Stockholms universitet

STATISTISKA INSTITUTIONEN HT-2020

Written re-exam in Probability Theory, 7.5 ECTS credits
Tuesday, 5th of January 2021, 13:00 – 18:30
Time allowed: FIVE hours + 30 minutes for upload
Examination: Home Exam

You are asked to answer below stated questions as well as motivate your solutions. The total amount of points is 100. Grades are assigned as follows: **A** (91+), **B** (75-90), **C** (66-74), **D** (58-65), **E** (50-57), **Fx** (30-49), and **F** (0-29)

You are **allowed** to use ALL available material but do it individually: no help from other person.

The teacher reserves the right to examine the students **orally/ZOOM-based** on the answers provided.

1. (8 points) Let X and Y be *i.i.d* standard normal random variables, and $U:=X+Y;\ V:=X-Y$
   a) Calculate the joint pdf: $f_{U,V}(u,v)$
   b) Prove that U and V are independent

2. (12 points)

a) Let random variable X have a pdf $f(x)=\begin{cases}\frac{x+2}{3}, & 1<x<4\\ 0, & otherwise\end{cases}$

Find a monotone function $u(x)$ such that the random variable $Y=u(X)$ has a *uniform(0,1)* distribution.

b) Check that below stated function is a pdf and find the moment generating function corresponding to $f(x)=\frac{1}{2\beta}\exp\{\frac{-|x-a|}{\beta}\},-\infty<x,\alpha<\infty,\beta>0$. Make sure to provide detailed explanations.

3. (12 points) Let the distribution of Y conditional on $X=x$ be $N(x, x^2)$ $[Y\,|\,X=x]\sim N(x,x^2)$ and the marginal distribution of X be $U(0,3)$. Find $E[Y]$, $Var(Y)$ and $Cov(X, Y)$.

4. (12 points) One tosses two dice: the outcomes are numbers from 1 to 6. Let X be the "outcome" on the first dice and Y is the minimum of the two. Find joint distribution of (X,Y) as calculate $E[X]$, $E[Y]$, $Var(X)$, $Var(Y)$, and $Cov(X,Y)$.

5. (10 points) Let $f(x,y)=\begin{cases}x+y, & 0<x<1,\quad 0<y<1\\ 0, & otherwise\end{cases}$

   a) Find the probability density function of $X+Y$
   b) Calculate $P(X+Y\le 0.5)$

6. (12 points) Let the joint pdf of (X,Y) be $f(x,y)=1,\ 0<y<1,\ y<x<y+1$. Find
   a) marginals of $3*X$ and Y and their first two moments;
   b) $Corr(3*X,Y)$

7. (12 points) A and B are hiking and agree to meet at a certain place between 13:00 and 18:00. Let us suppose they arrive at the meeting place independently and randomly and that both make it during agreed time interval. Find the distribution of the length of time that A waits for B. (If B arrives before A, define A's waiting time as zero).

8. (12 points)) Let $f(x,y) = \begin{cases} 8xy(1-x^2), & 0<x<1,\ 0<y<1 \\ 0, & otherwise \end{cases}$

   Find the probability density function of $X*Y$.

9. (10 points) Let us assume that sequence of random variable X_n converges in distribution to a constant c. Show that it also converges in probability to the same constant c. In other words, converges in probability and convergence in distribution are equivalent in this particular case. (Hint: start with writing the limiting distribution explicitly as a cdf).

Good Luck