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Inference Theory

WRITTEN EXAMINATION
Tuesday February 25, 2025

Allowed tools: Calculator, handwritten notes in terms of one A4 paper, where it is allowed to write on both sides, formulas (provided during the exam).

Passing rate: 50% of overall total, which is 100 points. There are five questions worth 20 points each.

For the maximum number of points on each problem detailed, well motivated and clear solutions are required.

1. Let $X_1, \dots, X_n$ be a random sample generated from a probability mass function
$$f(x|\theta) = \frac{1}{1+\theta}\Big(\frac{\theta}{1+\theta}\Big)^x, \quad x = 0, 1, 2, \dots$$
Investigate whether the MLE $\hat{\theta} = \bar{X}$ is the best unbiased estimator of θ. (You do not need to show that $\bar{X}$ is the MLE.)

2. X has probability density function
$$f(x|\theta) = \frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}, \quad x > 0, \quad \theta > 0.$$
Let $x_1, \dots, x_n$ be an observed random sample from this distribution.
   (a) Is this an exponential family of distributions? Why/why not?
   (b) Derive the MLE of θ.
   (c) Suppose n is large. Derive an approximate $100(1-\alpha)\%$ two-sided confidence interval expression for θ.

3. Let $X_1, \dots, X_n$ be a random sample of X with probability density function
$$f(x|\theta, \mu) = \frac{1}{\mu}e^{-\frac{1}{\mu}(x-\theta)}, \quad x \geq \theta, \quad \mu > 0, \quad \theta > 0.$$
   (a) Derive a two-dimensional sufficient statistic for (θ, μ).
   (b) Use the result in (a) to determine unbiased estimators of θ and μ.

4. We have one observation from a random variable with probability density function
$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1.$$
Here we have only two possible values of θ: 2 and 3. We want to test
$$H_0 : \theta = 2 \text{ vs. } H_1 : \theta = 3.$$
Determine the most powerful test of H_0 vs. H_1 with $\alpha = 0.05$. Also, compute the power of the test for $\theta = 3$. (The power will be low, since we have only one variable.)

5. Let $X_1, \ldots, X_n$ be a random sample generated from a Poisson distribution with parameter λ, that is
$$f(x|\lambda) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x = 0, 1, 2, \ldots$$
Use that
$$\frac{S(\lambda)}{\sqrt{I_n(\lambda)}} \overset{\text{appr}}{\sim} N(0, 1),$$
where $S(\lambda) = \frac{\partial}{\partial \lambda} log L(\lambda|\mathbf{X})$ to construct an approximate $100(1-\alpha)\%)$ confidence interval for λ. (You may stop where you observe that you have arrived at a quadratic equation to be solved.)