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Department of Statistics
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EXAM IN STATISTICAL INFERENCE
2022-01-14

Time: 8.00 – 13.00.
Allowed tools: Pocket calculator, tables and formulas attached
The exam consists of five problems. Clear, detailed and well motivated solutions to the problems are expected.
Examples of solutions will be published at Athena shortly after the exam.

Problem 1. (20 p)

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a population with one of the following densities
a)

$$f(x;\theta) = \theta x^{\theta-1},\ 0 < x < 1,\ \theta > 0.$$

This is the *beta* $(\theta, 1)$ distribution.
b)

$$f(x;\theta) = \theta a x^{a-1} e^{-\theta x^a},\ x > 0,\ \theta > 0,\ a > 0,$$

where a is a fixed and known number. This is the Weibull distribution.
c)

$$f(x;\theta) = \frac{\theta a^\theta}{x^{\theta+1}},\ x > a,\ \theta > 0,\ a > 0,$$

where a is a fixed and known number, This is the Pareto distribution.
In each case, find a real-valued sufficient statistic for θ.

Problem 2. (20 p)

A psychologist wishes to study whether color influences an individual's choice of an item. She therefore conduct an experiment in which cinnamon buns are placed in bowls with three different colors (red, blue, and green) and a person is supposed to chose a bun from one of the three bowls. Let p_R, p_B, and p_G be the probabilities of choosing a bun from the red, the blue and the green bowl, respectively.

a) Assume that choices from N persons are recorded and that the persons' choices are independent of each other. Based on these data, derive the maximum likelihood estimator of p_R, p_B, and p_G.

b) Derive the expected values and variances of the maximum likelihood estimator of p_R, p_B, and p_G.

Problem 3 (20 p)

A tennis player counts the number of times she can hit the ball correctly. Suppose that the probability of a correct hit is constant $1 - p$ for all hits, so that the number of correct hits and a failure hit, X, follows a geometric distribution with pmf

$$f(x; p) = (1 - p)^{x-1} p, \ 0 < p < 1, \ x = 1, 2, 3, \dots$$

a) Based on n observations, $x_1, x_2, \dots, x_n$, find the maximum likelihood estimator of p.

b) Find the asymptotic distribution of the maximum likelihood estimator, possibly by using the Delta Method.

c) Derive the Fisher information $I(p)$ and verify that the variance of the maximum likelihood estimator of p asymptotically attains $1/I(p)$.

Problem 4. (20 p)

The psychologist in Problem 2 wishes to test the hypothesis that persons choose the bowls with equal probabilities, i.e. she wishes to test the hypothesis

$$H_0 : p_R = p_B = p_G$$

versus at least one of the probabilities differ from at least one of the others.
a) Construct the test statistic for the likelihood ratio test for testing this hypothesis.
b) Derive a large sample version of the likelihood ratio test statistic and state its asymptotic distribution.

Problem 5. (20 p)

For evaluating the precision of a measurement instrument, the instrument is applied for measuring a known standard. Suppose that the measurements, $X_1, X_2, \dots, X_n$ are stochastically independent and the measurement errors are normally distributed so that each observation is $N(\mu, \sigma^2)$ with μ known. Because a more serious error is to judge the precision adequate when it is not, we wish to test $H_0 : \sigma \geq \sigma_0$ versus $H_A : \sigma < \sigma_0$, where σ_0^{-1} represents a minimum tolerable precision.
a) Derive a most powerful test for testing this hypothesis.

b) Find the sampling distribution for the test statistic (or a transformation of it) derived in problem a. This distribution should be valid for finite samples. Using this distribution, indicate how a critical value is found.

Formelblad inferensteori HT-2021

Gamma function:

$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$; $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$; $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$; If α is an integer: $\Gamma(n) = (n-1)!$

Conditional pdf:

$$f(y|x) = \frac{f(x,y)}{f_X(x)} \quad ; \quad f(x|y) = \frac{f(x,y)}{f_y(y)}$$

Variance:

$$\text{Var } X = \text{E } X^2 - (\text{E } X)^2,$$

Mean Squared Error:

$$\text{E}_\theta(W-\theta)^2 = \text{Var}_\theta W + (\text{E}_\theta W - \theta)^2 = \text{Var}_\theta W + (\text{Bias}_\theta W)^2$$

Univariate transformation:

$$f_Y(y) = \begin{cases} \sum_{i=1}^k f_X(g_i^{-1}(y)) \left|\frac{d}{dy} g_i^{-1}(y)\right|, & y \in Y \\ 0, & Otherwise \end{cases}$$

Theorem 3.4.2 *If X is a random variable with pdf or pmf of the form (3.4.1), then*

(3.4.4) $$\text{E}\left(\sum_{i=1}^k \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\frac{\partial}{\partial \theta_j} \log c(\boldsymbol{\theta});$$

(3.4.5) $$\text{Var}\left(\sum_{i=1}^k \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\frac{\partial^2}{\partial \theta_j^2} \log c(\boldsymbol{\theta}) - \text{E}\left(\sum_{i=1}^k \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(X)\right).$$

Theorem 5.5.24 (Delta Method) *Let Y_n be a sequence of random variables that satisfies $\sqrt{n}(Y_n - \theta) \to \text{n}(0, \sigma^2)$ in distribution. For a given function g and a specific value of θ, suppose that $g'(\theta)$ exists and is not 0. Then*

(5.5.10) $$\sqrt{n}[g(Y_n) - g(\theta)] \to \text{n}(0, \sigma^2[g'(\theta)]^2) \text{ in distribution.}$$

Theorem 6.2.6 (Factorization Theorem) *Let $f(\mathbf{x}|\theta)$ denote the joint pdf or pmf of a sample $\mathbf{X}$. A statistic $T(\mathbf{X})$ is a sufficient statistic for θ if and only if there exist functions $g(t|\theta)$ and $h(\mathbf{x})$ such that, for all sample points $\mathbf{x}$ and all parameter points θ,*

$$f(\mathbf{x}|\theta) = g(T(\mathbf{x})|\theta)h(\mathbf{x}). \tag{6.2.3}$$

Theorem 6.2.10 *Let $X_1, \ldots, X_n$ be iid observations from a pdf or pmf $f(x|\boldsymbol{\theta})$ that belongs to an exponential family given by*

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta}) \exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right),$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_d)$, $d \leq k$. Then

$$T(\mathbf{X}) = \left(\sum_{j=1}^{n} t_1(X_j), \ldots, \sum_{j=1}^{n} t_k(X_j)\right)$$

is a sufficient statistic for $\boldsymbol{\theta}$.

Theorem 6.2.13 *Let $f(\mathbf{x}|\theta)$ be the pmf or pdf of a sample $\mathbf{X}$. Suppose there exists a function $T(\mathbf{x})$ such that, for every two sample points $\mathbf{x}$ and $\mathbf{y}$, the ratio $f(\mathbf{x}|\theta)/f(\mathbf{y}|\theta)$ is constant as a function of θ if and only if $T(\mathbf{x}) = T(\mathbf{y})$. Then $T(\mathbf{X})$ is a minimal sufficient statistic for θ.*

Definition 6.2.21 Let $f(t|\theta)$ be a family of pdfs or pmfs for a statistic $T(\mathbf{X})$. The family of probability distributions is called *complete* if $\mathrm{E}_\theta g(T) = 0$ for all θ implies $P_\theta(g(T) = 0) = 1$ for all θ. Equivalently, $T(\mathbf{X})$ is called a *complete statistic*.

Theorem 6.2.24 (Basu's Theorem) *If $T(\mathbf{X})$ is a complete and minimal sufficient statistic, then $T(\mathbf{X})$ is independent of every ancillary statistic.*

Method of moments:

Let $X_1, \ldots, X_n$ be a sample from a population with pdf or pmf $f(x|\theta_1, \ldots, \theta_k)$. Method of moments estimators are found by equating the first k sample moments to the corresponding k population moments, and solving the resulting system of simultaneous equations. More precisely, define

$$\begin{aligned} m_1 &= \frac{1}{n}\sum_{i=1}^{n} X_i^1, \quad \mu'_1 = \mathrm{E}X^1, \\ m_2 &= \frac{1}{n}\sum_{i=1}^{n} X_i^2, \quad \mu'_2 = \mathrm{E}X^2, \\ &\vdots \\ m_k &= \frac{1}{n}\sum_{i=1}^{n} X_i^k, \quad \mu'_k = \mathrm{E}X^k. \end{aligned} \tag{7.2.1}$$

Maximum Likelihood Estimation:

$$L(\theta|\mathbf{x}) = L(\theta_1, \ldots, \theta_k|x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i|\theta_1, \ldots, \theta_k). \tag{7.2.3}$$

Theorem 7.2.10 (Invariance property of MLEs) *If $\hat{\theta}$ is the MLE of θ, then for any function $\tau(\theta)$, the MLE of $\tau(\theta)$ is $\tau(\hat{\theta})$.*

Theorem 7.3.9 (Cramér–Rao Inequality) *Let $X_1, \ldots, X_n$ be a sample with pdf $f(\mathbf{x}|\theta)$, and let $W(\mathbf{X}) = W(X_1, \ldots, X_n)$ be any estimator satisfying*

$$\frac{d}{d\theta} \mathrm{E}_\theta W(\mathbf{X}) = \int_{\mathcal{X}} \frac{\partial}{\partial\theta} [W(\mathbf{x}) f(\mathbf{x}|\theta)] \; d\mathbf{x}$$

(7.3.4) *and*

$$\mathrm{Var}_\theta W(\mathbf{X}) < \infty.$$

Then

$$\mathrm{Var}_\theta \; (W(\mathbf{X})) \geq \frac{\left(\frac{d}{d\theta} \mathrm{E}_\theta W(\mathbf{X})\right)^2}{\mathrm{E}_\theta \left(\left(\frac{\partial}{\partial\theta} \log f(\mathbf{X}|\theta)\right)^2\right)}. \tag{7.3.5}$$

Corollary 7.3.10 (Cramér–Rao Inequality, iid case) *If the assumptions of Theorem 7.3.9 are satisfied and, additionally, if $X_1, \ldots, X_n$ are iid with pdf $f(x|\theta)$, then*

$$\mathrm{Var}_\theta \, W(\mathbf{X}) \geq \frac{\left(\frac{d}{d\theta} \mathrm{E}_\theta W(\mathbf{X})\right)^2}{n\mathrm{E}_\theta \left(\left(\frac{\partial}{\partial\theta} \log f(X|\theta)\right)^2\right)}.$$

Lemma 7.3.11 *If $f(x|\theta)$ satisfies*

$$\frac{d}{d\theta} \mathrm{E}_\theta \left(\frac{\partial}{\partial\theta} \log f(X|\theta)\right) = \int \frac{\partial}{\partial\theta} \left[\left(\frac{\partial}{\partial\theta} \log f(x|\theta)\right) f(x|\theta)\right] \, dx$$

(true for an exponential family), then

$$\mathrm{E}_\theta \left(\left(\frac{\partial}{\partial\theta} \log f(X|\theta)\right)^2\right) = -\mathrm{E}_\theta \left(\frac{\partial^2}{\partial\theta^2} \log f(X|\theta)\right).$$

Corollary 7.3.15 (Attainment) *Let $X_1, \ldots, X_n$ be iid $f(x|\theta)$, where $f(x|\theta)$ satisfies the conditions of the Cramér–Rao Theorem. Let $L(\theta|\mathbf{x}) = \prod_{i=1}^n f(x_i|\theta)$ denote the likelihood function. If $W(\mathbf{X}) = W(X_1, \ldots, X_n)$ is any unbiased estimator of $\tau(\theta)$, then $W(\mathbf{X})$ attains the Cramér–Rao Lower Bound if and only if*

$$a(\theta)[W(\mathbf{x}) - \tau(\theta)] = \frac{\partial}{\partial\theta} \log L(\theta|\mathbf{x}) \tag{7.3.12}$$

for some function $a(\theta)$.

Theorem 7.3.17 (Rao–Blackwell) *Let W be any unbiased estimator of $\tau(\theta)$, and let T be a sufficient statistic for θ. Define $\phi(T) = \mathrm{E}(W|T)$. Then $\mathrm{E}_\theta \phi(T) = \tau(\theta)$ and $\mathrm{Var}_\theta\, \phi(T) \leq \mathrm{Var}_\theta\, W$ for all θ; that is, $\phi(T)$ is a uniformly better unbiased estimator of $\tau(\theta)$.*

Theorem 7.3.23 *Let T be a complete sufficient statistic for a parameter θ, and let $\phi(T)$ be any estimator based only on T. Then $\phi(T)$ is the unique best unbiased estimator of its expected value.*

Likelihood ratio test statistic:

Definition 8.2.1 The *likelihood ratio test statistic* for testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_0^c$ is

$$\lambda(\mathbf{x}) = \frac{\sup_{\Theta_0} L(\theta|\mathbf{x})}{\sup_{\Theta} L(\theta|\mathbf{x})}.$$

Theorem 8.3.12 (Neyman–Pearson Lemma) *Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where the pdf or pmf corresponding to θ_i is $f(\mathbf{x}|\theta_i), i = 0, 1$, using a test with rejection region R that satisfies*

$$\mathbf{x} \in R \quad \text{if} \quad f(\mathbf{x}|\theta_1) > k f(\mathbf{x}|\theta_0)$$

(8.3.1) *and*

$$\mathbf{x} \in R^c \quad \text{if} \quad f(\mathbf{x}|\theta_1) < k f(\mathbf{x}|\theta_0),$$

for some $k \geq 0$, and

(8.3.2) $$\alpha = P_{\theta_0}(\mathbf{X} \in R).$$

Then

a. *(Sufficiency) Any test that satisfies (8.3.1) and (8.3.2) is a UMP level α test.*

b. *(Necessity) If there exists a test satisfying (8.3.1) and (8.3.2) with $k > 0$, then every UMP level α test is a size α test (satisfies (8.3.2)) and every UMP level α test satisfies (8.3.1) except perhaps on a set A satisfying $P_{\theta_0}(\mathbf{X} \in A) = P_{\theta_1}(\mathbf{X} \in A) = 0$.*

Neyman-pearsons lemma (NPL)

Antag att vi har två enkla hypoteser:

H0: theta=theta_0 och HA: theta=theta_A

För ett givet alfa så definieras det starkaste testet av:

Förkasta H0 om L(theta_0)/L(Theta_A) < C

Där värdet på C väljs så att P(X RR;theta=theta0)=alfa

Pivot:

Definition 9.2.6 A random variable $Q(\mathbf{X},\theta) = Q(X_1,\ldots,X_n,\theta)$ is a *pivotal quantity* (or *pivot*) if the distribution of $Q(\mathbf{X},\theta)$ is independent of all parameters. That is, if $\mathbf{X} \sim F(\mathbf{x}|\theta)$, then $Q(\mathbf{X},\theta)$ has the same distribution for all values of θ.

Pivotmetoden för konfidensintervall
Q(X;theta) är en pivot om det är en funktion av observationerna X och av parametern theta, men dess fördelning får inte bero på theta.

Allmänt kan vi då bilda ett konfidensintervall genom:
P(a <= Q(X,theta) <= b) = 1-alfa

Finns oändligt många möjligheter att välja a och b men oftast vill man välja det så att intervallet b-a är så litet så möjligt.

Theorem 10.1.12 (Asymptotic efficiency of MLEs) *Let $X_1, X_2, \ldots,$ be iid $f(x|\theta)$, let $\hat\theta$ denote the MLE of θ, and let $\tau(\theta)$ be a continuous function of θ. Under the regularity conditions in Miscellanea 10.6.2 on $f(x|\theta)$ and, hence, $L(\theta|\mathbf{x})$,*

$$\sqrt{n}[\tau(\hat\theta) - \tau(\theta)] \to \mathrm{n}[0, v(\theta)],$$

where $v(\theta)$ is the Cramér–Rao Lower Bound. That is, $\tau(\hat\theta)$ is a consistent and asymptotically efficient estimator of $\tau(\theta)$.

ARE – Asymptotic Relative Efficiency:

Definition 10.1.16 If two estimators W_n and V_n satisfy

$$\sqrt{n}[W_n - \tau(\theta)] \to \mathrm{n}[0, \sigma_W^2]$$
$$\sqrt{n}[V_n - \tau(\theta)] \to \mathrm{n}[0, \sigma_V^2]$$

in distribution, the *asymptotic relative efficiency* (ARE) of V_n with respect to W_n is

$$\mathrm{ARE}(V_n, W_n) = \frac{\sigma_W^2}{\sigma_V^2}.$$

Theorem 10.3.1 (Asymptotic distribution of the LRT—simple H_0) *For testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, suppose $X_1, \ldots, X_n$ are iid $f(x|\theta)$, $\hat{\theta}$ is the MLE of θ, and $f(x|\theta)$ satisfies the regularity conditions in Miscellanea 10.6.2. Then under H_0, as $n \to \infty$,*

$$-2 \log \lambda(\mathbf{X}) \to \chi_1^2 \text{ in distribution,}$$

where χ_1^2 is a χ^2 random variable with 1 degree of freedom.

Table of Common Distributions

Discrete Distributions

Bernoulli(p)

pmf	$P(X = x\|p) = p^x(1-p)^{1-x};\quad x = 0, 1;\quad 0 \le p \le 1$
mean and variance	$\mathrm{E}X = p, \quad \mathrm{Var}\, X = p(1-p)$
mgf	$M_X(t) = (1-p) + pe^t$

Binomial(n, p)

pmf	$P(X = x\|n, p) = \binom{n}{x} p^x (1-p)^{n-x};\quad x = 0, 1, 2, \ldots, n;\quad 0 \le p \le 1$
mean and variance	$\mathrm{E}X = np, \quad \mathrm{Var}\, X = np(1-p)$
mgf	$M_X(t) = [pe^t + (1-p)]^n$
notes	Related to Binomial Theorem (Theorem 3.2.2). The *multinomial* distribution (Definition 4.6.2) is a multivariate version of the binomial distribution.

Discrete uniform

pmf	$P(X = x\|N) = \frac{1}{N};\quad x = 1, 2, \ldots, N;\quad N = 1, 2, \ldots$
mean and variance	$\mathrm{E}X = \frac{N+1}{2}, \quad \mathrm{Var}\, X = \frac{(N+1)(N-1)}{12}$
mgf	$M_X(t) = \frac{1}{N}\sum_{i=1}^{N} e^{it}$

Geometric(p)

pmf	$P(X = x\|p) = p(1-p)^{x-1};\quad x = 1, 2, \ldots;\quad 0 \le p \le 1$
mean and variance	$\mathrm{E}X = \frac{1}{p}, \quad \mathrm{Var}\, X = \frac{1-p}{p^2}$

mgf	$M_X(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$
notes	$Y = X - 1$ is negative binomial$(1, p)$. The distribution is *memoryless*: $P(X > s \mid X > t) = P(X > s - t)$.

Hypergeometric

pmf	$P(X = x \mid N, M, K) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}; \quad x = 0, 1, 2, \ldots, K;$ $M - (N - K) \le x \le M; \quad N, M, K \ge 0$
mean and variance	$\mathrm{E}X = \frac{KM}{N}, \quad \mathrm{Var}\, X = \frac{KM}{N}\frac{(N-M)(N-K)}{N(N-1)}$
notes	If $K \ll M$ and N, the range $x = 0, 1, 2, \ldots, K$ will be appropriate.

Negative binomial(r, p)

pmf	$P(X = x \mid r, p) = \binom{r+x-1}{x} p^r (1-p)^x; \quad x = 0, 1, \ldots; \quad 0 \le p \le 1$
mean and variance	$\mathrm{E}X = \frac{r(1-p)}{p}, \quad \mathrm{Var}\, X = \frac{r(1-p)}{p^2}$
mgf	$M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$
notes	An alternate form of the pmf is given by $P(Y = y \mid r, p) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$, $y = r, r+1, \ldots$. The random variable $Y = X + r$. The negative binomial can be derived as a gamma mixture of Poissons. (See Exercise 4.34.)

Poisson(λ)

pmf	$P(X = x \mid \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}; \quad x = 0, 1, \ldots; \quad 0 \le \lambda < \infty$
mean and variance	$\mathrm{E}X = \lambda, \quad \mathrm{Var}\, X = \lambda$
mgf	$M_X(t) = e^{\lambda(e^t - 1)}$

Continuous Distributions

Beta(α, β)

pdf	$f(x\|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1, \quad \alpha > 0, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}\, X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
mgf	$M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right) \frac{t^k}{k!}$
notes	The constant in the beta pdf can be defined in terms of gamma functions, $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. Equation (3.2.18) gives a general expression for the moments.

Cauchy(θ, σ)

pdf	$f(x\|\theta,\sigma) = \frac{1}{\pi\sigma} \frac{1}{1+\left(\frac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x < \infty; \quad -\infty < \theta < \infty, \quad \sigma > 0$
mean and variance	do not exist
mgf	does not exist
notes	Special case of Student's t, when degrees of freedom = 1. Also, if X and Y are independent $\mathrm{n}(0,1)$, X/Y is Cauchy.

Chi squared(p)

pdf	$f(x\|p) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2}; \quad 0 \le x < \infty; \quad p = 1, 2, \ldots$
mean and variance	$\mathrm{E}X = p, \quad \mathrm{Var}\, X = 2p$
mgf	$M_X(t) = \left(\frac{1}{1-2t}\right)^{p/2}, \quad t < \frac{1}{2}$
notes	Special case of the gamma distribution.

Double exponential(μ, σ)

pdf	$f(x\|\mu,\sigma) = \frac{1}{2\sigma} e^{-\|x-\mu\|/\sigma}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\, X = 2\sigma^2$
mgf	$M_X(t) = \frac{e^{\mu t}}{1-(\sigma t)^2}, \quad \|t\| < \frac{1}{\sigma}$
notes	Also known as the *Laplace* distribution.

Exponential(β)

pdf	$f(x\|\beta) = \frac{1}{\beta}e^{-x/\beta}, \quad 0 \le x < \infty, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \beta, \quad \mathrm{Var}\, X = \beta^2$
mgf	$M_X(t) = \frac{1}{1-\beta t}, \quad t < \frac{1}{\beta}$
notes	Special case of the gamma distribution. Has the *memoryless* property. Has many special cases: $Y = X^{1/\gamma}$ is *Weibull*, $Y = \sqrt{2X/\beta}$ is *Rayleigh*, $Y = \alpha - \gamma \log(X/\beta)$ is *Gumbel*.

F

pdf	$f(x\|\nu_1, \nu_2) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \frac{x^{(\nu_1-2)/2}}{\left(1+\left(\frac{\nu_1}{\nu_2}\right)x\right)^{(\nu_1+\nu_2)/2}};$ $0 \le x < \infty; \quad \nu_1, \nu_2 = 1, \ldots$
mean and variance	$\mathrm{E}X = \frac{\nu_2}{\nu_2-2}, \quad \nu_2 > 2,$ $\mathrm{Var}\, X = 2\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}, \quad \nu_2 > 4$
moments (mgf does not exist)	$\mathrm{E}X^n = \frac{\Gamma\left(\frac{\nu_1+2n}{2}\right)\Gamma\left(\frac{\nu_2-2n}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_2}{\nu_1}\right)^n, \quad n < \frac{\nu_2}{2}$
notes	Related to chi squared ($F_{\nu_1,\nu_2} = \left(\frac{\chi^2_{\nu_1}}{\nu_1}\right) / \left(\frac{\chi^2_{\nu_2}}{\nu_2}\right)$, where the χ^2s are independent) and t ($F_{1,\nu} = t^2_\nu$).

Gamma(α, β)

pdf	$f(x\|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}e^{-x/\beta}, \quad 0 \le x < \infty, \quad \alpha, \beta > 0$
mean and variance	$\mathrm{E}X = \alpha\beta, \quad \mathrm{Var}\, X = \alpha\beta^2$
mgf	$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < \frac{1}{\beta}$
notes	Some special cases are exponential ($\alpha = 1$) and chi squared ($\alpha = p/2$, $\beta = 2$). If $\alpha = \frac{3}{2}$, $Y = \sqrt{X/\beta}$ is *Maxwell*. $Y = 1/X$ has the *inverted gamma distribution*. Can also be related to the Poisson (Example 3.2.1).

Logistic(μ, β)

pdf	$f(x\|\mu, \beta) = \frac{1}{\beta} \frac{e^{-(x-\mu)/\beta}}{[1+e^{-(x-\mu)/\beta}]^2}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\, X = \frac{\pi^2\beta^2}{3}$

mgf	$M_X(t) = e^{\mu t}\Gamma(1-\beta t)\Gamma(1+\beta t), \quad \lvert t\rvert < \frac{1}{\beta}$
notes	The cdf is given by $F(x\mid\mu,\beta) = \frac{1}{1+e^{-(x-\mu)/\beta}}$.

Lognormal(μ, σ^2)

pdf	$f(x\mid\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\frac{e^{-(\log x-\mu)^2/(2\sigma^2)}}{x}, \quad 0 \le x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$
mean and variance	$\mathrm{E}X = e^{\mu+(\sigma^2/2)}, \quad \mathrm{Var}\,X = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$
moments (mgf does not exist)	$\mathrm{E}X^n = e^{n\mu+n^2\sigma^2/2}$
notes	Example 2.3.5 gives another distribution with the same moments.

Normal(μ, σ^2)

pdf	$f(x\mid\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\,X = \sigma^2$
mgf	$M_X(t) = e^{\mu t+\sigma^2t^2/2}$
notes	Sometimes called the *Gaussian* distribution.

Pareto(α, β)

pdf	$f(x\mid\alpha,\beta) = \frac{\beta\alpha^\beta}{x^{\beta+1}}, \quad a < x < \infty, \quad \alpha > 0, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \frac{\beta\alpha}{\beta-1}, \quad \beta > 1, \quad \mathrm{Var}\,X = \frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)}, \quad \beta > 2$
mgf	does not exist

t

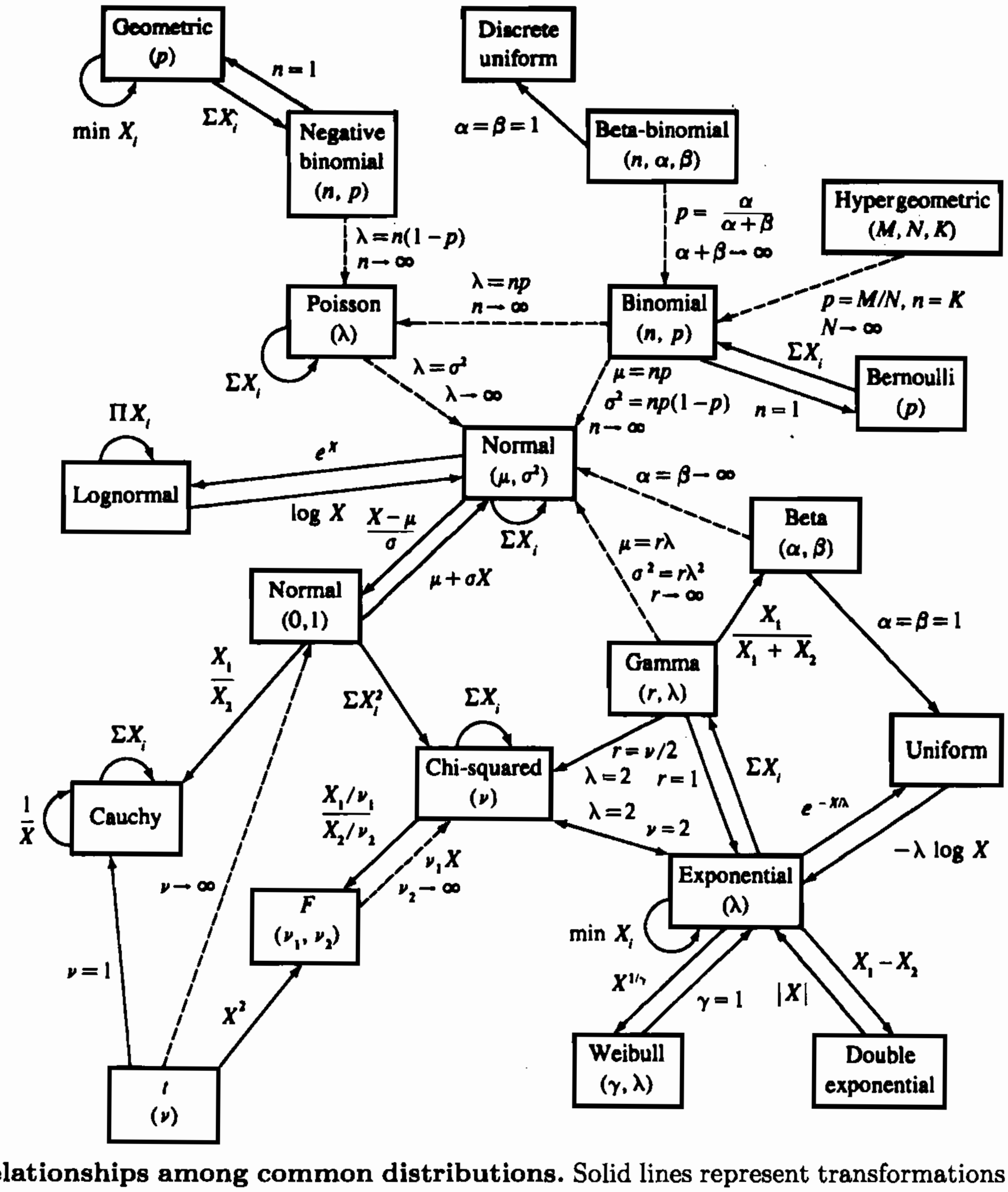
pdf	$f(x\mid\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\frac{1}{\left(1+\left(\frac{x^2}{\nu}\right)\right)^{(\nu+1)/2}}, \quad -\infty < x < \infty, \quad \nu = 1, \ldots$
mean and variance	$\mathrm{E}X = 0, \quad \nu > 1, \quad \mathrm{Var}\,X = \frac{\nu}{\nu-2}, \quad \nu > 2$
moments (mgf does not exist)	$\mathrm{E}X^n = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{\nu-n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\nu^{n/2}$ if $n < \nu$ and even, $\mathrm{E}X^n = 0$ if $n < \nu$ and odd.
notes	Related to F $(F_{1,\nu} = t_\nu^2)$.

Uniform(a, b)

pdf	$f(x\|a,b) = \frac{1}{b-a}, \quad a \leq x \leq b$
mean and variance	$\mathrm{E}X = \frac{b+a}{2}, \quad \mathrm{Var}\, X = \frac{(b-a)^2}{12}$
mgf	$M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$
notes	If $a = 0$ and $b = 1$, this is a special case of the beta ($\alpha = \beta = 1$).

Weibull(γ, β)

pdf	$f(x\|\gamma,\beta) = \frac{\gamma}{\beta} x^{\gamma-1} e^{-x^{\gamma}/\beta}, \quad 0 \leq x < \infty, \quad \gamma > 0, \quad \beta > 0$
mean and variance	$\mathrm{E}X = \beta^{1/\gamma}\Gamma\left(1+\frac{1}{\gamma}\right), \quad \mathrm{Var}\, X = \beta^{2/\gamma}\left[\Gamma\left(1+\frac{2}{\gamma}\right) - \Gamma^2\left(1+\frac{1}{\gamma}\right)\right]$
moments	$\mathrm{E}X^n = \beta^{n/\gamma}\Gamma\left(1+\frac{n}{\gamma}\right)$
notes	The mgf exists only for $\gamma \geq 1$. Its form is not very useful. A special case is exponential ($\gamma = 1$).



Relationships among common distributions. Solid lines represent transformations and special cases, dashed lines represent limits. Adapted from Leemis (1986).

TABELL 1. Normalfördelningen, standardiserad

$\Phi(z) = P(Z \leq z)$ där $Z \in N(0, 1)$.

För negativa värden, utnyttja att $\Phi(-z) = 1 - \Phi(z)$.

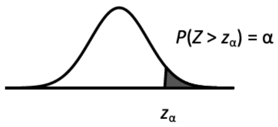
$$\Phi(z) = P(Z \leq z) \text{ with } Z \in \mathrm{N}(0, 1)$$

z	0	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	.5	.50399	.50798	.51197	.51595	.51994	.52392	.5279	.53188	.53586
.1	.53983	.5438	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
.3	.61791	.62172	.62552	.6293	.63307	.63683	.64058	.64431	.64803	.65173
.4	.65542	.6591	.66276	.6664	.67003	.67364	.67724	.68082	.68439	.68793
.5	.69146	.69497	.69847	.70194	.7054	.70884	.71226	.71566	.71904	.7224
.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.7549
.7	.75804	.76115	.76424	.7673	.77035	.77337	.77637	.77935	.7823	.78524
.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.8665	.86864	.87076	.87286	.87493	.87698	.879	.881	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.9032	.9049	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.9222	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.9452	.9463	.94738	.94845	.9495	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.9608	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.9732	.97381	.97441	.975	.97558	.97615	.9767
2	.97725	.97778	.97831	.97882	.97932	.97982	.9803	.98077	.98124	.98169
2.1	.98214	.98257	.983	.98341	.98382	.98422	.98461	.985	.98537	.98574
2.2	.9861	.98645	.98679	.98713	.98745	.98778	.98809	.9884	.9887	.98899
2.3	.98928	.98956	.98983	.9901	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.9918	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.9943	.99446	.99461	.99477	.99492	.99506	.9952
2.6	.99534	.99547	.9956	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.9972	.99728	.99736
2.8	.99744	.99752	.9976	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.999
3.1	.99903	.99906	.9991	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.9994	.99942	.99944	.99946	.99948	.9995
3.3	.99952	.99953	.99955	.99957	.99958	.9996	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.9997	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.9998	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.9999	.9999	.9999	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997
4	.99997	.99997	.99997	.99997	.99997	.99997	.99998	.99998	.99998	.99998

TABELL 2. Normalfördelningens kvantiler, standardiserad

$Z \in N(0, 1)$. Vilket värde har z_α om $P(Z > z_\alpha) = \alpha$ där α är en given sannolikhet.

Utnyttja även $\Phi(-z) = 1 - \Phi(z)$ för $P(Z \leq -z_\alpha)$.



α	z_α
0,25	0,6745
0,10	1,2816
0,05	1,6449
0,025	1,9600
0,010	2,3263
0,005	2,5758
0,0025	2,8070
0,0010	3,0902
0,0005	3,2905
0,00025	3,4808
0,00010	3,7190
0,00005	3,8906
0,000025	4,0556
0,000010	4,2649
0,000005	4,4172