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OBS! Läs noga igenom anvisningarna i tentamen, t.ex. hur du ska skriva svaren.
Det är ditt ansvar som student att följa de anvisningar som ges.

NOTE! Read the examination instructions carefully, e.g. how to write the answers.
It is your responsibility as a student to follow the given instructions.

Skriv din anonymiseringskod och dagens datum på allt material du lämnar in.
(*Enter your anonymization code and todays date on all submitted materials*)

Anonymiseringskod (*Anonymization code*)	3	1	1	-	0	0	1	4	-	M	M	N
Datum (*Date* YYYY-MM-DD)	2022-11-30						Plats nr. (*Seat No.*)	22				

Kurs/Kurskod (*Course/Course code*)	ST721A
Kursmoment (*Course component*)	PROBABILITY THEORY

Fylls i av tentamensvärd (*To be filled in by invigilator*)

Direkt i skrivning: (kryss)		Svarsblankett: (kryss)		Lösa svarsblad: (antal)	10
Lämnat in blankt: (kryss)		Dator: (kryss)			

Inlämningstid: 18:57 Signatur tentamensvärd: RR

Fylls i av lärare/examinator (*To be filled in by teacher/examinator*)

Betyg:	A	Poäng:	100

Signatur rättande lärare/examinator: ______________________

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Uppgg.nr.: (Task no.) 1

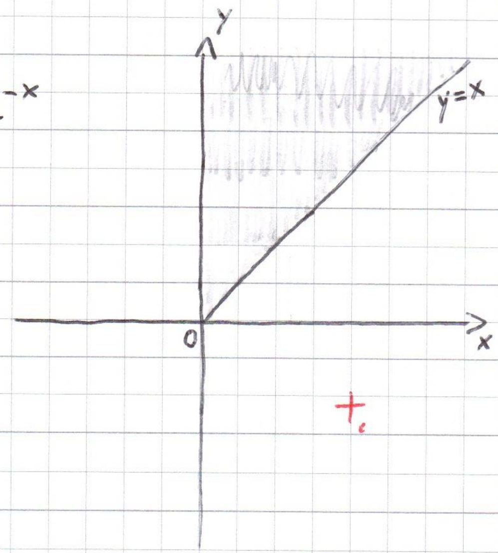
Lärarens kommentar: (Teacher's note)

EXERCISE 1

$f_{XY}(x,y) = e^{-y}$, $0<x<y<\infty$ → why it is a pdf?

a)

$f_X(x) = \int_x^{+\infty} e^{-y} \cdot dy = -[e^{-y}]_x^{+\infty} = -(0 - e^{-x}) = e^{-x}$



There fore:

$f_X(x) = e^{-x}$, $0<x<+\infty$

$f_{XY}(x,y) = e^{-y} \cdot I_{[0<x<y<+\infty]}(x,y)$ ok

+.

b) $f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)}$

$f_{Y|X}(y|x) = \dfrac{e^{-y}}{e^{-x}} = e^{-y-(-x)} = e^{-y+x}$, $x<y<+\infty$ ok

$E[Y|X] = \int_x^{+\infty} \underset{f(x)}{y} \cdot \underset{g'(x)}{e^{-y+x}} \cdot dy = [y \cdot (-e^{-y+x})]_x^{+\infty} - \int_x^{+\infty} -(e^{-y+x}) \cdot dy =$

$= x + \int_x^{+\infty} e^{-y+x} \cdot dy = x - [e^{-y+x}]_x^{+\infty} = x - (0-1) = x+1$ ok

$E[Y^2|X] = \int_x^{+\infty} \underset{f(x)}{y^2} \cdot \underset{g'(x)}{e^{-y+x}} \cdot dy = [y^2(-e^{-y+x})]_x^{+\infty} - \int_x^{+\infty} 2y \cdot (-e^{-y+x})\, dy =$

$= x^2 + 2\underbrace{\int_x^{+\infty} y \cdot e^{-y+x} \cdot dy}_{x+1} = x^2 + 2(x+1)$

There fore:

$V[Y|X] = E[Y^2|X] - E[Y|X]^2$

$V[Y|X] = x^2 + 2(x+1) - (x+1)^2 = \cancel{x^2} + \cancel{2x} + 2 - \cancel{x^2} - \cancel{2x} - 1 = 1$ ok

c) it continues in the back →

Poäng: (Points)

Tentamensservice/SU/typ_c

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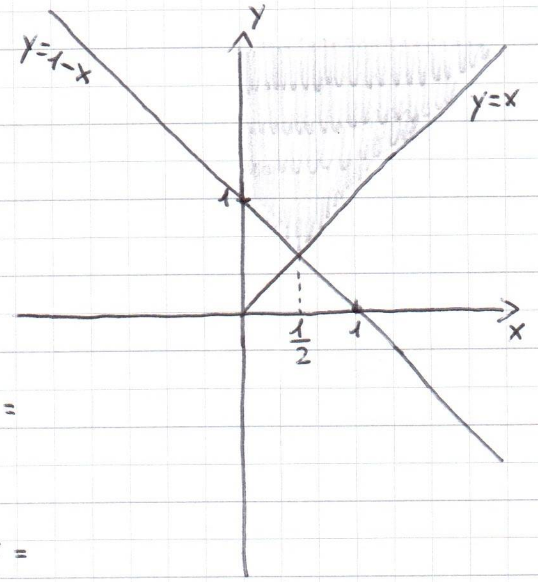
Lärarens kommentar: (*Teacher's note*)

c) $P(X+Y>1)$

We need to calculate the integral on the shaded area.



$P(X+Y>1) = P(Y>1-X)$

$P(Y>1-X) = 1 - P(Y \leq 1-X)$

$P(Y \leq 1-X) = \int_0^{1/2}\int_x^{1-x} e^{-y} \cdot dy \cdot dx =$

$= \int_0^{1/2} -[e^{-y}]_x^{1-x} \cdot dx = \int_0^{1/2} -(e^{-1+x} - e^{-x}) \cdot dx =$

$\int_0^{1/2} e^{-x} - e^{x-1} \cdot dx = \int_0^{1/2} e^{-x} \cdot dx - \int_0^{1/2} e^{x-1} \cdot dx =$

$-[e^{-x}]_0^{1/2} - [e^{x-1}]_0^{1/2} = -(e^{-1/2}-1) - (e^{-1/2}-e^{-1}) = -e^{-1/2}+1-e^{-1/2}+e^{-1} =$

$= -2e^{-1/2}+1+e^{-1}$

$P(Y>1-X) = 1-(-2e^{-1/2}+1+e^{-1}) = \cancel{1}+2\cdot e^{-1/2}\cancel{-1}-e^{-1} = 2\cdot e^{-1/2}-e^{-1}$

OK

Poäng: (*Points*)

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Uppg.nr.: (Task no.) 2

Lärarens kommentar: (Teacher's note)

EXERCISE 2

$$f(x,y) = \begin{cases} x+y & , 0<x<1 , 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

a)

$U = X + Y \qquad V = X$

$x+y=u \qquad x=v$

$v+y=u \qquad x=v$

$y = u-v \qquad x=v$

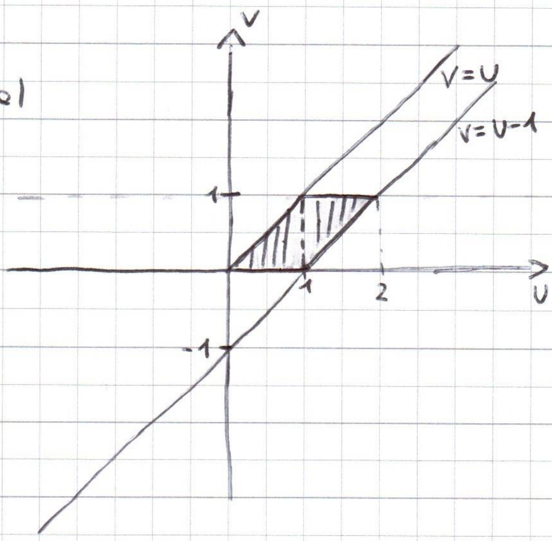
$$J = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = |-1| = 1$$

$f_{UV}(u,v) = v + (u-v) = u \qquad 0<v<1 , 0<u-v<1$

OK

$0<v<1 , v<u<1+v$

To find the probability density function of $X+Y$, we need to calculate the integral with respect to v. We can see from the domain that the area should be splitted in two parts. Therefore:



Shaded area is the domain of f_{UV}

$$f_U(u) = \begin{cases} \int_0^u u \cdot dv & 0<u<1 \\ \int_{u-1}^1 u \cdot dv & 1<u<2 \end{cases}$$

$$f_U(u) = \begin{cases} u\cdot[v]_0^u & 0<u<1 \\ u\cdot[v]_{u-1}^1 & 1<u<2 \end{cases}$$

good

(+)

$$f_U(u) = \begin{cases} u^2 & 0<u<1 \\ -u^2+2u & 1<u<2 \end{cases}$$

How about for $u\leq 0$ and $u \geq 2$?

b) it continues in the back ⟶

Poäng: (Points)

Tentamensservice/SU/typ_c

b) $P(X+Y \leq 1.5) = P(U \leq 1.5)$

$$P(U \leq 1.5) = \int_0^1 u^2 \cdot du + \int_1^{1.5} (-u^2+2u) \cdot du = \frac{1}{3}[u^3]_0^1 + \int_1^{1.5} (-u^2+2u)\, du =$$

$$\frac{1}{3} + \int_1^{1.5} -u^2 \cdot du + \int_1^{1.5} 2u \cdot du = \frac{1}{3} - \frac{1}{3}[u^3]_1^{1.5} + [u^2]_1^{1.5} = \frac{1}{3} - \frac{1}{3}\left[\left(\frac{3}{2}\right)^3 - 1\right] + \left[\left(\frac{3}{2}\right)^2 - 1\right] =$$

$$= \frac{19}{24}$$

OK

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Lärarens
kommentar:
(Teacher's
note)

Poäng:
(Points)

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Uppg.nr.: (Task no.) 3

Lärarens kommentar: (Teacher's note)

EXERCISE 3

X = outcome on the first dice

Y = max of the two

Y \ X	1	2	3	4	5	6	
1	$\frac{1}{36}$	0	0	0	0	0	$\frac{1}{36}$
2	$\frac{1}{36}$	$\frac{2}{36}$	0	0	0	0	$\frac{3}{36}$
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{3}{36}$	0	0	0	$\frac{5}{36}$
4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{4}{36}$	0	0	$\frac{7}{36}$
5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	0	$\frac{9}{36}$
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{6}{36}$	$\frac{11}{36}$
	$\frac{6}{36}$	$\frac{6}{36}$	$\frac{6}{36}$	$\frac{6}{36}$	$\frac{6}{36}$	$\frac{6}{36}$	1

OK

$E[X] = 1\cdot\frac{6}{36} + 2\cdot\frac{6}{36} + 3\cdot\frac{6}{36} + 4\cdot\frac{6}{36} + 5\cdot\frac{6}{36} + 6\cdot\frac{6}{36} =$

$= \frac{6}{36} + \frac{6}{18} + \frac{6}{12} + \frac{6}{9} + 5\cdot\frac{6}{36} + 1 = \frac{7}{2}$ OK

$E[Y] = 1\cdot\frac{1}{36} + 2\cdot\frac{3}{36} + 3\cdot\frac{5}{36} + 4\cdot\frac{7}{36} + 5\cdot\frac{9}{36} + 6\cdot\frac{11}{36} = \frac{161}{36}$ OK

$E[X^2] = 1\cdot\frac{6}{36} + 4\cdot\frac{6}{36} + 9\cdot\frac{6}{36} + 16\cdot\frac{6}{36} + 25\cdot\frac{6}{36} + 36\cdot\frac{6}{36} = \frac{91}{6}$

$E[Y^2] = 1\cdot\frac{1}{36} + 4\cdot\frac{3}{36} + 9\cdot\frac{5}{36} + 16\cdot\frac{7}{36} + 25\cdot\frac{9}{36} + 36\cdot\frac{11}{36} = \frac{199}{9}$

$V[X] = E[X^2] - E[X]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$ OK

$V[Y] = E[Y^2] - E[Y]^2 = \frac{199}{9} - \left(\frac{161}{36}\right)^2 = \frac{2735}{1296}$

To calculate the covariance, we need $E[X\cdot Y]$:

$E[X\cdot Y] = 1\cdot 1\cdot\frac{1}{36} + 1\cdot 2\frac{1}{36} + 2\cdot 2\cdot\frac{2}{36} + 1\cdot 3\frac{1}{36} + 2\cdot 3\cdot\frac{1}{36} + 3\cdot 3\cdot\frac{3}{36} + \ldots$

it continues in the back →

Poäng: (Points)

In conclusion:

$$E[X \cdot Y] = \frac{154}{9}$$

$$\operatorname{cov}(X,Y) = E[X \cdot Y] - E[X] \cdot E[Y] = \frac{154}{9} - \frac{7}{2} \cdot \frac{161}{36} = \frac{35}{24}$$

ok

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Poäng: (*Points*)

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Lärarens kommentar: (Teacher's note)

EXERCISE 4:

$X \sim N(0,1) \qquad Y \sim N(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad -\infty < x < +\infty$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-y^2/2} \qquad -\infty < y < +\infty$$

a) First of all, we need to find $f_{XY}(x,y)$. We know that X and Y are independent, so:

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-y^2/2} \qquad -\infty < x < +\infty,\ -\infty < y < +\infty$$

$U = X + Y$

$V = X - Y$

$x + y = u \qquad x - y = v$

$x = u - y \qquad u - y - y = v$

$x = u - y \qquad y = \frac{u-v}{2}$

$x = \frac{u+v}{2} \qquad y = \frac{u-v}{2}$

$$J = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

OK

$$f_{UV}(u,v) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\left(\frac{u+v}{2}\right)^2/2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\left(\frac{u-v}{2}\right)^2/2} \cdot \frac{1}{2} \qquad -\infty < u < +\infty,\ -\infty < v < +\infty$$

OK

b) it continues in the back $\longrightarrow$

Poäng: (Points)

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Lärarens kommentar: (Teacher's note)

b-c) We can see that:

$$f_{UV}(u,v) = \frac{1}{2\pi} \cdot e^{\frac{-(u^2+2uv+v^2)}{8}} \cdot e^{\frac{-(u^2-2uv+v^2)}{8}} \cdot \frac{1}{2} =$$

$$= \frac{1}{4\pi} \cdot e^{\frac{-u^2 \cancel{-2uv} - v^2 - u^2 \cancel{+2uv} - v^2}{8}} = \frac{1}{4\pi} \cdot e^{\frac{-2u^2-2v^2}{8}} =$$

$$= \frac{1}{4\pi} e^{-u^2/4} \cdot e^{-v^2/4}$$

Since we can express $f_{UV}(u,v)$ as:

$f_{UV}(u,v) = c \cdot g_U(u) \cdot h_V(v)$

U and V are independent.

If U and V are independent, then $cov(U,V)=0$. This because $cov(X,Y) = E[X \cdot Y] - E[X] \cdot E[Y]$. If X and Y are independent:

$cov[X,Y] = E[X] \cdot E[Y] - E[X] \cdot E[Y] = 0$

This means that:

$$corr(U,V) = \frac{cov(U,V)}{\sigma_U \cdot \sigma_V} = \frac{0}{\sigma_U \cdot \sigma_V} = 0$$

ok

Poäng: (Points)

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Uppg.nr.: (Task no.) 5

Lärarens kommentar: (Teacher's note)

EXERCISE 5

$f(x,y)=1 \quad , 0<y<1 \quad , y<x<y+1$ is it a pdf ?

a)

$U=2X$

$V=Y$ these are just marginals !

see pp 170-171 for an easier way

$2x=U \qquad y=V$

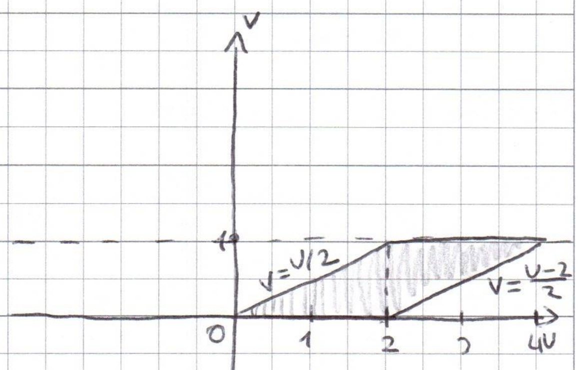
$x=\frac{U}{2} \qquad y=V$

$$J=\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

$f_{UV}(u,v)=1\cdot\frac{1}{2} \qquad 0<v<1 \;,\; v<\frac{u}{2}<v+1$

$0<v<1 \;,\; 2v<u<2(v+1)$

$$f_U(u)=\begin{cases}\int_0^{u/2}\frac{1}{2}\,dv & 0<u<2\\ \int_{\frac{u-2}{2}}^{1}\frac{1}{2}\,dv & 2<u<4\end{cases}$$



$$f_U(u)=\begin{cases}\frac{u}{4} & 0<u<2\\ \frac{4-u}{4} & 2<u<4\end{cases}$$

$f_V(v)=\int_{2v}^{2v+2}\frac{1}{2}\,du=\frac{1}{2}[u]_{2v}^{2v+2}=\frac{1}{2}(\cancel{2v}+2-\cancel{2v})=1$

$0<v<1$

So $U=2X$ and $V=Y$

$E[U]=\int_0^2 u\cdot\frac{u}{4}\,du+\int_2^4\frac{u\cdot(4-u)}{4}\,du=\frac{1}{4}\int_0^2 u^2\,du+\frac{1}{4}\int_2^4 4u-u^2\,du=$

$=\frac{1}{4}\frac{1}{3}[u^3]_0^2+\frac{1}{4}\left(2[u^2]_2^4-\frac{1}{3}[u^3]_2^4\right)=\frac{1}{12}\cdot 8+\frac{1}{4}\left[2(16-4)-\frac{1}{3}(64-8)\right]=$

$=2$ it continue in the back →

Poäng: (Points)

$$E[U^2] = \int_0^2 u^2 \cdot \frac{u}{4} \cdot du + \int_2^4 \frac{u^2(4-u)}{4} \cdot du =$$

$$= \frac{1}{4}\int_0^2 u^3 \cdot du + \frac{1}{4}\int_2^4 4u^2 - u^3 \cdot du = \frac{1}{16}[u^4]_0^2 + \frac{1}{4}\left(\frac{4}{3}[u^3]_2^4 - \frac{1}{4}[u^4]_2^4\right) =$$

$$= \frac{1}{16} 16 + \frac{1}{4}\left(\frac{4}{3}(56) - \frac{1}{4}(240)\right) = \frac{14}{3}$$

$$E[V] = \int_0^1 v \cdot dv = \frac{1}{2}[v^2]_0^1 = \frac{1}{2}$$

$$E[V^2] = \int_0^1 v^2 \cdot dv = \frac{1}{3}[v^3]_0^1 = \frac{1}{3}$$

$Y \sim U[0,1]$ get immediately

therefore:

$$V[U] = \frac{14}{3} - 2^2 = \frac{2}{3}$$

$$V[V] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

b)

$$corr(U,V) = \frac{cov(U,V)}{\sigma_U \cdot \sigma_V}$$

$$cov(U,V) = E[U \cdot V] - E[U] \cdot E[V]$$

$$E[U \cdot V] = \int_0^1 \int_{2v}^{2v+2} u \cdot v \cdot \frac{1}{2} \cdot du \cdot dv =$$

$$= \frac{1}{2}\int_0^1 v \cdot \frac{1}{2}[u^2]_{2v}^{2v+2} \cdot du \cdot dv =$$

$$= \frac{1}{2}\int_0^1 v \cdot \frac{1}{2}[(2v+2)^2 - (2v)^2] \cdot dv =$$

$$= \frac{1}{4}\int_0^1 v \cdot (\cancel{4v^2} + 8v + 4 - \cancel{4v^2}) \cdot dv = \frac{1}{4}\int_0^1 v(8v+4) \cdot dv =$$

$$\frac{1}{\cancel{4}}\int_0^1 \cancel{4}v(2v+1) \cdot dv = \int_0^1 2v^2 + v \cdot dv = \frac{2}{3}[v^3]_0^1 + \frac{1}{2}[v^2]_0^1 = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}$$

Thus:

$$cov(U,V) = \frac{7}{6} - 2 \cdot \frac{1}{2} = \frac{7}{6} - 1 = \frac{1}{6}$$

see p.171. and $cor(2X,Y) =$ $= cor(X,Y)$.

$$corr(U,V) = \frac{\frac{1}{6}}{\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{1}{12}}} = \frac{\sqrt{2}}{2}$$

ok

Should be $\frac{1}{\sqrt{2}}$

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Uppgg.nr.: (Task no.) 6

Lärarens kommentar: (Teacher's note)

EXERCISE 6:

a) $f(x) = \begin{cases} \frac{x+1}{2}, & -1<x<1 \\ 0 & \text{otherwise} \end{cases}$ is it a pdf (?)

We know from a theorem that U(X) should be $F_X(X)$. Therefore we need to calculate the cumulative distribution function of $f(x)$:

mistake (?)

$$\int_{-1}^{x} \frac{t+1}{2} \cdot dt = \frac{1}{2}\int_{-1}^{x} t+1 \cdot dt = \frac{1}{2}\left(\frac{1}{2}[t^2]_{-1}^{x} + [t]_{-1}^{x}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{2}[x^2-1] + [x+1]\right) = \frac{1}{2}\left(\frac{1}{2}x^2 - \cancel{\frac{1}{2}} + \frac{1}{2}x + \cancel{\frac{1}{2}}\right) = \frac{1}{4}x^2 + \frac{1}{4}x$$

Therefore:

$$F_X(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{4}x^2 + \frac{1}{4}x & -1<x<1 \\ 1 & x \ge 1 \end{cases}$$

→ some mistake as (±) for $x=1$ you get $F_X(x) = \frac{1}{2} \neq 1$

OK

b) $X \sim \text{Gamma}(\alpha, \beta)$

$$M_X(t) = \int_0^{+\infty} e^{tx} \cdot \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\beta} \cdot dx = \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha}\int_0^{+\infty} x^{\alpha-1} \cdot e^{-\frac{x}{\beta}+tx} \cdot dx =$$

$$= \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha}\int_0^{+\infty} x^{\alpha-1} \cdot e^{-x\left(\frac{1}{\beta}-t\right)} \cdot dx = \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha}\int_0^{+\infty} x^{\alpha-1} \cdot e^{-x\left(\frac{1-\beta t}{\beta}\right)} \cdot dx =$$

$$= \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha}\int_0^{+\infty} x^{\alpha-1} \cdot e^{-x/\left(\frac{\beta}{1-\beta t}\right)} \cdot dx =$$

$$\frac{1}{\cancel{\Gamma(\alpha)}\cdot\beta^\alpha} \cdot \cancel{\Gamma(\alpha)} \cdot \left(\frac{\beta}{1-\beta t}\right)^\alpha = \frac{\cancel{\beta^\alpha}}{(1-\beta t)^\alpha} \cdot \frac{1}{\cancel{\beta^\alpha}} = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < \frac{1}{\beta}$$ OK

We need to calculate:

$E[X]$ and $E[X^2]$

it continues in the back →

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(*Teacher's note*)

$$E[X] = \frac{d}{dt} M_X(t)\Big|_{t=0}$$

$$E[X] = \frac{d}{dt}\left(\frac{1}{1-\beta t}\right)^{\alpha}\Big|_{t=0} = \frac{d}{dt}\left[\frac{1}{(1-\beta t)^{\alpha}}\right]\Big|_{t=0} = \frac{d}{dt}(1-\beta t)^{-\alpha}\Big|_{t=0} = -\alpha(1-\beta t)^{-\alpha-1}\cdot(-\beta)\Big|_{t=0}$$

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = -\alpha\cdot(-\beta) = \alpha\cdot\beta$$

$$E[X^2] = \frac{d}{dt}\left[\alpha\cdot\beta(1-\beta t)^{-\alpha-1}\right]\Big|_{t=0} = \alpha\cdot\beta\cdot(-\alpha-1)(1-\beta t)^{-\alpha-2}\cdot(-\beta)\Big|_{t=0}$$

$$\frac{d^2}{dt^2} M_X(t) = \alpha^2\cdot\beta^2 + \alpha\cdot\beta^2$$

So:

$$E[X] = \alpha\cdot\beta$$

$$E[X^2] = \alpha^2\cdot\beta^2 + \alpha\cdot\beta^2$$

$$V[X] = E[X^2] - E[X]^2 = \cancel{\alpha^2\cdot\beta^2} + \alpha\cdot\beta^2 - \cancel{\alpha^2\cdot\beta^2} = \alpha\cdot\beta^2$$ OK

Poäng:
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Lärarens kommentar: (Teacher's note)

EXERCISE 7:

$$f(x_1,x_2,x_3,x_4)=\begin{cases}\frac{3}{4}(x_1^2+x_2^2+x_3^2+x_4^2), & 0<x_i<1, \ i=1,2,3,4\\ 0 & \text{otherwise}\end{cases}$$

To show that it is a joint pdf:

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{3}{4}(x_1^2+x_2^2+x_3^2+x_4^2)\,dx_1\,dx_2\,dx_3\,dx_4=$$

$$=\frac{3}{4}\int_0^1\int_0^1\int_0^1 \frac{1}{3}[x_1^3]_0^1+x_2^2[x_1]_0^1+x_3^2[x_1]_0^1+x_4^2[x_1]_0^1\,dx_2\,dx_3\,dx_4=$$

$$=\frac{3}{4}\int_0^1\int_0^1\int_0^1 \frac{1}{3}+x_2^2+x_3^2+x_4^2\,dx_2\,dx_3\,dx_4=$$

$$=\frac{3}{4}\int_0^1\int_0^1 \frac{1}{3}[x_2]_0^1+\frac{1}{3}[x_2^3]_0^1+x_3^2[x_2]_0^1+x_4^2[x_2]_0^1\,dx_3\,dx_4=$$

$$=\frac{3}{4}\int_0^1\int_0^1 \frac{1}{3}+\frac{1}{3}+x_3^2+x_4^2\,dx_3\,dx_4=\frac{3}{4}\int_0^1 \frac{1}{3}[x_3]_0^1+\frac{1}{3}[x_3]_0^1+\frac{1}{3}[x_3^3]_0^1+x_4^2[x_3]_0^1\,dx_4=$$

$$=\frac{3}{4}\int_0^1 \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+x_4^2\,dx_4=\frac{3}{4}\int_0^1 1+x_4^2\,dx_4=\frac{3}{4}\left([x_4]_0^1+\frac{1}{3}[x_4^3]_0^1\right)=$$

$$=\frac{3}{4}\left(1+\frac{1}{3}\right)=\frac{\cancel{3}}{\cancel{4}}\cdot\frac{\cancel{4}}{\cancel{3}}=1$$ OK

a) $$P\left(X_1<\frac{1}{2}, X_2<\frac{3}{4}, X_4>\frac{1}{2}\right)=\int_{1/2}^1\int_0^1\int_0^{3/4}\int_0^{1/2}\frac{3}{4}(x_1^2+x_2^2+x_3^2+x_4^2)\,dx_1\,dx_2\,dx_3\,dx_4=$$

$$\frac{3}{4}\int_{1/2}^1\int_0^1\int_0^{3/4}\frac{1}{3}[x_1^3]_0^{1/2}+x_2^2[x_1]_0^{1/2}+x_3^2[x_1]_0^{1/2}+x_4^2[x_1]_0^{1/2}\,dx_2\,dx_3\,dx_4=$$

$$=\frac{3}{4}\int_{1/2}^1\int_0^1\int_0^{3/4}\frac{1}{24}+\frac{1}{2}x_2^2+\frac{1}{2}x_3^2+\frac{1}{2}x_4^2\,dx_2\,dx_3\,dx_4=$$

$$=\frac{3}{4}\int_{1/2}^1\int_0^1\frac{1}{24}[x_2]_0^{3/4}+\frac{1}{6}[x_2^3]_0^{3/4}+\frac{1}{2}x_3^2[x_2]_0^{3/4}+\frac{1}{2}x_4^2[x_2]_0^{3/4}\,dx_3\,dx_4=$$

$$=\frac{3}{4}\int_{1/2}^1\int_0^1\frac{1}{32}+\frac{9}{128}+\frac{3}{8}x_3^2+\frac{3}{8}x_4^2\,dx_3\,dx_4=$$

$$=\frac{3}{4}\int_{1/2}^1\int_0^1\frac{13}{128}+\frac{3}{8}x_3^2+\frac{3}{8}x_4^2\,dx_3\,dx_4=$$

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$$= \frac{3}{4}\int_{1/2}^{1} \frac{13}{128}[x_3]_0^1 + \frac{1}{8}[x_3^3]_0^1 + \frac{3}{8}x_4^2[x_3]_0^1 \cdot dx_4 =$$

$$= \frac{3}{4}\int_{1/2}^{1} \frac{13}{128} + \frac{1}{8} + \frac{3}{8}x_4^2 \cdot dx_4 = \frac{3}{4}\int_{1/2}^{1} \frac{29}{128} + \frac{3}{8}x_4^2 \cdot dx_4 =$$

$$= \frac{3}{4}\left(\frac{29}{128}[x_4]_{1/2}^1 + \frac{1}{8}[x_4^3]_{1/2}^1\right) = \frac{3}{4}\left(\frac{29}{128}\left(1-\frac{1}{2}\right) + \frac{1}{8}\left(1-\frac{1}{8}\right)\right) =$$

$$= \frac{171}{1024} \qquad \frac{151}{1024}$$

ok

b) $$\underline{f_{X_1 X_2}(x_1,x_2)} = \int_0^1\int_0^1 \frac{3}{4}(x_1^2+x_2^2+x_3^2+x_4^2) \cdot dx_3 \cdot dx_4 =$$

$$\frac{3}{4}\int_0^1\int_0^1 x_1^2+x_2^2+x_3^2+x_4^2 \cdot dx_3 \cdot dx_4 = \frac{3}{4}\int_0^1 x_1^2[x_3]_0^1 + x_2^2[x_3]_0^1 + \frac{1}{3}[x_3^3]_0^1 + x_4^2[x_3]_0^1 \cdot dx_4 =$$

$$= \frac{3}{4}\int_0^1 x_1^2+x_2^2+\frac{1}{3}+x_4^2 \cdot dx_4 = \frac{3}{4}\left(x_1^2[x_4]_0^1 + x_2^2[x_4]_0^1 + \frac{1}{3}[x_4]_0^1 + \frac{1}{3}[x_4^3]_0^1\right) =$$

$$= \frac{3}{4}\left(x_1^2+x_2^2+\frac{1}{3}+\frac{1}{3}\right) = \frac{3}{4}\left(x_1^2+x_2^2+\frac{2}{3}\right) = \underbrace{\frac{3}{4}x_1^2+\frac{3}{4}x_2^2+\frac{1}{2}} \quad 0<x_1<1,\ 0<x_2<1$$

OK

$$\boxed{E[X_1 \cdot X_2]} = \int_0^1\int_0^1 x_1 \cdot x_2 \cdot \left(\frac{3}{4}x_1^2 + \frac{3}{4}x_2^2 + \frac{1}{2}\right) \cdot dx_1 \cdot dx_2 =$$

$$= \int_0^1\int_0^1 \frac{3}{4}x_1^3 \cdot x_2 + \frac{3}{4}x_1 \cdot x_2^2 + \frac{1}{2}x_1 \cdot x_2 \cdot dx_1 \cdot dx_2 =$$

$$= \int_0^1 \frac{3}{16}x_2[x_1^4]_0^1 + \frac{3}{8}x_2^2[x_1^2]_0^1 + \frac{1}{2}x_2[x_1^2]_0^1 \cdot dx_2 =$$

$$= \int_0^1 \frac{3}{16}x_2 + \frac{3}{8}x_2^2 + \frac{1}{2}x_2 \cdot dx_2 = \int_0^1 \frac{3}{8}x_2^2 + \frac{11}{16}x_2 \cdot dx_2 = \frac{1}{8}[x_2^3]_0^1 + \frac{11}{12}[x_2^2]_0^1 =$$

$$= \frac{1}{8} + \frac{11}{12} = \boxed{\frac{25}{24}} \qquad \frac{5}{16}$$

c) $$f(x_3,x_4 \mid x_1,x_2) = \frac{f(x_1,x_2,x_3,x_4)}{f(x_1,x_2)}$$

$$f(x_3,x_4 \mid x_1,x_2) = \frac{\cancel{\frac{3}{4}}(x_1^2+x_2^2+x_3^2+x_4^2)}{\cancel{\frac{3}{4}}\left(x_1^2+x_2^2+\frac{2}{3}\right)} = \frac{x_1^2+x_2^2+x_3^2+x_4^2}{x_1^2+x_2^2+\frac{2}{3}}$$

ok

$$f\left(x_3,x_4 \mid x_1=\frac{1}{3}, x_2=\frac{2}{3}\right) = \frac{\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+x_3^2+x_4^2}{\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+\frac{2}{3}} = \frac{\frac{5}{9}+x_3^2+x_4^2}{\frac{11}{9}} \qquad 0<x_3<1,\ 0<x_4<1$$

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$$P\left(X_3 > \tfrac{3}{4}, X_4 > \tfrac{1}{2} \mid X_1 = \tfrac{1}{3}, X_2 = \tfrac{2}{3}\right) = \int_{1/2}^{1}\int_{3/4}^{1} \frac{\frac{5}{9} + x_3^2 + x_4^2}{\frac{11}{9}} \cdot dx_3 \cdot dx_4 =$$

$$= \frac{9}{11}\int_{1/2}^{1}\int_{3/4}^{1} \frac{5}{9} + x_3^2 + x_4^2 \cdot dx_3 \cdot dx_4 =$$

$$= \frac{9}{11}\int_{1/2}^{1} \frac{5}{9}[x_3]_{3/4}^{1} + \frac{1}{3}[x_3^3]_{3/4}^{1} + x_4^2[x_3]_{3/4}^{1} \cdot dx_4 =$$

$$= \frac{9}{11}\int_{1/2}^{1} \frac{5}{36} + \frac{37}{192} + x_4^2\left(1 - \frac{3}{4}\right) \cdot dx_4 = \frac{9}{11}\int_{1/2}^{1} \frac{191}{576} + \frac{1}{4}x_4^2 \cdot dx_4 =$$

$$\frac{9}{11}\left(\frac{191}{576}(x_4)_{1/2}^{1} + \frac{1}{12}[x_4^3]_{1/2}^{1}\right) = \frac{9}{11}\left(\frac{191}{1152} + \frac{7}{96}\right) = \boxed{\frac{25}{128}} \quad \boxed{\frac{203}{1408}}$$

Lärarens kommentar: (Teacher's note)

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Lärarens kommentar: (*Teacher's note*)

EXERCISE 8:

$$f(x,y) = \begin{cases} c \cdot xy(1-x^2), & 0<x<1, \ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^1\int_0^1 c\,xy(1-x^2)\cdot dx \cdot dy = 1$$ OK

$$\int_0^1\int_0^1 cxy - cx^3 y \cdot dx \cdot dy = c\int_0^1\int_0^1 xy - x^3y \cdot dx \cdot dy =$$

$$= c\int_0^1 \frac{y}{2}[x^2]_0^1 - \frac{y}{4}[x^4]_0^1 \cdot dy = c\int_0^1 \frac{y}{2} - \frac{y}{4} \cdot dy = c\int_0^1 \frac{1}{4}y \cdot dy =$$

$$= \frac{1}{8}[y^2]_0^1 = \frac{1}{8}\cdot c$$

There fore:

$$\frac{1}{8}\cdot c = 1 \iff c = 8$$ //OK

$U = X \cdot Y \qquad V = X$ OK

$x = v \qquad xy = u$

$x = v \qquad vy = u$

$x = v \qquad y = \frac{u}{v}$

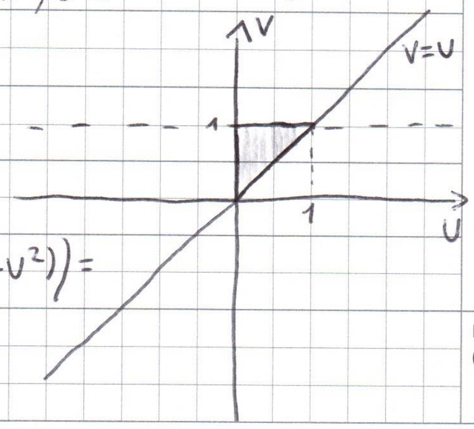
$$J = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & '' \end{vmatrix} = \left|-\frac{1}{v}\right| = \frac{1}{v}$$

$$f_{UV}(u,v) = 8\cancel{v}\cdot\frac{u}{\cancel{v}}(1-v^2)\frac{1}{v} = 8u(1-v^2)\cdot\frac{1}{v} \qquad 0<v<1, \ 0<u<v$$

In order to find the pdf of $X \cdot Y$, we need to calculate the following integral:

$$f_U(u) \overset{?}{=} \int_u^1 8u(1-v^2)\cdot\frac{1}{v}\cdot dv = 8u\int_u^1 \frac{1}{v} - v \cdot dv =$$

OK marginal



$$= 8u\cdot\left([\ln(v)]_u^1 - \frac{1}{2}[v^2]_u^1\right) = 8u\left(0 - \ln(u) - \frac{1}{2}(1-v^2)\right) =$$

$$= \cancel{8u} - 8u\cdot\ln(u) - 8u\cdot\frac{1}{2}(1-v^2) =$$

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$-8u \cdot \ln(u) - 8u \cdot \frac{1}{2} + 8u^3$ $\quad 0 < u < 1.$

if you take $\int_0^1 (\quad) \, du \neq 1$ //

$\pm$

TG

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Lärarens kommentar: (Teacher's note)

EXERCISE 9:

Convergence in probability:

$\lim_{n\to+\infty} P(\omega : |X_n - X| < \varepsilon) = 1$, $\forall \varepsilon > 0$, or equivalently:

$\lim_{n\to+\infty} P(\omega : |X_n - X| \geq \varepsilon) = 0$

Therefore:

$\lim_{n\to+\infty} P(|\bar{X}_n - \mu| \geq \varepsilon) = 0$

$P(|\bar{X}_n - \mu| \geq \varepsilon) = P((\bar{X}_n - \mu)^2 \geq \varepsilon^2)$, since both quantities are positive.

Now we can use the Chebyshev's inequality:

$$P((\bar{X}_n - \mu)^2 \geq \varepsilon^2) \leq \frac{E[(\bar{X}_n - \mu)^2]}{\varepsilon^2}$$

$$E[(\bar{X}_n - \mu)^2] = V[\bar{X}_n] = \frac{\sigma^2}{n}$$

Thus

$P((\bar{X}_n - \mu)^2 \geq \varepsilon^2) \leq \frac{\sigma^2}{n \cdot \varepsilon^2} \to$ when $n \to +\infty$, this therm $\to 0$.

VG

Poäng: (Points)

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Regler i skrivsalen

- Följ tentamensvärds anvisningar.
- Väskor och ytterkläder ska placeras på anvisad plats.
- Placera ID-handling väl synlig på bordet framför dig.
- Ingen student får lämna skrivsalen under de första 30 minuterna.
- Endast en student i taget får besöka toaletten. Vid toalettbesök skriv ditt namn och klockslag på avsedd lista. Efter toalettbesöket ska du åter ange klockslag på listan.
- Elektronisk utrustning som mobiltelefon eller Smartwatch ska vara avstängd och placerad på anvisad plats.
- Under tentamen gäller tystnad – det är förbjudet att prata, eller på annat sätt kommunicera, med andra studenter under pågående tentamen.
- Innan tentamenshandlingarna lämnas in; skriv sidnummer, anonymiseringskod och datum på alla inlämnade papper.

Om något är oklart – fråga gärna tentamensvärden. Lycka till!

Rules in the examination hall

- Follow the invigilator's instructions.
- Bags and outerwear must be placed at the designated place.
- Place your ID document clearly visible on the table in front of you.
- No student may leave the examination hall for the first 30 minutes.
- Only one student at a time may visit the toilet. Before visiting the toilet, write your name and time on the intended list. After the toilet visit, enter the time on the list again.
- Electronic equipment such as a mobile phone or Smartwatch must be switched off and placed at the designated place.
- During the exam, silence applies – you are not allowed to talk, or otherwise communicate, with other students during the exam.
- Before submitting the examination documents; remember to write the page number, anonymization code, and date on all papers.

Please do not hesitate to ask the invigilator if anything is unclear. Good luck!

Tentamensservice
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