Written Exam in Probability Theory, 7.5 ECTS credits
Wednesday, 30th of November 2022, 14:00 – 19:00

Examination: On-campus Exam

You are asked to answer below stated questions and motivate your solutions. Grades are assigned as follows: **A** (91+), **B** (75-90), **C** (66-74), **D** (58-65), **E** (50-57), **Fx** (30-49), and **F** (0-29)

You are **allowed** to use any calculator. Other supplementary material is attached to your exam questions.

The teacher reserves the right to further examine the students on the answers provided.

1. (12 points) *Let* continuous random vector *(X,Y)* have a joint pdf: $f(x,y) = e^{-y}$, $0 < x < y < \infty$. Compute the following:

   a) $f_X(x)$ and write explicitly *f(x,y)* as one formula that contains both domain and function expression of the pdf
   b) $f(y|\,x) = (Y \mid X = x)$; $E[Y \mid X = x]$; $Var(Y \mid X = x)$
   c) $P(X+Y>1)$

2. (12 points) Let $f(x,y) = \begin{cases} x+y, & 0<x<1, \quad 0<y<1 \\ 0, & otherwise \end{cases}$
   a) Find the probability density function of $X+Y$
   b) Calculate $P(X + Y \le 1.5)$

3. (12 points) One tosses two dice: the outcomes are the numbers from 1 to 6. Let X be the "outcome" on the first dice and Y is the max of the two. Find the joint distribution of (X,Y) and calculate $E[X]$, $E[Y]$, $Var(X)$, $Var(Y)$, and $Cov(X,Y)$

4. (12 points) Let X and Y be *i.i.d* standard normal random variables, and $U{:=}X+Y$; $V{:=}X-Y$
   a) Calculate the joint pdf: $f_{U,V}(u,v)$
   b) Calculate the $Corr(U,V)$
   c) Are the r.v. U and V independent? Support your statement by probabilistic argument.

5. (12 points) Let the joint pdf of (X,Y) be $f(x,y)=1$, $0<y<1$, $y<x<y+1$
   a) Find pdf's of $2X$ and Y explicitly and calculate their means and variances
   b) Further, find $Corr(2X,Y)$

6. (10 points)

a) Let r.v. X have a pdf $f(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1 \\ 0, & otherwise \end{cases}$

Find a monotone function $u(x)$ such that the random variable $Y=u(X)$ has a *uniform(0,1)* distribution

b) Derive a moment generating function of random variable *X:= Gamma(α, β)* as defined in appendix. Use derived mgf to calculate *Var[X]*

7. (15 points)) Let *n=4* and

$$f(x_1, x_2, x_3, x_4) = \begin{cases} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2), & 0 < x_i < 1,\ i = 1,2,3,4 \\ 0, & otherwise \end{cases}$$

Show that the above function is a joint pdf. Calculate

a) $P(X_1 < \frac{1}{2}, X_2 < \frac{3}{4}, X_4 > \frac{1}{2})$

b) Marginal $f(x_1, x_2)$ and $E([X_1 * X_2])$

c) Find $f\left(x_3, x_4 \mid x_1 = \frac{1}{3}, x_2 = \frac{2}{3}\right)$ and $P\left(X_3 > \frac{3}{4}, X_4 > \frac{1}{2} \mid X_1 = \frac{1}{3}, X_2 = \frac{2}{3}\right)$

8. (12 points) Let $f(x, y) = \begin{cases} c * xy(1 - x^2), & 0 < x < 1,\ 0 < y < 1 \\ 0, & otherwise \end{cases}$

When you have found the right constant "c" for the above function to be a bivariate pdf, find a probability density function of *X*Y.*

9. (10 points) (Weak Law of Large Numbers) Let $X_1, X_2, \ldots$ be iid random variables with mean μ and finite variance σ^2. Prove, that sample mean of the sample converges in probability: $\overline{X_n} \overset{p}{\Rightarrow} \mu$.

Good Luck