,\ 0 < y < 1 \\ 0, & otherwise \end{cases}$

   When you have found the right constant "c" for the above function to be a bivariate pdf, find a probability density function of *X*Y*.

9. (10 points) (Weak Law of Large Numbers) Let $X_1, X_2, \ldots$ be iid random variables with mean μ and finite variance σ^2. Prove, that sample mean of the sample converges in probability: $\overline{X_n} \overset{p}{\Rightarrow}$ μ.

Good Luck