Stockholms universitet

STATISTISKA INSTITUTIONEN HT-2022

Written Re-Exam in Probability Theory, 7.5 ECTS credits
Tuesday, 3rd of January 2023, 14:00 – 19:00

Examination: On-campus Exam

You are asked to answer below stated questions and motivate your solutions. Grades are assigned as follows: **A** (91+), **B** (75-90), **C** (66-74), **D** (58-65), **E** (50-57), **Fx** (30-49), and **F** (0-29)

You are **allowed** to use any calculator. Other supplementary material is attached to your exam questions.

The teacher reserves the right to further examine the students on the answers provided.

1. (12 points) Let X and Y be *i.i.d* normal random variables with pdf $N(1,4)$, and $U:=X+Y;\ V:=X-Y$

   a) Calculate the joint pdf: $f_{U,V}(u,v)$
   b) Calculate the $Corr(U,V)$

2. (12 points) Let $f(x,y)=\begin{cases} 6xy^2, & 0<x<1, \quad 0<y<1 \\ 0, & otherwise \end{cases}$

   a) Show that $f(x,y)$ defines a proper density function
   b) Calculate $P(X+Y \geq 0.9)$
   c) Calculate $[P(0.5 < X < 1) - P(0 < X < 0.5)]$. When you get the answer, discuss what sign it has and argue if it is reasonable and intuitively expected.

3. (12 points) Let $f(x,y)=\begin{cases} x+y, & 0<x<1, \quad 0<y<1 \\ 0, & otherwise \end{cases}$

   a) Find the probability density function of $X+Y$
   b) Calculate $P(X+Y > 1.5)$

4. (12 points)

   a) Let random variable X have a pdf $f(x)=\begin{cases} \frac{x+2}{3}, & 1<x<4 \\ 0, & otherwise \end{cases}$

   Find a monotone function $u(x)$ such that the random variable $Y=u(X)$ has a *uniform(0,1)* distribution.

   b) Check that the below stated function is a pdf and find the moment generating function corresponding to $f(x)=\frac{1}{2\beta}\exp\{\frac{-|x-a|}{\beta}\}, -\infty < x, \alpha < \infty, \beta > 0$. Make sure to provide detailed explanations.

c) Let $\bar{X}_1$ and $\bar{X}_2$ be respective means of two independent samples of size n drawn from a population having variance σ^2. Find the value of n such that $P(\left|\bar{X}_1 - \bar{X}_2\right| < \sigma) \approx 0.9$.

Please, justify your calculations. How your calculation of the above probability would change if both samples were quadrupled (4 times larger = "*4n*"). Provide calculation and give intuitive explanation for your answer.

5. (12 points) Let the joint pdf of (X,Y) be $f(x,y)=1,\ 0<y<1,\ y<x<y+1$

   a) Find pdf's of $3X$ and $2Y$ explicitly and calculate their means and variances
   b) Further, find $Corr(3X,2Y)$

6. (12 points) A and B are hiking and agree to meet at a certain place on a certain day (24 hours). Let us suppose they arrive at the meeting place independently and randomly during these 24 hours. Find the distribution of the length of time that A waits for B. (If B arrives before A, define A's waiting time as zero)

7. (15 points) Let

$$f(x_1, x_2, x_3, x_4) = \begin{cases} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2), & 0 < x_i < 1,\ i = 1,2,3,4 \\ 0, & otherwise \end{cases}$$

Show that the above function is a joint pdf. Calculate

   a) $P(X_2 < \frac{3}{4}, X_4 > \frac{1}{2})$
   b) Marginal $f(x_2, x_3)$ and $E([X_2 * X_3])$
   c) Find $f\left(x_1, x_4 \mid x_2 = \frac{1}{3}, x_3 = \frac{2}{3}\right)$ and $P\left(X_1 > \frac{3}{4}, X_4 > \frac{1}{2} \mid X_2 = \frac{1}{3}, X_3 = \frac{2}{3}\right)$

8. (12 points) Let $f(x,y) = \begin{cases} c * xy(1 - y^2), & 0 < x < 1,\ 0 < y < 1 \\ 0, & otherwise \end{cases}$

   When you have found the right constant "c" for the above function to be a bivariate pdf, find a probability density function of $X*Y$.

9. (10 points) Let us assume that sequence of random variable X_n converges in distribution to a constant c. Show that it also converges in probability to the same constant c. In other words, converges in probability and convergence in distribution are equivalent in this particular case. (Hint: start with writing the limiting distribution explicitly as a cdf)

Good Luck