Stockholms universitet

OBS! Läs noga igenom anvisningarna i tentamen, t.ex. hur du ska skriva svaren. Det är ditt ansvar som student att följa de anvisningar som ges.

NOTE! Read the examination instructions carefully, e.g. how to write the answers. It is your responsibility as a student to follow the given instructions.

Skriv din anonymiseringskod och dagens datum på allt material du lämnar in.
(*Enter your anonymization code and todays date on all submitted materials*)

Anonymiseringskod (*Anonymization code*)	~~8~~	~~8~~	~~8~~	-	~~0~~	~~0~~	0	~~5~~	-	G	C	X

Datum (*Date* YYYY-MM-DD)	2021-11-29	Plats nr. *(Seat No.)*	13

Kurs/Kurskod (*Course/Course code*)	ST721A
Kursmoment (*Course component*)	ST721A 11SE

Fylls i av tentamensvärd (*To be filled in by invigilator*)

Direkt i skrivning: (kryss)		Svarsblankett: (kryss)		Lösa svarsblad: (antal)	9

Lämnat in blankt: (kryss)		Dator: (kryss)	

Inlämningstid: 15::50 Signatur tentamensvärd: MW

Fylls i av lärare/examinator (*To be filled in by teacher/examinator*)

Betyg:	(91)	Poäng:	(A)

Signatur rättande lärare/examinator: AA

Regler i skrivsalen

- Följ tentamensvärds anvisningar.
- Väskor och ytterkläder ska placeras på anvisad plats.
- Placera ID-handling väl synlig på bordet framför dig.
- Ingen student får lämna skrivsalen under de första 30 minuterna.
- Endast en student i taget får besöka toaletten. Vid toalettbesök skriv ditt namn och klockslag på avsedd lista. Efter toalettbesöket ska du åter ange klockslag på listan.
- Elektronisk utrustning som mobiltelefon eller Smartwatch ska vara avstängd och placerad på anvisad plats.
- Under tentamen gäller tystnad – det är förbjudet att prata, eller på annat sätt kommunicera, med andra studenter under pågående tentamen.
- Innan tentamenshandlingarna lämnas in; skriv sidnummer, anonymiseringskod och datum på alla inlämnade papper.

Om något är oklart – fråga gärna tentamensvärden. Lycka till!

Rules in the examination hall

- Follow the invigilator's instructions.
- Bags and outerwear must be placed at the designated place.
- Place your ID document clearly visible on the table in front of you.
- No student may leave the examination hall for the first 30 minutes.
- Only one student at a time may visit the toilet. Before visiting the toilet, write your name and time on the intended list. After the toilet visit, enter the time on the list again.
- Electronic equipment such as a mobile phone or Smartwatch must be switched off and placed at the designated place.
- During the exam, silence applies – you are not allowed to talk, or otherwise communicate, with other students during the exam.
- Before submitting the examination documents; remember to write the page number, anonymization code, and date on all papers.

Please do not hesitate to ask the invigilator if anything is unclear. Good luck!

Tentamensservice
tentamensservice@su.se | www.su.se/tentamensservice

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1. $f(x,y) = \begin{cases} x+y, & 0<x<1, \quad 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

we first control that it is a proper pdf.

we see that it is never less than 0 by definition.

we now test that it integrates to 1.

$$\int_0^1\int_0^1 x+y\,dx\,dy = \int_0^1 \frac{x^2}{2} + xy\Big|_0^1 dy = \int_0^1 \frac{1}{2}+y\,dy = \frac{y}{2}+\frac{y^2}{2}\Big|_0^1 =$$

$= \frac{1}{2}+\frac{1}{2} = 1$, it is a proper pdf ok

a) we now find pdf of $X+Y$, we use substitution.

$U = X+Y$ this gives $0<v<1$

$V = X$ $0<U-V<1$

ok

1	2	3	4	5	6	7	8	9	2
12	10	12	14	11	10	5	11	0	(85)

we are now to find $f(u)$.

we rewrite the second limit as;

$U-1<v<1$ and $0<v<U$

we now integrate over the two domains of v.

$$f_{x+y}(u) = \int_0^U U\,dv = UV\Big|_0^U = U^2$$

and

$$f_{x+y}(u) = \int_{U-1}^1 U\,dv = UV\Big|_{U-1}^1 = U-U(U-1) = U-U^2+U = 2U-U^2$$

we get the pdf; $f_{x+y}(u) = \begin{cases} U^2, & 0<u<1 \\ 2U-U^2, & 1<u<2 \\ 0, & \text{otherwise} \end{cases}$ ok

b) we are now to calculate $P(X+Y \le 1)$

we use our pdf from a).

$$P(X+Y \le 1) = \int_0^1 U^2\,du = \frac{U^3}{3}\Big|_0^1 = \frac{1}{3}$$ ok

Uppg.nr.:
(*Task no.*)

1

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

2.

X_1 = "outcome of dice one"

X_2 = "outcome of dice two"

Y = "Max of the two"

we first write all the possible combinations of X_1 and X_2

X_1 \	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

we now write the pdf for (X_1, Y)

$f(x,y) =$

Y \ X_1	1	2	3	4	5	6	$f_Y(y) =$
1	$\frac{1}{36}$	0	0	0	0	0	$\frac{1}{36}$
2	$\frac{1}{36}$	$\frac{2}{36}$	0	0	0	0	$\frac{3}{36}$
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{3}{36}$	0	0	0	$\frac{5}{36}$
4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{4}{36}$	0	0	$\frac{7}{36}$
5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	0	$\frac{9}{36}$
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{6}{36}$	$\frac{11}{36}$
$f_X(x) =$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

ok

we now calculate what is asked for.

$E(X) = \sum_1^6 x_i \cdot P(X=x) = 3,5$ ok

$E(X^2) = \sum_1^6 x_i^2 \cdot P(X=x) = 15,16\ldots 7 \approx \approx 15,167$

$E(Y) = \sum_1^6 y_i \cdot P(X=x) = 4,472\ldots \approx 4,472$

$E(Y^2) = \sum_1^6 y_i^2 \cdot P(X=x) = 21,972\ldots \approx \approx 21,972$

$Var(X) = E(X^2) - (E(X))^2 = 2,917$

$Var(Y) = E(Y^2) - (E(Y))^2 = 1,973216 \approx 1,973$

$E(XY) = \sum_1^{21} x_i \cdot y_i \cdot P(X=x, Y=y) = 17,111\ldots \approx 17,111$

$Cov(X,Y) = E(XY) - E(X)E(Y) = 1,459$ ok

VG

Poäng: (Points)

Uppg.nr.:
(*Task no.*)

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

3

we have $Y|X \sim N(X, X^2)$

$X \sim U(0,4)$

we calculate variance and expected value using formulas,

$E(Y) = E(E(Y|X)) = EX = \frac{4+0}{2} = 2$ OK

$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)) =$

$= EX^2 + VarX = \frac{16}{3} + \frac{4}{3} = \frac{20}{3}$ OK

we calculate EX^2 separately.

$EX^2 = \int_0^4 \frac{x^2}{4} dx = \frac{1}{4}\int_0^4 x^2 dx = \frac{1}{4} \cdot \left(\frac{x^3}{3}\Big|_0^4\right) = \frac{16}{3}$ OK - 20/3 ?

For the $Cov(X,Y)$ we need $E(XY)$.

$E(XY) = E(XE(Y|X)) = E(X \cdot X) = E(X^2)$

So we get;

$Cov(X,Y) = E(X^2) - E(X)E(Y) = E(X^2) - E(X) \cdot E(X) =$

$= E(X^2) - (E(X))^2 = \frac{4}{3}$ OK -

Uppg.nr.:
(*Task no.*)

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

Uppg.nr.: (*Task no.*) 4

Lärarens kommentar: (*Teacher's note*)

$E(2x) = 2E(x) = 2 \cdot \frac{1}{2} = 1$

$E(2x^2) = 2 \cdot E(x^2) = 2 \cdot \int_0^1 x^2 dx = 2 \cdot \left(\frac{x^3}{3}\Big|_0^1\right) = 2 \cdot \frac{1}{3} = \frac{2}{3}$

$Var(2x) = 4 \cdot Var(x) = 4 \cdot \left(\frac{1}{3} - \frac{1}{4}\right) = 4 \cdot \frac{1}{12} = \frac{1}{3}$ ok –

b) We know that $Corr(2x, Y) = Corr(X, Y)$ so we calculate that.

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

we need to calculate $E(X,Y)$

$$E(XY) = \int_0^1 \int_y^{y+1} xy \, dx \, dy = \int_0^1 \frac{x^2}{2} y \Big|_y^{y+1} dy = \int_0^1 \frac{(y+1)^2}{2} \cdot y - \frac{y^3}{2} dy =$$

$$= \int_0^1 \frac{y^3 + 2y^2 + y}{2} - \frac{y^3}{2} dy = \int_0^1 y^2 + \frac{y}{2} dy = \frac{y^3}{3} + \frac{y^2}{4}\Big|_0^1 = \frac{7}{12}$$

$$Cov(X,Y) = \frac{7}{12} - 1 \cdot \frac{1}{2} = \frac{1}{12}$$

This gives

$$Corr(2x, Y) = Corr(X,Y) = \frac{1/12}{\sqrt{\frac{1}{12}} \cdot \sqrt{\frac{1}{6}}} = \frac{1/12}{\sqrt{\frac{1}{72}}} = \frac{\sqrt{\frac{1}{144}}}{\sqrt{\frac{1}{72}}} = \sqrt{\frac{1}{144}} \cdot \sqrt{\frac{72}{1}} = \sqrt{\frac{72}{144}} =$$

ok

$$= \frac{1}{\sqrt{2}}$$

Poäng: (*Points*)

Uppg.nr.: (Task no.) 4

Lärarens kommentar: (Teacher's note)

4.

$$f(x,y)=\begin{cases}1, & 0<y<1,\ y<x<y+1\\ 0, & \text{otherwise}\end{cases}$$

we first test that it is proper.

we see that it obviously is never less than 0 by definition.

we now test to see that it integrates to 1.

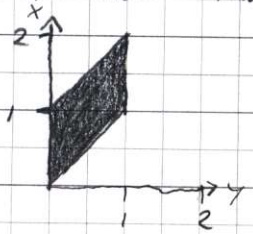
$$\int_0^1\int_y^{y+1} 1\,dx\,dy = \int_0^1 x\Big|_y^{y+1}\,dy = \int_0^1 y+1-y\,dy = \int_0^1 1\,dy = y\Big|_0^1 = 1 \quad \text{ok}$$

It is proper.

a) we first find for X.

$$f_x(x)=\int_0^1 1\,dy = y\Big|_0^1 = 1, \quad X \text{ is } U(0,1) \text{ ok.}$$

We now find for Y. For this we utilize a drawing.



To find the marginals of y we will integrate x over $y<x<2$ and $0<x<y$

we get $f_y(y)$ for 2 different limits.

$$f_y(y)=\int_y^2 1\,dx = x\Big|_y^2 = 2-y, \quad 1<y<2$$

$$f_y(y)=\int_0^y 1\,dx = x\Big|_0^y = y, \quad 0<y<1$$

I believe, you flip-flopped $X \leftrightarrow Y$

$U[0,1]$, not X

so we $f_x(2x)=\begin{cases}\frac{1}{2}, & 0<x<2\\ 0, & \text{otherwise}\end{cases}$

overall ok

and $f_y(y)=\begin{cases}y, & 0<y<1\\ 2-y, & 1<y<2\\ 0, & \text{otherwise}\end{cases}$ ok.

$$E(Y)=\int_0^1 y^2\,dy+\int_1^2 2y-y^2\,dy = \frac{y^3}{3}\Big|_0^1 + y^2-\frac{y^3}{3}\Big|_1^2 = \frac{1}{3}+4-\frac{8}{3}-1+\frac{1}{3}=1$$

$$E(Y^2)=\int_0^1 y^3\,dy+\int_1^2 2y^2-y^3\,dy = \frac{y^4}{4}\Big|_0^1 + \frac{2y^3}{3}-\frac{y^4}{4}\Big|_1^2 = \frac{1}{4}+\frac{16}{3}-4-\frac{2}{3}+\frac{1}{4}=\frac{7}{6}$$

$$\text{Var}(Y)=\frac{7}{6}-1^2=\frac{1}{6}$$

V.G.V.

P.T.O

Poäng: (Points)

Uppg.nr.: (Task no.) 5

Lärarens kommentar: (Teacher's note)

5.

X has pdf $f(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

a)

we first test that it is proper.

we see that it is never less than 0 by definition.

we now integrate to see if it goes to one.

$$\int_{-1}^{1} \frac{x+1}{2}\,dx = \frac{1}{2}\int_{-1}^{1} x+1\,dx = \frac{1}{2}\left(\frac{x^2}{2}+x\Big|_{-1}^{1}\right) = \frac{1}{2}\left(\frac{1}{2}+1-\frac{1}{2}+1\right) = \frac{1}{2}\cdot 2 = 1$$ ok

It is a proper pdf.

The monotone $u(x)$ so that $Y \sim U(0,1)$ is the primitive.

$$u(x) = F(x) = \begin{cases} 0, & x < -1 \\ \frac{x^2+2x}{4}, & -1 < x < 1 \\ 1, & x > 1 \end{cases}$$ ← something wrong? $x=1 \mapsto F(x) = \frac{3}{4}$ ±

b)

we will now find the mgf of $X \sim \Gamma(\alpha, \beta)$.

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\beta}, \quad 0 \le x < \infty, \quad \alpha, \beta > 0$$

$$M_x(t) = Ee^{tx} = \int_0^\infty e^{tx} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\beta}\,dx =$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \int_0^\infty x^{\alpha-1} \cdot e^{tx - \frac{x}{\beta}}\,dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \Gamma(\alpha) \cdot \left(\frac{\beta}{1-\beta t}\right)^\alpha =$$

$$= \left(\frac{\beta}{\beta - \beta^2 t}\right)^\alpha = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta \text{ ok.}$$

we now find $E(X)$ using the MGF

$$E(X) = M_x'(0) = \alpha\left(\frac{1}{1-\beta t}\right)^{\alpha-1} \cdot \beta t\Big|_{t=0} = \alpha \cdot 1 \cdot \beta = \alpha\beta$$ ok +

VG

Poäng: (Points)

Uppg.nr.:
(*Task no.*)

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

Uppg.nr.: (Task no.) 6

Lärarens kommentar: (Teacher's note)

6.

A and B are hiking, they arrive at a place independently and randomly within a span of 24 hours.

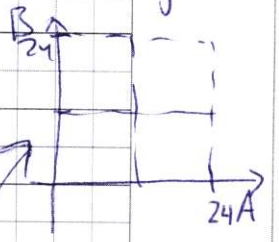
X is the time A waits for B.

A and B are uniformly distributed on the square ok

$(0,24)\times(0,24]$

we first calculate if B arrives before A. $X=0$

Can you draw the setting?



$$P(X=x)=\int_0^{24}\int_0^{a}\frac{1}{576}\,db\,da=\frac{1}{576}\int_0^{24}\int_0^{a}1\,db\,da=\frac{1}{576}\cdot\int_0^{24}b\Big|_0^{a}\,da=$$

$$=\frac{1}{576}\int_0^{24}a\,da=\frac{1}{576}\cdot\frac{a^2}{2}\Big|_0^{24}=\frac{1}{576}\cdot 288=\frac{1}{2}$$

we now calculate for if A arrives before B.

$$P(0<X<24)=1-P(B-A>X)=1-\int_0^{24-X}\int_{a+X}^{24}\frac{1}{576}\,da\,db=$$

$$=1-\frac{1}{576}\int_0^{24-X}\int_{a+X}^{24}1\,db\,da=1-\frac{1}{576}\int_0^{24-X}b\Big|_{a+X}^{24}\,da=$$

$$=1-\frac{1}{576}\int_0^{24-X}24-a-x\,da=1-\frac{1}{576}\left(24a-\frac{a^2}{2}-ax\Big|_0^{24-X}\right)$$

$$=1-\frac{1}{576}\left(24(24-x)-\frac{(24-x)^2}{2}-(24-x)x\right)=$$

$$=1-\frac{1}{576}\left(576-24x-288+24x-\frac{x^2}{2}-24x+x^2\right)$$

$$=1-\frac{1}{576}\left(288-24x+\frac{x^2}{2}\right)=1-\frac{1}{2}+\frac{x}{24}-\frac{x^2}{576}=$$

$$=\frac{1}{2}+\frac{x}{24}-\frac{x^2}{576}$$

, what is the "domain" $x=24$

$\frac{1}{2}+1-1=\frac{1}{2}$??

$$F_X(x)=\begin{cases}0, & \\ 1/2, & \\ \quad, & \\ \quad, & \\ 1, & \end{cases}$$

± Overall, ok

Poäng: (Points)

Uppg.nr.:
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Lärarens kommentar:
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Poäng:
(*Points*)

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Uppg.nr.: (Task no.) 9

Lärarens kommentar: (Teacher's note)

Poäng: (Points)

9.

We have a sequence of r.v. X_n it converges in distribution to a constant,

$$P(X_n \leq x) \to \begin{cases} 0 \text{ if } x < c \\ 1 \text{ if } x > c \end{cases}$$

This is equivalent to

$P(|X_n - c| > \epsilon) \to 0$ for every $\epsilon > 0$

Showing that in this particular case it also converges in probability.

(?)

Uppg.nr.:
(*Task no.*)

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)