Stockholms universitet

STATISTISKA INSTITUTIONEN HT-2020

Written re-exam in Probability Theory, 7.5 ECTS credits
Tuesday, 1st of December 2020, 13:00 – 18:30
Time allowed: FIVE hours + 30 minutes for upload
Examination: Home Exam

You are asked to answer below stated questions as well as motivate your solutions. The total amount of points is 100. Grades are assigned as follows: **A** (91+), **B** (75-90), **C** (66-74), **D** (58-65), **E** (50-57), **Fx** (30-49), and **F** (0-29)

You are **allowed** to use ALL available material but do it individually: no help from other person.

The teacher reserves the right to examine the students **orally/ZOOM-based** on the answers provided.

1. (15 points)) Let the distribution of Y conditional on $X = x$ be $N(x, x^2)$ $[Y \mid X = x] \sim N(x, x^2)$ and the marginal distribution of X be $U(0,2)$
   a) Find $E[Y]$, $Var(Y)$ and $Cov(X, Y)$
   b) Prove that $\frac{Y}{X}$ and X are independent

   Note: Solutions manual contains a very short answer to part b) and part of it is miss-leading. Please, attempt to do better than that. Grading will be done on how well you explain your answer.

2. (12 points)

   a) (8 points) One tosses two dice: the outcomes are numbes from 1 to 6. Let X be the "outcome" on the first dice and Y is the max of the two. Find joint distribution of (X,Y) as calculate $E[X]$, $E[Y]$, $Var(X)$, $Var(Y)$, and $Cov(X,Y)$.
   b) (4 points) Let $\bar{X}_1$ and $\bar{X}_2$ be respective means of two independent samples of size n drawn from a population having variance σ^2. Find the value of n such that $P(\left|\bar{X}_1 - \bar{X}_2\right| < \sigma) \approx 0.9$. Please, justify your calculations. How your calculation of the above probability would change if both samples were quadrupled (4 times larger = "$4n$"). Provide calculation and give intuitive explanation for your answer.

3. (12 points) Let $f(x, y) = \begin{cases} x + y, & 0 < x < 1, \quad 0 < y < 1 \\ 0, & otherwise \end{cases}$
   a) Find the probability density function of $X+Y$
   b) Calculate $P(X + Y \leq 1)$

4. (15 points) Let the joint pdf of (X,Y) be $f(x,y)=1$, $0<y<1$, $y<x<y+1$. Find
   a) marginals of $2*X$ and Y and their first two moments;
   b) $Corr(2*X,Y)$

5. (12 points)

a) Let random variable X have a pdf $f(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1 \\ 0, & otherwise \end{cases}$

Find a monotone function *u(x)* such that the random variable *Y=u(X)* has a *uniform(0,1)* distribution.

b) Show that $F_X(x) = \begin{cases} \frac{e^x}{4}, & x < 0 \\ 1 - \frac{e^{-x}}{4}, & x \ge 0 \end{cases}$ is a cdf and find $F_X^{-1}(y)$

6. (12 points) *A* and *B* are hiking and agree to meet at a certain place on a certain day (24 hours). Let us suppose they arrive at the meeting place independently and randomly during these 24 hours. Find the distribution of the length of time that *A* waits for *B*. (If B arrives before A, define A's waiting time as zero).

7. (12 points)) Let $f(x,y) = \begin{cases} 24xy(1-x^2), & 0 < x < 1,\ 0 < y < 1 \\ 0, & otherwise \end{cases}$

Find the probability density function of $X*Y$.

8. (10 points) Let us assume that sequence of random variable X_n converges in distribution to a constant *c*. Show that it also converges in probability to the same constant *c*. In other words, converges in probability and convergence in distribution are equivalent in this particular case. (Hint: start with writing the limiting distribution explicitly as a cdf).

Good Luck