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> **OBS! Läs noga igenom anvisningarna i tentamen, t.ex. hur du ska skriva svaren. Det är ditt ansvar som student att följa de anvisningar som ges.**
>
> ***NOTE! Read the examination instructions carefully, e.g. how to write the answers. It is your responsibility as a student to follow the given instructions.***

Skriv din anonymiseringskod och dagens datum på allt material du lämnar in.
(*Enter your anonymization code and todays date on all submitted materials*)

Anonymiseringskod (*Anonymization code*)	311-0005-XZR		
Datum (*Date YYYY-MM-DD*)	2023-11-27	Plats nr. (*Seat No.*)	30

Kurs/Kurskod (*Course/Course code*)	ST721A
Kursmoment (*Course component*)	Sannolikhetsteori

Fylls i av tentamensvärd (*To be filled in by invigilator*)

Direkt i skrivning: (kryss)		Svarsblankett: (kryss)		Lösa svarsblad: (antal)	12
Lämnat in blankt: (kryss)		Dator: (kryss)			

Inlämningstid: 13:03 Signatur tentamensvärd: [signature]

Fylls i av lärare/examinator (*To be filled in by teacher/examinator*)

Betyg:	B	Poäng:	81

Signatur rättande lärare/examinator: AA

1	2	3	4	5	6	7	8	9	Σ	Grade
2	12	12	4	12	10	13	10	6	81	

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Uppg.nr.: (Task no.) 2

Lärarens kommentar: (Teacher's note)

Outcomes:

$X = \{1,2,3,4,5,6\}$ $Y = \{1,2,3,4,5,6\}$

$f_{XY}(x,y)$:

Y \ X	1	2	3	4	5	6	$f_Y(y)$
1	1/36	0	0	0	0	0	1/36
2	2/36	1/36	0	0	0	0	3/36
3	2/36	2/36	1/36	0	0	0	5/36
4	2/36	2/36	2/36	1/36	0	0	7/36
5	2/36	2/36	2/36	2/36	1/36	0	9/36
6	2/36	2/36	2/36	2/36	2/36	1/36	11/36
$f_X(x)$	11/36	9/36	7/36	5/36	3/36	1/36	1

OK

$E[X] = \sum_{i=1}^{n} x_i \cdot f_X(x) =$
$= 1\cdot\frac{11}{36} + 2\cdot\frac{9}{36} + 3\cdot\frac{7}{36} + 4\cdot\frac{5}{36} + 5\cdot\frac{3}{36} + 6\cdot\frac{1}{36} = \boxed{2,528}$

$E[Y] = \sum_{i=1}^{n} y_i \cdot f_Y(y) =$
$= 1\cdot\frac{1}{36} + 2\cdot\frac{3}{36} + 3\cdot\frac{5}{36} + 4\cdot\frac{7}{36} + 5\cdot\frac{9}{36} + 6\cdot\frac{11}{36} = \boxed{4.472}$

$Var[X] = E[X^2] - E[X]^2$

$E[X^2] = \sum_{i=1}^{n} x_i^2 \cdot f_X(x) = 1^2\cdot\frac{11}{36} + 2^2\cdot\frac{9}{36} + 3^2\cdot\frac{7}{36} + 4^2\cdot\frac{5}{36} + 5^2\cdot\frac{3}{36} + 6^2\cdot\frac{1}{36} =$
$= 8,361 \Rightarrow Var[X] = 8.361 - 2,528^2 = \boxed{1.971}$

$E[Y^2] = \sum_{i=1}^{n} y_i^2 \cdot f_Y(y) = 1^2\cdot\frac{1}{36} + 2^2\cdot\frac{3}{36} + 3^2\cdot\frac{5}{36} + 4^2\cdot\frac{7}{36} + 5^2\cdot\frac{9}{36} + 6^2\cdot\frac{11}{36} =$
$= 21,972 \Rightarrow Var[Y] = 21,972 - 4.472^2 = \boxed{1.971}$

equal! reasonable?

To solve $Cov[X,Y]$ we need to know $E[X\cdot Y]$

$E[X\cdot Y] = \sum_{i=1}^{n} x_i \cdot y_i \cdot f_{XY}(x,y) = 1\cdot1\cdot\frac{1}{36} + 1\cdot2\cdot\frac{2}{36} + 1\cdot3\cdot\frac{2}{36} + \ldots + 6\cdot6\cdot\frac{1}{36} =$
$= 12,250$

$Cov[X,Y] = E[X\cdot Y] - E[X]\cdot E[Y] = \boxed{0,945}$

$Corr = \frac{0,945}{\sqrt{1,971\cdot1,971}} = 0,48$

They have a positive correlation which means the outcome of X increases when outcome of Y increases and vice versa.

How one interprets corr ≈ 1/2 ?

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$f_{XY}(x,y) = e^{-y}, \quad 0<x<y<\infty$ pdf?

$f_{XY}(x,y) = e^{-y} \cdot I_{[0<x<y<\infty]}(x,y)$ ok

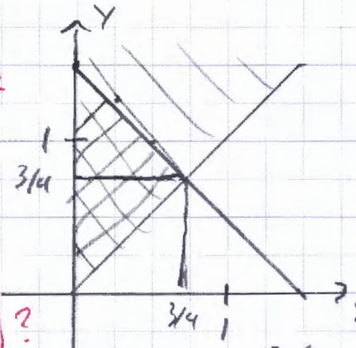

Diagram 3.1

(a) $f_X(x) = \int_x^\infty e^{-y}\,dy = 1-e^{-y}\Big|_x^\infty = -0 + e^{-x}$

$= e^{-x}, \quad 0<x<\infty$ ok

(b) $f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{e^{-y}}{e^{-x}} \quad x<y<\infty \quad \boxed{0<x<\infty}$?

$E[Y|X=x] = \int_x^\infty y \cdot \frac{e^{-y}}{e^{-x}}\,dy = \int_x^\infty y \cdot e^{-y+x}\,dy =$

$= y \cdot -e^{-y+x}\Big|_x^\infty - \int_x^\infty -e^{-y+x}\,dy = x - 1\,e^{-y+x}\Big|_x^\infty$

$= x-(0-1) = x+1$ ok

To solve $Var[Y|X=x]$ we need to find $E[Y^2|X=x]$

$E[Y^2|X=x) = \int_x^\infty y^2 \cdot e^{-y+x}\,dy = |y^2 \cdot -e^{-y+x}|_x^\infty - \int_x^\infty 2y \cdot -e^{-y+x}\,dy$

$= x^2 - \left(|2y \cdot e^{-y+x}|_x^\infty - \int_x^\infty 2 \cdot e^{-y+x}\,dy\right)$

$= x^2 + 2x + |2 \cdot -e^{-y+x}|_x^\infty = x^2 - 2x + 2$

$Var[Y|X=x] = x^2+2x+2-(x+1)^2 = x^2+2x+2-x^2-2x-1 = 1$ ok

(c) $P(X+Y \leq \boxed{1,5})$?? (See diagram 3.1) $Y \leq 1,5-X$

$P(X+Y\leq 1,5) = \int_0^{3/4}\int_x^{1,5-x} e^{-y}\,dy\,dx = \int_0^{3/4} 1-e^{-y}\Big|_x^{1,5-x} dx = \int_0^{3/4}(-e^{x-1,5}+e^{-x})\,dx$

$= \int_0^{3/4} -e^{x-1,5}\,dx + \int_0^{3/4} e^{-x}\,dx = |-e^{x-1,5}|_0^{3/4} + |-e^{-x}|_0^{3/4}$

$= -e^{-3/4} + e^{-1,5} - e^{-3/4} + 1 = 0,278$

OBS! Wrong exercise, see next page for $P(X+Y>1)$!

$\int_0^\infty\int_x^\infty e^{-y}\,dy\,dx = \int_0^\infty 1-e^{-y}\Big|_x^\infty dx = \int_0^\infty e^{-x}\,dx = 1-e^{-x}\Big|_0^\infty = 1$

It is a pdf, forgot this at first!

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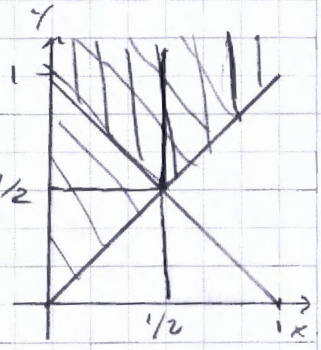
Lärarens kommentar: (Teacher's note)

$P(X+Y>1)$:

We need two double integrals A and B



$$A: \int_0^{1/2}\int_{1-x}^{\infty} e^{-y}\,dy\,dx = \int_0^{1/2} \left|-e^{-y}\right|_{1-x}^{\infty} dx$$

$$= \int_0^{1/2} e^{x-1}\,dx = \left|e^{x-1}\right|_0^{1/2} = e^{-1/2} - e^{-1}$$

$$B: \int_{1/2}^{\infty}\int_{x}^{\infty} e^{-y}\,dy\,dx = \int_{1/2}^{\infty} \left|-e^{-y}\right|_x^{\infty} dx = \int_{1/2}^{\infty} e^{-x}\,dx$$

$$= \left|-e^{-x}\right|_{1/2}^{\infty} = e^{-1/2}$$

$$P(X+Y>1) = A+B = e^{-1/2} - e^{-1} + e^{-1/2} = 2e^{-1/2} - e^{-1} = 0{,}845$$

OK

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Lärarens kommentar: (Teacher's note)

$$f_{XY}(x,y) = \begin{cases} x+y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

is it pdf?

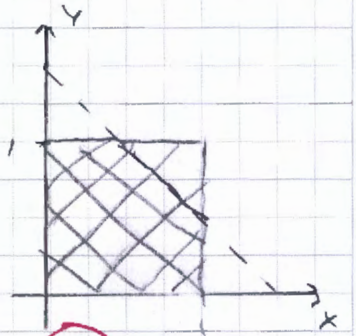
$$\int_0^1\int_0^1 (x+y)\,dy\,dx = \int_0^1 \left| yx+\frac{y^2}{2}\right|_0^1 dx = \int_0^1 (x+\tfrac{1}{2})dx = \left|\frac{x^2}{2}+\frac{x}{2}\right|_0^1 = 1$$ OK

a) $U = X+Y$, $V = Y$ OK

$g_1(x,y) = u = x+y \Rightarrow h_1(u,v) = x = u-v$, $0<u<2$

$g_2(x,y) = v = y \Rightarrow h_2(u,v) = y = v$, $0<v<1$

$$|J| = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$ OK



(1)

$$f_{U,V}(u,v) = \begin{cases} u, & 0<u<v+1,\ 0<v<1 \\ 0, & \text{otherwise} \end{cases}$$

$u > v-1$

$$\int_0^1\int_0^{v+1} u\,du\,dv = \int_0^1 \left|\frac{u^2}{2}\right|_0^{v+1} dv = \int_0^1 \frac{v^2+2v+1}{2}\,dv$$

$$= \left|\frac{v^3}{6}+\frac{1v^2}{2}+\frac{v}{2}\right|_0^1 = \frac{1}{6}+\frac{1}{2}+\frac{1}{2}.$$ not 1 mistake somewhere

b) $P(X+Y\leq 1{,}5) = P(U\leq 1{,}5)$

$$= \int_0^{1,5}\int_0^1 u\,dv\,du = \int_0^{1,5} |u^2|_0^1\,du = \int_0^{1,5} 1\,du = |u^2|_0^{1,5} = 3/2$$

(0)

positive mistake somewhere.

good attempt but problems with domain

Answer:

$$f_U(u) = \begin{cases} 0, & u\leq 0 \\ u^2, & 0<u<1 \\ 2u-u^2, & 1<u<2 \\ 0, & u\geq 2 \end{cases}$$

In b): substitute

Answer: $\frac{19}{24}$ //

Poäng: (Points)

Tentamensservice/SU/typ_b

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Lärarens kommentar: (Teacher's note)

$X \sim N(0,1) \quad Y \sim N(0,1)$

$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-Y^2/2}$ ok

a) $U := X + Y, \quad V := X - Y$

$g_1(x,y) = u = x + y \Rightarrow h_1(u,v) = x = u - x + v \Rightarrow x = \frac{u+v}{2}$

$g_2(x,y) = v = x - y \Rightarrow h_2(u,v) = y = \frac{u+v}{2} - v = \frac{u-v}{2}$

$|J| = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \left| -\frac{1}{4} - \frac{1}{4} \right| = \left| -\frac{1}{2} \right|$ ok

$f_{UV}(u,v) = \frac{1}{2\pi} \cdot \exp\left(-\left(\frac{u+v}{2}\right)^2/2\right) \cdot \exp\left(-\left(\frac{u-v}{2}\right)^2/2\right) \cdot \frac{1}{2} =$

b-c)

$= \frac{1}{4\pi} \cdot \exp\left\{-\frac{u^2+2vu+v^2}{8} - \frac{u^2-2vu+v^2}{8}\right\}$

$= \frac{1}{4\pi} \cdot \exp\left\{\frac{-u^2-2vu-v^2-u^2+2vu-v^2}{8}\right\}$

$= \frac{1}{4\pi} \cdot \exp\left\{\frac{-2u^2-2v^2}{8}\right\} = \frac{1}{4\pi} \cdot e^{-u^2/4} \cdot e^{-v^2/4}$

which can be written as:

$\frac{1}{2\sqrt{2\pi}} \cdot e^{-u^2/4} \cdot \frac{1}{2\sqrt{2\pi}} \cdot e^{-v^2/4}$ which shows us that U and V are independent and $U \sim N(0,2)$ and $V \sim N(0,2)$ ok

Two independent variables have zero correlation

$Corr(U,V) = 0$

ok

Poäng: (Points)

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Uppg.nr.: (Task no.) 6

Lärarens kommentar: (Teacher's note)

a) $f(x) = \begin{cases} \frac{x+1}{2} & -1 < x < 1 \\ 0 & \text{, otherwise} \end{cases}$

$$\int_{-1}^{1} \frac{x+1}{2}\,dx = \left| \frac{x^2}{4} + \frac{x}{2} \right|_{-1}^{1} = \frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} = 1 \quad \text{pdf!}$$ ok

$$u(x) = \int_{-1}^{x} \frac{t+1}{2}\,dt = \left| \frac{t^2}{4} + \frac{t}{2} \right|_{-1}^{x} = \frac{x^2}{4} + \frac{x}{2} - \frac{1}{4} + \frac{1}{2} = \frac{x^2+2x+1}{4}$$

$$u(x) = \begin{cases} 1 & x \geq 1 \\ \frac{x^2+2x+1}{4} & 0 < x < 1 \\ 0 & x < 1 \end{cases}$$ OK

b) $X := \text{Gamma}(\alpha, \beta) \quad 0 \leq x < \infty \quad , \alpha, \beta > 0$

$$M_X(t) = \int_0^\infty \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\beta} \cdot e^{tx}\,dx = \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha} \cdot \int_0^\infty x^{\alpha-1} \cdot e^{-x\left(\frac{1}{\beta} - t\right)}\,dx$$

$$= \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha} \cdot \int_0^\infty x^{\alpha-1} \cdot e^{-x\left(\frac{1}{\frac{\beta}{1-\beta t}}\right)}\,dx$$

set $\beta_1 = \frac{\beta}{1-\beta t}$ and substitute the integral

$$\frac{1}{\Gamma(\alpha)\cdot\beta^\alpha} \cdot \Gamma(\alpha)\cdot\beta_1^\alpha = \left(\frac{\beta_1}{\beta}\right)^\alpha = \left(\frac{\frac{\beta}{1-\beta t}}{\beta}\right)^\alpha = \left(\frac{1}{1-\beta t}\right)^\alpha$$ ok

$t < \frac{1}{\beta}$

$\text{Var}[X] = E[X^2] - E[X]^2$

$M_X'(t) = \alpha\beta \cdot (1-\beta t)^{-\alpha-1}$ = d β

$M_X''(t) = -\alpha-1 \cdot (\alpha\beta - \beta \cdot (1-\beta t)^{-\alpha-2})$

$\Rightarrow E[X^2] = \alpha^2\beta^2 + \alpha\beta^2$ ok

$\text{Var}[X] = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$

ok

Poäng: (Points)

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Uppg.nr.: (Task no.) 7

Lärarens kommentar: (Teacher's note)

$$f(x_1,x_2,x_3,x_4) = \begin{cases} 3/4(x_1^2+x_2^2+x_3^2+x_4^2), & 0<x_i<1,\ i=1,2,3,4 \\ 0, & \text{otherwise} \end{cases}$$

a) $\int_0^1\int_0^1\int_0^1\int_0^1 3/4\,(x_1^2+x_2^2+x_3^2+x_4^2)\,dx_1\,dx_2\,dx_3\,dx_4$

$= \int_0^1\int_0^1\int_0^1 3/4 \left| \frac{x_1^3}{3} + x_1x_2^2 + x_1x_3^2 + x_1x_4^2 \right|_0^1 dx_2\,dx_3\,dx_4$

$= \int_0^1\int_0^1\int_0^1 3/4 \left(\frac{1}{3} + x_2^2 + x_3^2 + x_4^2\right) dx_2\,dx_3\,dx_4$

keep doing this till we get:

$3/4\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) = \frac{3}{4}\cdot\frac{4}{3} = 1$ ok

b) $P(X_1<1/2,\ X_2<3/4,\ X_4>1/2) = \int_{1/2}^1\int_0^1\int_0^{3/4}\int_0^{1/2} 3/4\,(x_1^2+x_2^2+x_3^2+x_4^2)\,dx_1\,dx_2\,dx_3\,dx_4$

$= \int_{1/2}^1\int_0^1\int_0^{3/4} 3/4 \left| \frac{x_1^3}{3} + x_1x_2^2 + x_1x_3^2 + x_1x_4^2 \right|_0^{1/2} dx_2\,dx_3\,dx_4$

$= \int_{1/2}^1\int_0^1\int_0^{3/4} 3/4 \left(\frac{1}{24} + \frac{x_2^2}{2} + \frac{x_3^2}{2} + \frac{x_4^2}{2}\right) dx_2\,dx_3\,dx_4$

$= \int_{1/2}^1\int_0^1 3/4 \left| \frac{x_2}{24} + \frac{x_2^3}{6} + \frac{x_2x_3^2}{2} + \frac{x_2x_4^2}{2} \right|_0^{3/4} dx_3\,dx_4$

$= \int_{1/2}^1\int_0^1 3/4 \left(\frac{3}{72} + \frac{27}{384} + \frac{3x_3^2}{8} + \frac{3x_4^2}{8}\right) dx_3\,dx_4$

$= \int_{1/2}^1 3/4 \left| \frac{3x_3}{72} + \frac{27x_3}{384} + \frac{3x_3^3}{24} + \frac{3x_3x_4^2}{8} \right|_0^1 dx_4$

$= \int_{1/2}^1 3/4 \left(\frac{3}{72} + \frac{27}{384} + \frac{3}{24} + \frac{3x_4^2}{8}\right) dx_4$

$= 3/4 \left| \frac{3x_4}{72} + \frac{27x_4}{384} + \frac{3x_4}{24} + \frac{3x_4^3}{24} \right|_{1/2}^1$

$= 3/4\left(\frac{3}{72} - \frac{3}{144} + \frac{27}{384} - \frac{27}{768} + \frac{3}{24} - \frac{3}{48} + \frac{3}{24} - \frac{3}{192}\right)$

$= 3/4\left(\frac{3}{144} + \frac{27}{768} + \frac{3}{48} + \frac{3}{24} - \frac{3}{192}\right) = 0{,}1709$ ok

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b)

$$f_{X_1 X_2}(x_1,x_2) = \int_0^1\int_0^1 3/4\,(x_1^2+x_2^2+x_3^2+x_4^2)\,dx_3\,dx_4$$

$$= \int_0^1 3/4 \left| x_3 x_1^2 + x_3 x_2^2 + \frac{x_3^3}{3} + x_3 x_4^2 \right|_0^1 dx_4$$

$$= \int_0^1 3/4 \left(x_1^2+x_2^2+\frac{1}{3}+x_4^2\right) dx_4$$

$$= 3/4 \left| x_4 x_1^2 + x_4 x_2^2 + \frac{x_4}{3} + \frac{x_4^3}{3} \right|_0^1$$

$$= 3/4\,(x_1^2+x_2+2/3) = \boxed{3/4\,(x_1^2+x_2^2)+\frac{1}{2}}$$ OK

$$E[X_1 \cdot X_2] = \int_0^1\int_0^1 x_1 x_2 \cdot 3/4\,(x_1^2+x_2^2+2/3)\,dx_1\,dx_2$$

$$= \int_0^1 3/4 \int_0^1 \left(x_1^3 x_2 + x_1 x_2^3 + \frac{2}{3} x_1 x_2\right) dx_1\,dx_2$$

$$= \int_0^1 3/4 \left| \frac{x_1^4 x_2}{4} + \frac{x_1^2 x_2^3}{2} + \frac{2 x_1^2 x_2}{6} \right|_0^1 dx_2$$

$$= \int_0^1 3/4 \left(\frac{x_2}{4} + \frac{x_2^3}{2} + \frac{2 x_2}{6} \right) dx_2$$

$$= 3/4 \left| \frac{x_2^2}{8} + \frac{x_2^4}{8} + \frac{2 x_2^2}{12} \right|_0^1 = 3/4\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{6}\right) = 0{,}3125$$

OK

c) next page

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Lärarens kommentar: (Teacher's note)

c)

$f(x_3, x_4 \mid x_1 = 1/3, x_2 = 2/3)$

$$= \frac{3/4(x_1^2 + x_2^2 + x_3^2 + x_4^2)}{3/4(x_1^2 + x_2^2 + 2/3)} = \frac{\frac{1}{9} + \frac{4}{9} + x_3^2 + x_4^2}{\frac{1}{9} + \frac{4}{9} + \frac{6}{9}}$$

$$= \frac{5/9 + x_3^2 + x_4^2}{11/9}$$ OK

$P(X_3 > 3/4, X_4 > 1/2 \mid X_1 = 1/3, X_2 = 2/3)$

$$= \int_{1/2}^{1}\int_{3/4}^{1} \left(\frac{5/9 + x_3^2 + x_4^2}{11/9}\right) dx_3\, dx_4 = \int_{1/2}^{1} \frac{9}{11} \left| \frac{5x_3}{9} + \frac{x_3^3}{3} + x_3 x_4^2 \right|_{3/4}^{1} dx_4$$

$$= \int_{1/2}^{1} \frac{9}{11}\left(\frac{5}{9} - \frac{15}{36} + \frac{1}{3} + \frac{27}{192} + x_4^2 - \frac{3}{4}x_4^2\right) dx_4$$

$$= \int_{1/2}^{1} 9/11 \left(0{,}6128 + \frac{x_4^2}{4}\right) dx_4 = 9/11 \left| 0{,}6128 x_4 + \frac{x_4^3}{12} \right|_{1/2}^{1}$$

$$= \frac{9}{11}\left(0{,}6128 - \frac{0{,}6128}{2} + \frac{1}{12} - \frac{1}{96}\right) = 0.31$$

Some mistake in calculations: see p.18

Poäng: (Points)

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Uppg.nr.: (Task no.) 8

Lärarens kommentar: (Teacher's note)

$$f(x,y)=\begin{cases} c\cdot xy(1-x^2), & 0<x,y<1 \\ 0, & \text{otherwise}\end{cases}$$

$$\int_0^1\int_0^1 (cx\cdot y - cx^3y)\,dy\,dx = \int_0^1 \left| \frac{cxy^2}{2} - \frac{cx^3y^2}{2}\right|_0^1$$

$$= \int_0^1 \left(\frac{cx}{2} - \frac{cx^3}{2}\right)dx = \left|\frac{cx^2}{4} - \frac{cx^4}{8}\right|_0^1 = \frac{c}{4} - \frac{c}{8} = 1$$

$\boxed{c=8}$ OK

$U = X\cdot Y \quad V = X$ OK

$g_1(x,y) = x\cdot y = u \Rightarrow h_1(u,v) = y = \frac{u}{v}$
$0<u<1$

$g_2(x,y) = x = v \Rightarrow h_2(u,v) = x = v$
$u<v<1$

$$|J| = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v}\end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2}\end{vmatrix} = \left|-\frac{1}{v}\right| = \frac{1}{v}$$ OK

$$f_{U,V}(u,v) = 8\cdot u(1-v^2)\cdot\frac{1}{v}$$

$$= 8u\left(\frac{1}{v} - \frac{v^2}{v}\right) = 8u\left(\frac{1}{v} - v\right)$$ ok for the formula but not for the domain!!

$$\int_0^1\int_u^1 8u\left(\frac{1}{v}-v\right)dv\,du = \int_0^1 \left|-\frac{8u}{v^2} - 4uv^2\right|_u^1 du$$ OK

$$= \int_0^1 \left(-8u - 4u + \frac{8}{u} - 4u^3\right)du$$

$$= \int_0^1 \left(-12u + \frac{8}{u} - 4u^3\right)du = \left|-6u^2 + \frac{8}{u^2} - u^4\right|_0^1 = -6+8-1 = 1$$

pdf?

The question was to find pdf for X·Y.
That implies: $f_U(u) \stackrel{?}{=}$ ⊖

⊕ good overall

Poäng: (Points)

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Uppg.nr.: (Task no.) 9

Lärarens kommentar: (Teacher's note)

$\bar{X}_n \xrightarrow{P} \mu \Rightarrow \lim_{n\to\infty} P(|\bar{X}_n - \mu| < \varepsilon) = 1, \quad \forall \varepsilon > 0$

$\Rightarrow \lim_{n\to\infty} P(|\bar{X}_n - \mu| \geq \varepsilon) = 0$

Square this! $(\bar{X}_n - \mu)^2$ will be positive

how they are connected in the proof?

$P((\bar{X}_n - \mu)^2 \geq \varepsilon^2) = P(Var[\bar{X}_n] \geq \varepsilon^2) = P(\frac{\sigma_x^2}{n} \geq \varepsilon^2)$

$\lim_{n\to\infty} P(\frac{\sigma_x^2}{n} \geq \varepsilon) = P(0 \geq \varepsilon) = 0$

that is not correct as there is no expectation on the left-hand side.

Overall: too many logical gaps in the proof. (±)

Poäng: (Points)

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Uppg.nr.: (Task no.) 1

Lärarens kommentar: (Teacher's note)

Is it a pdf (?)

$f(x,y)=1, \quad 0<y<1, \quad y<x<y+1$

$U=3X \quad V=Y$

See pp. 170-171

Why to do the transformation? + note that

$corr(3\cdot X, Y) = corr(X,Y)$

$g_1(x,y)=u=3x \Rightarrow h_1(u,v)=x=\frac{u}{3}$

$3v<u<3v+3$

$g_2(x,y)=v=y \Rightarrow h_2(u,v)=y=v$

$0<v<1$

$$|J|=\begin{vmatrix}\frac{\delta x}{\delta u} & \frac{\delta x}{\delta v}\\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v}\end{vmatrix}=\begin{vmatrix}1/3 & 0\\ 0 & 1\end{vmatrix}=\frac{1}{3}$$ OK

$$f_{UV}(u,v)\begin{cases}1/3, & 0<v<1, \ 3v<u<3v+3\\ 0, & \text{otherwise}\end{cases}$$

$$f(u)=\begin{cases}\int_3^6 1/3\, du & 3<u<6\\ \int_0^3 1/3\, du & 0<u<3\\ 0, & \text{otherwise}\end{cases}$$

$$=\begin{cases}\left|\frac{u}{3}\right|_3^6\\ \left|\frac{u}{3}\right|_0^3\\ 0,\end{cases}=\begin{cases}1 & 3<u<6\\ 1 & 0<u<3\\ 0 & \text{otherwise}\end{cases}$$

$$f_V(v)=\begin{cases}\int_0^1 1/3\, dv\\ 0 \text{ otherwise}\end{cases}=\begin{cases}\frac{1}{3}, & 0<v<1\\ 0, & \text{otherwise}\end{cases}$$

(−)

$E[u]=\int_0^6 u\, du=\left|\frac{u^2}{2}\right|_0^6=\frac{36}{2}=18$

$E[u^2]\int_0^6 u^2\, du=\left|\frac{u^3}{3}\right|_0^6=72$

$Var[u]=72-18^2$ is negative. Mistake somewhere

$E[v]=\int_0^1 v 1/3\, dv=\left|\frac{v^2}{6}\right|_0^1=\frac{1}{6}$

$E[v^2]=\int_0^1 v^2 1/3\, dv=\left|\frac{v^3}{9}\right|_0^1=\frac{1}{9} \Rightarrow Var(v)=0{,}0833$

$$corr(3X,Y)=\frac{E[u\cdot v]-E[u\cdot v]}{\sqrt{Var[u]\,Var[v]}}$$

Poäng: (Points)

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Regler i skrivsalen

- Följ tentamensvärds anvisningar.
- Väskor och ytterkläder ska placeras på anvisad plats.
- Placera ID-handling väl synlig på bordet framför dig.
- Ingen student får lämna skrivsalen under de första 30 minuterna.
- Endast en student i taget får besöka toaletten. Vid toalettbesök skriv ditt namn och klockslag på avsedd lista. Efter toalettbesöket ska du åter ange klockslag på listan.
- Elektronisk utrustning som mobiltelefon eller Smartwatch ska vara avstängd och placerad på anvisad plats.
- Under tentamen gäller tystnad – det är förbjudet att prata, eller på annat sätt kommunicera, med andra studenter under pågående tentamen.
- Innan tentamenshandlingarna lämnas in; skriv sidnummer, anonymiseringskod och datum på alla inlämnade papper.

Om något är oklart – fråga gärna tentamensvärden. Lycka till!

Rules in the examination hall

- Follow the invigilator's instructions.
- Bags and outerwear must be placed at the designated place.
- Place your ID document clearly visible on the table in front of you.
- No student may leave the examination hall for the first 30 minutes.
- Only one student at a time may visit the toilet. Before visiting the toilet, write your name and time on the intended list. After the toilet visit, enter the time on the list again.
- Electronic equipment such as a mobile phone or Smartwatch must be switched off and placed at the designated place.
- During the exam, silence applies – you are not allowed to talk, or otherwise communicate, with other students during the exam.
- Before submitting the examination documents; remember to write the page number, anonymization code, and date on all papers.

Please do not hesitate to ask the invigilator if anything is unclear. Good luck!

Tentamensservice
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