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OBS! Läs noga igenom anvisningarna i tentamen, t.ex. hur du ska skriva svaren. Det är ditt ansvar som student att följa de anvisningar som ges.

NOTE! Read the examination instructions carefully, e.g. how to write the answers. It is your responsibility as a student to follow the given instructions.

Skriv din anonymiseringskod och dagens datum på allt material du lämnar in. (*Enter your anonymization code and todays date on all submitted materials*)											
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Datum (*Date YYYY-MM-DD*)	2025-02-25	Plats nr. (*Seat No.*)	12

Kurs/Kurskod (*Course/Course code*)	ST 4301
Kursmoment (*Course component*)	Re-Exam

Fylls i av tentamensvärd (*To be filled in by invigilator*)

Direkt i skrivning: (kryss)		Svarsblankett: (kryss)		Lösa svarsblad: (antal)	12st.
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Inlämningstid: 18:55 Signatur tentamensvärd: [signature]

Fylls i av lärare/examinator (*To be filled in by teacher/examinator*)

Betyg:	A	Poäng:	88

Signatur rättande lärare/examinator: PGA

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Uppg.nr.: (Task no.) 1

Lärarens kommentar: (Teacher's note)

Question 1.

$$E\left(\frac{1}{n}\Sigma X\right) = \frac{1}{n} n E(x) = E(x)$$

We have that: $E(\overline{X}) = E(X) \Rightarrow E(X) = \dfrac{1 - \frac{1}{1+\theta}}{\frac{1}{1+\theta}} =$

$$= \frac{1+\theta-1}{1+\theta} \cdot \frac{1+\theta}{1} = \theta$$

$$E(X) = \frac{1-P}{P} \Rightarrow P = \frac{1}{1+\theta}$$

This is a Geometric PMF. Starting from zero! not from one as in the formula sheet

We check if $\overline{X}$ attains Cramér-Rao lower bound:

$$V(X) = \frac{1 - \frac{1}{1+\theta}}{\left(\frac{1}{1+\theta}\right)^2} = \frac{\theta(1+\theta)^2}{1+\theta} = \theta(1+\theta)$$

$$V(\overline{X}) = \frac{V(X)}{n} = \frac{\theta(1+\theta)}{n}$$

Since this is a geometric pmf this belongs to exponential family of distributions

$$f(x|\theta) = \underbrace{\left(\frac{1}{1+\theta}\right)}_{C(\theta)} \underbrace{e^{x \ln(\theta/1+\theta)}}_{e^{T_\cdot(x) W_\cdot(\theta)}} \underbrace{I_{\{x \in \mathbb{N} = 0,1,2\}}(x)}_{h(x)}$$

$\Leftarrow$ unbiased

$$V_\theta(W(\theta)) \geq \frac{\left(\frac{d}{d\theta} E_\theta(W(X))\right)^2}{-n E\left(\frac{\partial^2 l(\theta|X)}{\partial^2 \theta}\right)}$$

$$\left(\frac{d}{d\theta} E_\theta(W(x))\right)^2 = \left(\frac{d}{d\theta}\theta\right)^2 = 1^2 = 1$$

Setting up the likelihood function then taking the log.

$$L(\theta|X) = f(x|\theta) = \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^x$$

$$l(\theta|X) = \log\left(\frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^x\right) = x\log\theta - (x+1)\log(1+\theta)$$

Now we take the first and then the second derivative, cont. on the second page.

Poäng: (Points) 20

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

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Uppg.nr.: (*Task no.*) 1

Lärarens kommentar: (*Teacher's note*)

$$\frac{\partial \ell(\theta|X)}{\partial\theta} = \frac{X}{\theta} - \frac{(X+1)}{1+\theta}$$

$$\frac{\partial^2 \ell(\theta|X)}{\partial\theta^2} = -\frac{X}{\theta^2} + \frac{(X+1)}{(1+\theta)^2}$$

Now we have each component we plug it in the formula:

$$\frac{\frac{d}{d\theta}E_\theta(W(X))^2}{-nE\left(\frac{\partial^2\ell(\theta|X)}{\partial\theta^2}\right)} = \frac{1}{-nE\left(\frac{\partial^2\ell(\theta|X)}{\partial\theta^2}\right)} =$$

$$= // -E\left(\frac{\partial^2\ell(\theta|X)}{\partial\theta^2}\right) = \frac{E(X)}{\theta^2} - \frac{E(X+1)}{(1+\theta)^2} =$$

$$= \frac{1}{\theta^2}\theta - \frac{1}{(1+\theta)^2}(\theta+1) = \frac{1}{\theta(1+\theta)} \Rightarrow -nE\left(\frac{\partial^2\ell(\theta|X)}{\partial\theta}\right) = \frac{n}{\theta(1+\theta)} //$$

$$= \frac{1}{\frac{n}{\theta(1+\theta)}} = \frac{\theta(1+\theta)}{n} //$$

$$V_\theta(W(X)) \geq \frac{\theta(1+\theta)}{n}$$

Answer: Yes! the MLE of $\hat{\theta} = \overline{X}$ is UMVUE! :)

Poäng: (*Points*)

Tentamensservice/SU/typ_b

Lärarens kommentar: (*Teacher's note*)

Poäng: (*Points*)

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Uppg.nr.: (Task no.) 2

Lärarens kommentar: (Teacher's note)

We have the following PDF: $f(x|\theta) = \frac{x}{\theta^2} e^{-x^2/2\theta^2}$, $x > 0$, $\theta > 0$

a)

A distribution that belongs to the exponential family of distributions can be written on the following form:

$$f(x|\theta) = h(x)\, C(\theta)\, e^{\sum w_i(\theta) t_i(x)}$$

By using this formula we can decide if our PDF is an Exp. family of dist.

$$f(\underline{x}|\theta) = \prod_{i=1}^{n} \frac{x}{\theta^2} e^{-x^2/2\theta^2} = \underbrace{\frac{1}{\theta^{2n}}}_{C(\theta)} \underbrace{\prod_{i=1}^{n} x_i}_{h(x)} \underbrace{e^{-\frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2}}_{e^{\sum w_i(\theta) t_i(x)}}$$

Answer: Yes! This an exp. family of dist also $x > 0$ and $\theta > 0$ are not related or x does not depend on θ in its domain.

OK

b)

We set up the likelihood function:

$$L(\theta|\underline{x}) = f(\underline{x}|\theta) = \frac{1}{\theta^{2n}} \prod_{i=1}^{n} x_i\, e^{-\frac{1}{2\theta^2}\sum x_i^2}$$

Taking the log-function and applying it on our likelihood function:

$$\ell(\theta|\underline{x}) = \log(L(\theta|\underline{x})) = \log\left(\theta^{-2n} \prod_{i=1}^{n} x_i\, e^{-\frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2}\right) =$$

$$= -2n\log(\theta) + \sum_{i=1}^{n} \log(x_i) - \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2$$

Now we take the first-derivative and set it to zero.

$$\frac{\partial \ell(\theta|\underline{x})}{\partial\theta} = 0 \Rightarrow \frac{\partial \ell(\theta|x)}{\partial\theta} = \frac{-2n}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^{n} x_i^2 = 0$$

We solve for θ i.e. $\hat{\theta}$ the MLE!

$$\frac{1}{\theta^3}\sum_{i=1}^{n} x_i^2 = \frac{2n}{\theta} \Rightarrow \theta^2 = \frac{\sum x_i^2}{2n} \Rightarrow \hat{\theta} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}}$$

We will check if this is an MLE by looking at the second derivative

Next page ⟶

Poäng: (Points) 20

Tentamensservice/SU/typ_b

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

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Uppg.nr.: (Task no.) 2

cont. b)

$$\frac{\partial^2 l(\theta|\mathbf{x})}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(-2n\theta^{-1} + \theta^{-3}\sum_{i=1}^{n} x_i^2\right)$$

$$= 2n\theta^{-2} - 3\theta^{-4}\sum_{i=1}^{n} x_i^2$$

We cant really determine by just looking att the second derivative we need to plug in $\hat{\theta} = \theta$

$$2n\hat{\theta}^{-2} - 3\theta^{-4}\sum x_i^2$$

$$\frac{2n}{\left(\sqrt{\frac{\sum x_i^2}{2n}}\right)^2} - \frac{3}{\left(\sqrt{\frac{\sum x_i^2}{2n}}\right)^4}\sum x_i^2 = \frac{2n}{\frac{\sum x_i^2}{2n}} - \frac{3}{\left(\frac{\sum x_i^2}{2n}\right)^2}\sum x_i^2 =$$

$$= \frac{4n^2}{\sum x_i^2} - \frac{3\cdot 4n^2}{\sum x_i^2} = -\frac{8n^2}{\sum x_i^2} < 0$$

So this has a MLE, since $\left.\frac{\partial^2 l}{\partial\theta^2}\right|_{\hat{\theta}=\theta} < 0$

Answer: The MLE $\hat{\theta} = \sqrt{\frac{\sum x_i^2}{2n}}$ //

Ok

Lärarens kommentar: (Teacher's note)

Poäng: (Points)

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Tentamensservice/SU/typ_b

Lärarens kommentar:
(*Teacher's note*)

Poäng:
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Uppg.nr.: (*Task no.*) 2

Lärarens kommentar: (*Teacher's note*)

c) For this sub-question we can use asymptotic theory of MLE: (i.e. The MLE is asymptotically efficient under certain regular cond. (For exp-families)).

$$\frac{\tau(\hat\theta) - \tau(\theta)}{\sqrt{\hat V_\theta(\tau(\theta))}} \xrightarrow{d} N(0,1)$$

And the approximate confidence interval will look like:

$$I_{\hat\theta} = \tau(\hat\theta) \pm z_{\alpha/2} \cdot \sqrt{\hat V_\theta(\tau(\hat\theta))}$$

Fisher info.

Where $\hat V_\theta(\tau(\hat\theta)) = \frac{(\tau'(\theta))^2|_{\theta=\hat\theta}}{\underbrace{-nE\left(\frac{\partial^2 \ell(\theta|x)}{\partial\theta}\right)}_{\hat I_n(\hat\theta)}}$

From previous sub-question we have that MLE: $\hat\theta = \sqrt{\frac{\sum_i x_i^2}{2n}}$

And $\tau'(\theta)^2 = \left(\frac{d}{d\theta}(\hat\theta)\right)^2 = 1^2 = 1$

$L(\theta|x) = f(x|\theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}$

$\ell(\theta|x) = -2\log(\theta) + \log(x) - \frac{1}{2\theta^2}x^2$

$\frac{\partial \ell}{\partial\theta} = \frac{-2}{\theta} + \frac{1}{\theta^3}x^2 = -2\theta^{-1} + \theta^3 x^2$

$\frac{\partial^2 \ell}{\partial\theta^2} = 2\theta^{-2} - 3\theta^{-4}x^2$

$\hat V_\theta(\tau(\hat\theta)) = -nE\left(\frac{\partial^2 \ell}{\partial\theta^2}\right) = -n(2\theta^{-2} + 3\theta^{-4}E(x^2))$

So we need to calculate $E(x^2)$

We continue on next page

You can also use the observed information number.

Poäng: (*Points*)

Tentamensservice/SU/typ_b

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

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Uppg.nr.: (*Task no.*) 2

Lärarens kommentar: (*Teacher's note*)

Cont. c).

$$E(X^2) = \int_0^\infty x^2 \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} = \int_0^\infty \frac{x^3}{\theta^2} e^{-\frac{x^2}{2\theta^2}} dx =$$

$$= -x^2 \underbrace{e^{-\frac{x^2}{2\theta^2}}\Big|_0^\infty}_{\to 0} - \int_0^\infty -3x^2 e^{-x^2/2\theta^2} dx =$$

$$= 3\int_0^\infty x^2 e^{-x^2/2\theta^2} dx = 3\Big(\underbrace{\Big[-x e^{-\frac{x^2}{2\theta^2}}\Big]_0^\infty}_{\to 0} + \int_0^\infty 2x e^{-x^2/2\theta^2} dx$$

$$= 6\theta^2$$

$$n(-2\theta^{-2} + 3\theta^{-4} \cdot 6\theta^2) =$$

$$= n(18\theta^{-2} - 2\theta^{-2}) = n \cdot 16\theta^{-2}$$

$$I_{\hat\theta} = \hat\theta \pm z_{\alpha/2} \sqrt{\frac{1}{n16\theta^{-2}}} \Rightarrow \frac{1}{\frac{16n}{\theta^2}} = \frac{\theta^2}{16n}$$

$$\Rightarrow I_{\hat\theta}: \hat\theta \pm z_{\alpha/2} \sqrt{\frac{\hat\theta^2}{16n}}.$$

$$\Leftrightarrow I_{\hat\theta}: \hat\theta \pm z_{\alpha/2} \frac{\hat\theta}{4\sqrt{n}}$$

Answer: $I_{\hat\theta}: \hat\theta \pm z_{\alpha/2} \frac{\hat\theta}{4\sqrt{n}}$

$$\Rightarrow I_{\hat\theta}: \sqrt{\frac{\sum x_i^2}{2n}} \pm z_{\alpha/2} \frac{\sqrt{\sum x_i^2/2n}}{4\sqrt{n}}$$

Poäng: (*Points*)

Tentamensservice/SU/typ_b

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Uppg.nr.: (Task no.) 3

Lärarens kommentar: (Teacher's note)

a) $X_i \sim Exp(\mu, \theta)$, θ = location parameter

To find our two-dimensional statistic for (θ, μ) we employ the Factorization Theorem:

$$f(\underline{X}|\theta,\mu) = g(T(\theta)|\theta,\mu)\cdot h(x)$$

$$f(\underline{X}|\theta,\mu) = \prod_{i=1}^{n} \frac{1}{\mu} e^{-\frac{1}{\mu}(X_i-\theta)} = \frac{1}{\mu^n} e^{-\frac{1}{\mu}(\Sigma X_i - n\theta)}$$

Since we have the restriction $X \geq \theta \Rightarrow \underline{X_{(1)} = \theta}$ we will get:

$$f(\underline{X}|\theta,\mu) = \frac{1}{\mu^n} \cdot e^{-\frac{1}{\mu}(\Sigma X_i - n\theta)} I_{[\theta,\infty)}(\underline{X})$$

Where $h(x) = 1$, $g(T(\underline{X})|\theta,\mu) =$

$$= \frac{1}{\mu^n} e^{-\frac{1}{\mu}(\Sigma X_i - \theta)} I_{[\theta,\infty)}(x)$$

We can see from above that our two dimensional sufficient statistic is

$T(\underline{X}) = (\Sigma X_i, X_{(1)})$, But the sufficient is $\underline{X}_{(1)}$. ?

Poäng: (Points) 1,8

Lärarens kommentar: (*Teacher's note*)

Poäng: (*Points*)

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Uppg.nr.: (Task no.) 3

Lärarens kommentar: (Teacher's note)

← location parameter

b) $X \sim E(\mu, \theta)$, $Y = X - \theta \sim Exp(\mu)$

$\sum X_i = \sum Y_i + n\theta$, $\sum Y_i \sim Gamma(n, \mu)$

$E(\sum X) =$

$E(\sum Y_i + n\theta) = n\mu + n\theta = n(\mu + \theta)$

$\overline{X} = \frac{\sum X_i}{n} \Rightarrow E(\overline{X}) = E\left(\frac{\sum Y_i + n\theta}{n}\right) = \frac{E(\sum Y_i) + n\theta}{n}$

$= \frac{1}{n} n(\mu + \theta) = \mu + \theta$. (1)

$F_{X_{(1)}}(x_{(1)}) = P(X_{(1)} \leq x_{(1)}) = 1 - P(X_{(1)} > x_{(1)}) =$

$= \cancel{1} - (\cancel{1} - P(X \leq x_{(1)}))^n = (F_X(x_{(1)}))^n$

$F_X(x_{(1)}) = \int_\theta^{x_{(1)}} \frac{1}{\mu} e^{-\frac{1}{\mu}(x-\theta)} dx = \left[-e^{-\frac{1}{\mu}(x-\theta)}\right]_\theta^{x_{(1)}} =$

$= -e^{-\frac{1}{\mu}(x_{(1)}-\theta)} + e^{-\frac{1}{\mu}\underbrace{(\theta-\theta)}_{=0}} = 1 - e^{-\frac{1}{\mu}(x_{(1)}-\theta)} =$

$F_{X_{(1)}}(x_{(1)}) = \left(1 - e^{-\frac{1}{\mu}(x_{(1)}-\theta)}\right)^n = 1 - e^{-\frac{n}{\mu}(x_{(1)}-\theta)}$.

Now to get the pdf of $x_{(1)}$ we take the derivative:

$f_{X_{(1)}}(x_{(1)}) = \frac{d}{dx_{(1)}} F_{X_{(1)}}(x_{(1)}) = \frac{d}{dx_{(1)}}\left(1 - e^{-\frac{n}{\mu}(x_{(1)}-\theta)}\right) =$

$= \frac{n}{\mu} e^{-\frac{n}{\mu}(x_{(1)}-\theta)} \Rightarrow x_{(1)} \sim Exp(\frac{\mu}{n}, \theta)$ so we have the mean:

$E(X_{(1)}) = \frac{\mu}{n} + \theta$ (2)

We shall use (1) and (2) to determine unbiased estimators for θ and μ, with other word we have two equations and two unknowns.

Continues on the next page.

Poäng: (Points)

Tentamensservice/SU/typ_b

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

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Uppg.nr.: (Task no.) 3

Lärarens kommentar: (Teacher's note)

cont. b)

$$E(\overline{X}) = \mu + \theta \Rightarrow \mu = E(\overline{X}) - \theta$$

$$E(X_{(1)}) = \frac{\mu}{n} + \theta \Rightarrow \theta = E(X_{(1)}) - \frac{\mu}{n} = \text{// Plug in } \mu$$

$$E(X_{(1)}) = \frac{(E(\overline{X}) - \theta)}{n} + \theta$$

$$E(X_{(1)})n - E(\overline{X}) = (n-1)\theta$$

$$\theta = \frac{n \cdot E(X_{(1)}) - E(\overline{X})}{n-1}$$

$$\mu = E(\overline{X}) = E(\overline{X}) - \frac{nE(X_{(1)}) - E(\overline{X})}{n-1}$$

$$\Rightarrow \mu = \frac{(n-1)E(\overline{X}) - nE(X_{(1)}) + E(\overline{X})}{n-1}$$

$$\mu = \frac{nE(\overline{X}) - nE(X_{(1)})}{n-1} = \frac{n}{n-1}\left(E(\overline{X}) - E(X_{(1)})\right)$$

We have now our unbiased estimators

$$\hat{\theta} = \frac{n \cdot E(X_{(1)}) - E(\overline{X})}{n-1} = \frac{n X_{(1)} - \overline{X}}{n-1}$$

and

$$\hat{\mu} = \frac{n}{n-1}\left(\overline{X} - X_{(1)}\right)$$

Answer: Unbiased estimators is

$$\hat{\theta} = \frac{n X_{(1)} - \overline{X}}{n-1} \quad \text{and} \quad \hat{\mu} = \frac{n}{n-1}\left(\overline{X} - X_{(1)}\right)$$

Poäng: (Points)

Tentamensservice/SU/typ_b

Lärarens kommentar: (*Teacher's note*)

Poäng: (*Points*)

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Uppg.nr.: (Task no.) 4

We employ Neyman-Pearson lemma since we are dealing with a simple hypothesis

$H_0: \theta = 2$ vs $H_1: \theta = 3$

$f(x|\theta) = \theta x^{\theta-1}$

Since we have only one observation

$\frac{L(\theta_0|X)}{L(\theta_1|X)} < k \iff \frac{L(2|X)}{L(3|X)} < k$

$\Rightarrow \frac{2x^{2-1}}{3x^{3-1}} = \frac{2x}{3x^2} = \frac{2}{3x} < k$

$\Rightarrow X > k' \Rightarrow P_{H_0}(X > k') = 0.05$

$\iff \int_{k'}^{1} 2x \, dx = 2\left[\frac{x^2}{2}\right]_{k'}^{1} = 1 - k'$

So: $1 - k'^2 = 0.05 \Rightarrow -k'^2 = -0.95$

$k'^2 = 0.95 \Rightarrow k' = \sqrt{0.95}$

So $X > \sqrt{0.95} \approx 0.9746794$ //

and $\beta(\theta=3) = P(X > \sqrt{0.95} | \theta = 3) =$

$= \int_{\sqrt{0.95}}^{1} 3x^2 \, dx = x^3 \Big|_{\sqrt{0.95}}^{1} = 1 - (\sqrt{0.95})^3$

$= 1 - 0.95^{3/2} \approx 0.07405$

Answer:

The most powerful test gives $X > \sqrt{0.95} \approx 0.9747$

and the power is $1 - 0.95^{3/2} \approx 0.07405$

Lärarens kommentar: (Teacher's note)

Poäng: (Points) 2

Tentamensservice/SU/typ_b

Lärarens kommentar: (*Teacher's note*)

Poäng: (*Points*)

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Uppg.nr.: (Task no.) 5

Lärarens kommentar: (Teacher's note)

we have $f(x|\lambda) = \frac{\lambda^x}{x!} e^{-\lambda} \quad x = 0, 1, \ldots$

$$L(\lambda|X) = \prod^n \frac{\lambda^x}{x!} e^{-\lambda}$$

$$\ell(\lambda|X) = \underbrace{-\log\left(\prod^n \frac{1}{x!}\right)}_{\text{constant}} + \Sigma x \log\lambda - \lambda n$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{\Sigma x}{\lambda} - n \qquad \frac{\partial \ell}{\partial \lambda} = 0$$

$$\Rightarrow \lambda = \frac{\Sigma x}{n} = \bar{x}$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{\Sigma x}{\lambda^2}$$

$$I_n = -E\left(-\frac{\Sigma x}{\lambda^2}\right) =$$

$$= \frac{nE(x)}{\lambda^2} = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

$$S(\lambda) = \frac{\Sigma x}{\lambda} - n$$

$$\sqrt{I_n} = \frac{\sqrt{n}}{\lambda} \qquad \frac{\frac{\Sigma x}{\lambda} - n}{\sqrt{\frac{n}{\lambda}}} =$$

$$= \frac{\Sigma x - \lambda n}{\sqrt{\frac{n}{\lambda}}} \Rightarrow$$

$$\frac{\sqrt{n}(\bar{x} - \lambda)}{\lambda} \xrightarrow{d} N(0, 1)$$

$$\sqrt{n}(\bar{x} - \lambda) \xrightarrow{d} N(0, \lambda^2)$$

Poäng: (Points)

Tentamensservice/SU/typ_b

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

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Uppg.nr.: (Task no.) 5

Lärarens kommentar: (Teacher's note)

$$I_\lambda = \bar{x} \pm z_{\alpha/2} \cdot \sqrt{\frac{\lambda^2}{n}}$$

$$I_\lambda = \bar{x} \pm z_{\alpha/2} \frac{\lambda}{\sqrt{n}}$$

$$\parallel$$

$$I_\lambda : \lambda \pm z_{\alpha/2} \frac{\lambda}{\sqrt{n}}$$

Asymptotic MLE Pois. Exp. fam.

$\bar{x} = \lambda$

Answer: our $100(1-\alpha)\%$ CI is

$$I_\lambda : \bar{x} \pm z_{\alpha/2} \frac{\lambda}{\sqrt{n}} \Rightarrow \lambda \pm z_{\alpha/2} \frac{\lambda}{\sqrt{n}}$$

But this is not a conf. int. for λ.

Poäng: (Points) 0

Tentamensservice/SU/typ_b

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

Regler i skrivsalen

- Följ tentamensvärds anvisningar.
- Väskor och ytterkläder ska placeras på anvisad plats.
- Placera ID-handling väl synlig på bordet framför dig.
- Ingen student får lämna skrivsalen under de första 30 minuterna.
- Endast en student i taget får besöka toaletten. Vid toalettbesök skriv ditt namn och klockslag på avsedd lista. Efter toalettbesöket ska du åter ange klockslag på listan.
- Elektronisk utrustning som mobiltelefon eller Smartwatch ska vara avstängd och placerad på anvisad plats.
- Under tentamen gäller tystnad – det är förbjudet att prata, eller på annat sätt kommunicera, med andra studenter under pågående tentamen.
- Innan tentamenshandlingarna lämnas in; skriv sidnummer, anonymiseringskod och datum på alla inlämnade papper.

Om något är oklart – fråga gärna tentamensvärden. Lycka till!

Rules in the examination hall

- Follow the invigilator's instructions.
- Bags and outerwear must be placed at the designated place.
- Place your ID document clearly visible on the table in front of you.
- No student may leave the examination hall for the first 30 minutes.
- Only one student at a time may visit the toilet. Before visiting the toilet, write your name and time on the intended list. After the toilet visit, enter the time on the list again.
- Electronic equipment such as a mobile phone or Smartwatch must be switched off and placed at the designated place.
- During the exam, silence applies – you are not allowed to talk, or otherwise communicate, with other students during the exam.
- Before submitting the examination documents; remember to write the page number, anonymization code, and date on all papers.

Please do not hesitate to ask the invigilator if anything is unclear. Good luck!

Tentamensservice
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Stockholms universitet