Stockholms
universitet

OBS! Läs noga igenom anvisningarna i tentamen, t.ex. hur du ska skriva svaren. Det är ditt ansvar som student att följa de anvisningar som ges.

NOTE! Read the examination instructions carefully, e.g. how to write the answers. It is your responsibility as a student to follow the given instructions.

Skriv din anonymiseringskod och dagens datum på allt material du lämnar in.
(*Enter your anonymization code and todays date on all submitted materials*)

Anonymiseringskod (*Anonymization code*)	3	1	1	-	0	0	1	6	-	T	X	0

Datum (*Date* YYYY-MM-DD)	2022-02-11	Plats nr. (*Seat No.*)	6

Kurs/Kurskod (*Course/Course code*)	ST4301
Kursmoment (*Course component*)	Inferensteori

Fylls i av tentamensvärd (*To be filled in by invigilator*)

Direkt i skrivning: (kryss)		Svarsblankett: (kryss)		Lösa svarsblad: (antal)	5

Lämnat in blankt: (kryss)		Dator: (kryss)	

Inlämningstid: 12 : 32 Signatur tentamensvärd: [signature]

Fylls i av lärare/examinator (*To be filled in by teacher/examinator*)

Betyg:	B	Poäng:	80

Signatur rättande lärare/examinator: [signature]

Regler i skrivsalen

- Följ tentamensvärds anvisningar.
- Väskor och ytterkläder ska placeras på anvisad plats.
- Placera ID-handling väl synlig på bordet framför dig.
- Ingen student får lämna skrivsalen under de första 30 minuterna.
- Endast en student i taget får besöka toaletten. Vid toalettbesök skriv ditt namn och klockslag på avsedd lista. Efter toalettbesöket ska du åter ange klockslag på listan.
- Elektronisk utrustning som mobiltelefon eller Smartwatch ska vara avstängd och placerad på anvisad plats.
- Under tentamen gäller tystnad – det är förbjudet att prata, eller på annat sätt kommunicera, med andra studenter under pågående tentamen.
- Innan tentamenshandlingarna lämnas in; skriv sidnummer, anonymiseringskod och datum på alla inlämnade papper.

Om något är oklart – fråga gärna tentamensvärden. Lycka till!

Rules in the examination hall

- Follow the invigilator's instructions.
- Bags and outerwear must be placed at the designated place.
- Place your ID document clearly visible on the table in front of you.
- No student may leave the examination hall for the first 30 minutes.
- Only one student at a time may visit the toilet. Before visiting the toilet, write your name and time on the intended list. After the toilet visit, enter the time on the list again.
- Electronic equipment such as a mobile phone or Smartwatch must be switched off and placed at the designated place.
- During the exam, silence applies – you are not allowed to talk, or otherwise communicate, with other students during the exam.
- Before submitting the examination documents; remember to write the page number, anonymization code, and date on all papers.

Please do not hesitate to ask the invigilator if anything is unclear. Good luck!

Stockholms universitet

Datum: (Date YYYY-MM-DD)	2022-02-11					Kurs/Kurskod: (Course/Course code)	ST4301						Sidnr.: (Page no.) 1
Anonymiseringskod (Anonymization code)	3	1	1	-	0	0	1	6	-	T	X	0	

Uppg.nr.: (Task no.) 1

Lärarens kommentar: (Teacher's note)

① We have $f_{Y_i}(y_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i - (\beta_0+\beta_1 x_i))^2}$ so

$$f(\underset{\sim}{y};\underset{\sim}{\theta}) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\Sigma(y_i-(\beta_0+\beta_1 x_i))^2} =$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\Sigma(y_i^2 - 2y_i(\beta_0+\beta_1 x_i) + (\beta_0+\beta_1 x_i)^2)}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}(\Sigma y_i^2 - 2\beta_0 \Sigma y_i - 2\beta_1 \Sigma x_i y_i + \Sigma(\beta_0+\beta_1 x_i)^2)}$$

$= g(\underset{\sim}{T}(\underset{\sim}{y});\underset{\sim}{\theta})h(\underset{\sim}{y})$, where $\underset{\sim}{T}(\underset{\sim}{y}) = (\Sigma y_i, \Sigma y_i^2, \Sigma x_i y_i)$,

$$g(\underset{\sim}{t};\underset{\sim}{\theta}) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}(t_2 - 2\beta_0 t_1 - 2\beta_1 t_3 + \Sigma(\beta_0+\beta_1 x_i)^2)}$$

and $h(\underset{\sim}{y}) = 1$, so by the Factorization Theorem,

$\underset{\sim}{T}(\underset{\sim}{Y}) := (\Sigma Y_i, \Sigma Y_i^2, \Sigma x_i Y_i)$ is a sufficient statistic for $\underset{\sim}{\theta}$.

bra!

Poäng: (Points) 20

Uppg.nr.:
(*Task no.*)

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

Stockholms universitet

Datum: (Date YYYY-MM-DD)	2022-02-11	Kurs/Kurskod: (Course/Course code) ST4301	Sidnr.: (Page no.) 2
Anonymiseringskod (Anonymization code)	311-0016-TXO		

Uppg.nr.: (Task no.) 2

Lärarens kommentar: (Teacher's note)

② We have $L(\theta) = \binom{N}{y_1 \; y_2 \; N-y_1-y_2} \theta_1^{y_1} \theta_2^{y_2} (1-\theta_1-\theta_2)^{N-y_1-y_3}$, so

$$\ell(\theta) = \log \binom{N}{y_1 \; y_2 \; N-y_1-y_2} + y_1 \log \theta_1 + y_2 \log \theta_2 + (N-y_1-y_2)\log(1-\theta_1-\theta_2)$$

so $\frac{\partial \ell}{\partial \theta_1}(\theta) = \frac{y_1}{\theta_1} - \frac{N-y_1-y_2}{1-\theta_1-\theta_2}$ and $\frac{\partial \ell}{\partial \theta_2}(\theta) = \frac{y_2}{\theta_2} - \frac{N-y_1-y_2}{1-\theta_1-\theta_2}$, so

$$\begin{cases} \frac{\partial \ell}{\partial \theta_1}(\hat{\theta}) = 0 \\ \frac{\partial \ell}{\partial \theta_2}(\hat{\theta}) = 0 \end{cases} \iff \begin{cases} \hat{\theta}_1 = \frac{y_1}{N} \\ \hat{\theta}_2 = \frac{y_2}{N} \end{cases}, \text{ so}$$

$$L(\hat{\theta}) = \binom{N}{y_1 \; y_2 \; N-y_1-y_2} \left(\frac{y_1}{N}\right)^{y_1} \left(\frac{y_2}{N}\right)^{y_2} \left(1 - \frac{y_1}{N} - \frac{y_2}{N}\right)^{N-y_1-y_2}$$

$$= \binom{N}{y_1 \; y_2 \; N-y_1-y_2} \frac{y_1^{y_1} y_2^{y_2} (N-y_1-y_2)^{N-y_1-y_2}}{N^N}$$

We have $H_0 : \theta_1 = \theta_2 = \theta_3 \iff \theta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and

$$L((\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})) = \binom{N}{y_1 \; y_2 \; N-y_1-y_2} \left(\frac{1}{3}\right)^{y_1} \left(\frac{1}{3}\right)^{y_2} \left(\frac{1}{3}\right)^{N-y_1-y_2}$$

$$= \binom{N}{y_1 \; y_2 \; N-y_1-y_2} \frac{1}{3^N}$$, so the LRT is

$$\lambda(\chi) = \frac{L((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))}{L(\hat{\theta})} = \left(\frac{N}{3}\right)^N y_1^{y_1} y_2^{y_2} (N-y_1-y_2)^{N-y_1-y_2}$$

Det blev lite fel med nämnare och täljare (-4)

A large sample version of the LRT is given by

$$-2\log\lambda(Y) = -2\left(N \log \frac{N}{3} + y_1 \log y_1 + y_2 \log y_2 + (N-y_1-y_2)\log(N-y_1-y_2)\right)$$

Följdfel

which approaches χ^2_1 as $N \to \infty$.

✓ (-4)

Tentamensservice/SU/typ_a

Poäng: (Points) 12

Uppg.nr.:
(*Task no.*)

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

Stockholms universitet

Datum: (Date YYYY-MM-DD)	2022-02-11	Kurs/Kurskod: (Course/Course code) ST4301	Sidnr.: (Page no.)
Anonymiseringskod (Anonymization code)	311-0016-TXO		3

Uppg.nr.: (Task no.) 3

Lärarens kommentar: (Teacher's note)

③ We have $f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$

a) so $L(\lambda) = f(\underset{\sim}{x};\lambda) = \prod_i f(x_i;\lambda) = \prod_i \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\Pi(x_i!)}$

so $\ell(\lambda) = -n\lambda + \Sigma x_i \log\lambda - \Sigma \log(x_i!)$

so $\ell'(\lambda) = -n + \frac{\Sigma x_i}{\lambda}$, so $\ell'(\hat\lambda) = 0 \iff \hat\lambda = \frac{\Sigma x_i}{n} = \bar x$,

and $\ell''(\lambda) = -\frac{\Sigma x_i}{\lambda^2} < 0$, so $\hat\lambda$ is the MLE.

Thus, by the invariance property of MLEs, the MLE of $\theta = \exp(-\lambda)$ is given by $\hat\theta = \exp(-\hat\lambda) = e^{-\bar x}$. R

6

b) We have $E(X) = \lambda$ and $Var(X) = \lambda$, so by the CLT,

$\sqrt{n}(\bar x - \lambda) \xrightarrow{D} N(0,\lambda)$. Let $g(\mu) := e^{-\mu}$. Then, by the Delta method,

$\sqrt{n}(g(\bar x) - g(\lambda)) \xrightarrow{D} N(0, \lambda g'(\lambda)^2)$, i.e.,

$\sqrt{n}(\hat\theta - \theta) \xrightarrow{D} N(0, \lambda(-e^{-\lambda})^2) = N(0, \lambda e^{-2\lambda})$ R

7

c) We have $I_n(\lambda) = -E(-\ell''(\lambda|\underset{\sim}{X})) = -E\left(-\frac{\Sigma X_i}{\lambda^2}\right) = \frac{\Sigma E(X_i)}{\lambda^2} =$

$= \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$, so $I(\lambda) = \frac{1}{\lambda}$.

But from (b) we have $\sqrt{n}(\bar x - \lambda) \xrightarrow{D} N(0,\lambda) = N(0, \frac{1}{1/\lambda})$

$= N(0, \frac{1}{I(\lambda)})$, so $\hat\lambda$ asymptotically attaince variance $\frac{1}{I(\lambda)}$.

R

7

Poäng: (Points) 20

Uppg.nr.:
(*Task no.*)

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

Stockholms universitet

Datum: (Date YYYY-MM-DD)	2022-02-11	Kurs/Kurskod: (Course/Course code)	ST4301	Sidnr.: (Page no.)
Anonymiseringskod (Anonymization code)	311-0016-TX0			4

Uppgg.nr.: (Task no.) 4

4 a) We have $L(\theta) = \prod_i f(x_i) = (\prod x_i)\theta^{-2n} e^{-\frac{1}{2\theta^2}\sum x_i^2}$

so $l(\theta) = \sum(\log x_i) - 2n\log\theta - \frac{1}{2\theta^2}\sum x_i^2$

so $l'(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^3}\sum x_i^2$, so

$l'(\hat\theta) = 0 \iff \hat\theta = \sqrt{\frac{1}{2n}\sum x_i^2}$

(and $\lim_{\theta\to\pm\infty} L(\theta) = 0$, so it is a maximum) ?

(Note also that we get $L(\hat\theta) = (\prod x_i)\left(\frac{1}{2n}\sum x_i^2\right)^{-n} e^{-n}$, which will be used in (5)).

Lärarens kommentar: (Teacher's note) 8

b) Let $Y := g(X) := \frac{X}{\theta}$. Then $f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$
$= f_X(\theta y)\theta = \frac{\theta y}{\theta^2} e^{-\frac{(\theta y)^2}{2\theta^2}} \cdot \theta = y e^{-\frac{1}{2}y^2}$, which is independent of θ, so Y is a pivot. Hence, $\exists\, a, b \in \mathbb{R}_{>0}$: $P(a \le Y \le b) = 1-\alpha$. We want

ok men svår väg!

$\frac{\alpha}{2} = P(Y < a) = \int_0^a f_Y(y)\,dy = \int_0^a y e^{-\frac{1}{2}y^2}\,dy$ and

$\frac{\alpha}{2} = P(Y > b) = \int_b^\infty f_Y(y)\,dy = \int_b^\infty y e^{-\frac{1}{2}y^2}\,dy$. Jag vill att du löser det! (-6)

We solve these two equations for a and b in terms of α and get $P(a \le \frac{X}{\theta} \le b) = P(\frac{X}{b} \le \theta \le \frac{X}{a})$, to get a $100(1-\alpha)$ confidence interval $[\frac{X}{b}, \frac{X}{a}]$ for θ.

Poäng: (Points) 14

Uppg.nr.:
(*Task no.*)

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

Lärarens kommentar: (Teacher's note)

5 We have $L(\theta_1,\theta_2) = \prod_i f(x_i)\prod_j f(y_j) =$

a) $= (\prod x_i)\theta_1^{-2n} e^{-\frac{1}{2\theta_1^2}\sum x_i^2} (\prod y_j)\theta_2^{-2m} e^{-\frac{1}{2\theta_2^2}\sum y_j^2}$

So the LRT is

$$\lambda(\underset{\sim}{x},\underset{\sim}{y}) = \frac{\sup_{\theta_2} L(\theta_2,\theta_2)}{\sup_{\theta_1,\theta_2} L(\theta_1,\theta_2)} = \frac{\left(\frac{1}{2(n+m)}(\sum x_i^2+\sum y_j^2)\right)^{-(n+m)} e^{-(n+m)}}{\left(\frac{1}{2n}\sum x_i^2\right)^{-n} e^{-n}\left(\frac{1}{2m}\sum y_j^2\right)^{-m} e^{-m}}$$

$$= \frac{(2(n+m))^{n+m}(\sum x_i^2)^n(\sum y_j^2)^m}{(\sum x_i^2+\sum y_j^2)^{n+m}(2n)^n\ (2m)^m} = \frac{(n+m)^{n+m}(\sum x_i^2)^n(\sum y_j^2)^m}{n^n m^m(\sum x_i^2+\sum y_j^2)^{n+m}}$$

$$= \frac{(n+m)^{n+m}}{n^n m^m}\frac{(\sum x_i^2)^n(\sum y_j^2)^m}{(\sum x_i^2+\sum y_j^2)^n(\sum x_i^2+\sum y_j^2)^m} =$$

$$= \frac{(n+m)^{n+m}}{n^n m^m}\frac{1}{\left(1+\frac{\sum y_j^2}{\sum x_i^2}\right)^n\left(\frac{\sum x_i^2}{\sum y_j^2}+1\right)^m}$$

We reject H_0 if λ is small enough, which is equivalent to either $\frac{\sum y_j^2}{\sum x_i^2}$ or $\frac{\sum x_i^2}{\sum y_j^2}$ is large enough, which R

is the desired conclusion.

10

b) We have $\sum X_i^2, \sum Y_j^2 \sim \Gamma(5, 2\theta_2)$ under H_0.

We reject H_0 if $P\left(\frac{\sum X_i^2}{\sum Y_j^2} \geq \max\{T_{obs}, \frac{1}{T_{obs}}\}\right) \leq \alpha$

i.e. if $P\left(\frac{\sum X_i^2}{\sum Y_j^2} \geq 4.367\right) \leq 0.01$.

For this, we need to know the distribution of $\frac{\sum X_i^2}{\sum Y_j^2}$,

i.e., the distribution of the quotient of two independent $\Gamma(5, 2\theta_2)$-distributed random variables.

och vilken fördelning är det?

4p för en bra idé men den är inte helt framme vid målet

Poäng: (Points) 14

Uppg.nr.:
(*Task no.*)

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)