Stockholms universitet

**OBS! Läs noga igenom anvisningarna i tentamen, t.ex. hur du ska skriva svaren. Det är ditt ansvar som student att följa de anvisningar som ges.**

***NOTE! Read the examination instructions carefully, e.g. how to write the answers. It is your responsibility as a student to follow the given instructions.***

Skriv din anonymiseringskod och dagens datum på allt material du lämnar in.
(*Enter your anonymization code and todays date on all submitted materials*)

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|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Anonymiseringskod (*Anonymization code*) | 3 | 1 | 1 | - | 0 | 0 | 0 | 6 | - | H | B | G |

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|---|---|---|---|
| Datum (*Date* YYYY-MM-DD) | 2025-01-16 | Plats nr. (*Seat No.*) | 107 |

| | |
|---|---|
| Kurs/Kurskod (*Course/Course code*) | ST4301 |
| Kursmoment (*Course component*) | |

**Fylls i av tentamensvärd** (*To be filled in by invigilator*)

| | | | | | |
|---|---|---|---|---|---|
| Direkt i skrivning: (kryss) | | Svarsblankett: (kryss) | | Lösa svarsblad: (antal) | 7 |
| Lämnat in blankt: (kryss) | | Dator: (kryss) | | | |

Inlämningstid: 15:43 Signatur tentamensvärd: ZSA

**Fylls i av lärare/examinator** (*To be filled in by teacher/examinator*)

| | | | |
|---|---|---|---|
| Betyg: | A | Poäng: | 95 |

Signatur rättande lärare/examinator: PGA

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| Datum: (Date YYYY-MM-DD) | 2025-01-16 | Kurs/Kurskod: (Course/Course code) | ST4301 | Ark nr.: (Sheet no.) 1 |
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| Anonymiseringskod (Anonymization code) | 311-0006-HBG | | | |

Uppg.nr.: (Task no.) 1

Lärarens kommentar: (Teacher's note)

a/ $f(x|\theta) = \theta(\theta+1)\, x^{\theta-1}(1-x)$ $\begin{cases} \theta \in \mathbb{R}^+ \\ x \in [0,1]. \end{cases}$

$\Longleftrightarrow f(x|\theta) = \theta(\theta+1)\, e^{\log(x^{\theta-1}\cdot(1-x))}$

$f(x|\theta) = \theta(\theta+1)\, e^{(\theta-1)\log(x) + \log(1-x)}$

Let $\vec{x} = (x_1, x_2 \ldots x_n)$, vector of whole sample

$f(\vec{x}|\theta) = \theta^n(\theta+1)^n\, e^{(\theta-1)\Sigma\log(x) + \Sigma\log(1-x)}$

We can note that this fulfills general expression of an exponential family, and support does not depend on $\theta$.

Answer: Yes, it is exponential family.

b/ From inspection of pdf and by the factorization theorem: $T = \Sigma\log(x)$ is sufficient for $\theta$.

Answer: $\Sigma\log(x)$ is sufficient for $\theta$.

c/ Yes, the statistic is complete, since the pdf belongs to exponential family and $\theta \in \mathbb{R}^+$ is an open set, by a theorem in the book!

answer: Yes, $\Sigma\log(x)$ is a complete sufficient statistic!

Poäng: (Points) 2

Tentamensservice/SU/typ_a

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

Uppg.nr.: (Task no.) 2

Lärarens kommentar: (Teacher's note)

a/

$$L(\theta|\vec{x}) = f(\vec{x}|\theta) = \theta^n (\theta+1)^n e^{(\theta-1)\sum \log(x) + \sum \log(1-x)}$$

$$\ell(\theta|\vec{x}) = \log L(\theta|\vec{x}) = n\log(\theta) + n\log(\theta+1) + (\theta-1)\sum\log(x) + \sum\log(1-x)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \frac{n}{\theta+1} + \sum \log(x)$$

Find MLE by setting $\frac{\partial \ell}{\partial \theta} = 0$

$$\rightarrow \frac{n}{\theta} + \frac{n}{\theta+1} + \sum \log(x) = 0$$

$$\Longleftrightarrow \theta^2\left(\sum\log(x)\right) + \theta\left(\sum\log(x) + 2n\right) + n = 0$$

We can see that the MLE of $\theta$ is dependent on the sufficient statistic and the sample size, which is expected, since the pdf is exponential family!

Answer: $\theta^2(\sum\log(x)) + \theta(\sum\log(x) + 2n) + n = 0$

Gives MLE of $\theta$, $\hat{\theta}$. Yes equation is related to sufficient statistic!

b/ Since pdf is exp. family, we know from CLT:

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N\left(0, \frac{1}{I_n}\right)$$

with $I_n = -\frac{\partial^2 \ell}{\partial \theta^2} = -\left(-\frac{n}{\theta^2} - \frac{n}{(\theta+1)^2}\right) = \frac{n}{\theta^2} + \frac{n}{(\theta+1)^2}$

Algebra gives $\rightarrow$ $I_n = n\frac{(2\theta^2 + 2\theta + 1)}{\theta^2(\theta+1)^2}$

Answer: $\sqrt{n}\,\hat{\theta} \xrightarrow{d} N\left(\theta, \frac{\theta^2(\theta+1)^2}{\cancel{n}(2\theta^2+2\theta+1)}\right)$

Poäng: (Points) 1.5

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

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Uppg.nr.: (Task no.) 3

Lärarens kommentar: (Teacher's note)

a/ Method of moments (MoM) estimator is calculated by setting empirical and theoretical moments equal:

$$E(X) = \frac{1}{n}\Sigma x = \bar{x}$$

$$X \sim f(x|\theta) \sim Beta(\theta, 2) = \frac{\Gamma(\theta+2)}{\Gamma(\theta)\Gamma(2)} x^{\theta-1}(1-x)^{2-1}$$

$$= \frac{(\theta+1)!}{(\theta-1)!\cdot 1!} x^{\theta-1}(1-x)$$

$$= \theta(\theta+1)x^{\theta-1}(1-x)$$

From formula sheet: $E(X) = \frac{\theta}{\theta+2}$

$$\Rightarrow \frac{\theta}{\theta+2} = \bar{x} \iff \tilde{\theta} = \frac{2\bar{x}}{1-\bar{x}}$$ OK

Answer: $\tilde{\theta} = \frac{2\bar{x}}{1-\bar{x}}$, with $\bar{x} = \frac{1}{n}\sum_i x_i$

b/ $\sqrt{n}(\tilde{\theta}-\theta) = \sqrt{n}\left(\frac{2\bar{x}}{1-\bar{x}} - \theta\right) = \frac{\sqrt{n}}{1-\bar{x}}\left(2\bar{x} - (1-\bar{x})\theta\right)$

$$= \frac{\sqrt{n}}{1-\bar{x}}\left(2\bar{x} - \theta + \theta\bar{x}\right) = \frac{\sqrt{n}}{1-\bar{x}}\left(\bar{x}(2+\theta) - \theta\right)$$

$$= \frac{\sqrt{n}}{1-\bar{x}}(\theta+2)\left(\bar{x} - \frac{\theta}{\theta+2}\right) \quad (1)$$

Note: $E(\bar{X}) = \frac{\theta}{\theta+2}$

~~$\rightarrow$ (1) i=~~

Which means that we have:

$$\frac{\sqrt{n}(\theta+2)}{1-\bar{x}}\left(\bar{x} - E(\bar{X})\right)$$

Tentamensservice/SU/typ_a

Poäng: (Points) 20

Lärarens kommentar: (*Teacher's note*)

Poäng: (*Points*)

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Uppg.nr.: (Task no.) 3

Lärarens kommentar: (Teacher's note)

From CLT we know:

$$\sqrt{n}(\bar{X} - E(\bar{X})) \xrightarrow{d} N(0, V(\bar{X}))$$

From formula sheet: $V(\bar{X}) = \frac{2\theta}{(\theta+2)^2(3+\theta)}$

since $X \sim Beta(\theta, 2)$

From Slutsky's theorem I know:

$$1-\bar{X} \xrightarrow{P} 1 - \frac{\theta}{\theta+2}$$

In total this means:

$$\frac{(\theta+2)}{1-\bar{X}} \cdot \sqrt{n}(\bar{X} - E(\bar{X})) \xrightarrow{P} \frac{(\theta+2)}{1-\frac{\theta}{\theta+2}} \underbrace{\sqrt{n}(\bar{X} - E(\bar{X}))}_{N\left(0, \frac{2\theta}{(\theta+2)^2(\theta+3)}\right)}$$

$$\frac{\theta+2}{1-\frac{\theta}{\theta+2}} = \frac{\theta+2}{\frac{2}{\theta+2}} = \frac{(\theta+2)^2}{2}$$

I can get this inside $N(\cdot,\cdot)$ by squaring it, since $aZ \sim N(0, a^2)$, if $Z \sim N(0,1)$

so the variance would be:

$$\frac{2\theta \cdot (\theta+2)^4}{4(\theta+2)^2(\theta+3)} = \frac{\theta(\theta+2)^2}{2(\theta+3)}$$

__Answer:__ $\sqrt{n}(\tilde{\theta} - \theta) \to N\left(0, \frac{\theta(\theta+2)^2}{2(\theta+3)}\right)$

Poäng: (Points)

Tentamensservice/SU/typ_a

Lärarens kommentar: (*Teacher's note*)

Poäng: (*Points*)

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Uppg.nr.: (Task no.) 4

Lärarens kommentar: (Teacher's note)

$$f(x|\theta) = 2e^{-2(x-\theta)} \quad \begin{cases} x \geq \theta \\ \theta > 0 \end{cases}$$

Since we have $x \geq \theta$, this means that $\min(x) \geq \theta \iff x_{(1)} \geq \theta$

I will create CI by pivoting: $x_{(1)} - \theta$

Let $T = \min_i(X_i)$

$F_T(t) = P(T<t) = P(\min(X_i) < t) = 1 - P(\min_i(X_i) > t)$

$= 1 - P(X > t)^n = 1 - (1 - F_X(t))^n$

$$F_X(t) = \int_\theta^t 2e^{-2(x-\theta)}\,dx = 1 - e^{-2(t-\theta)}$$

$$\to F_T(t) = 1 - e^{-2n(t-\theta)} \quad \begin{cases} t \geq \theta \\ \theta > 0 \end{cases}$$

$f_T(t)$ is strictly decreasing, so the optimal CI has term $(\min(X) > \theta > \min(X) - a)$ where $a > 0$.

$\to 0{,}9 = P(\min(X) - a < \theta < \min(X)) = P(T - a < \theta) = P(T < a + \theta)$

$\iff 0{,}9 = 1 - e^{-n2(a+\theta-\theta)} = 1 - e^{-2na}$

$$\to a = -\frac{1}{2n}\log(0{,}1) = 0{,}19$$

$90\%$ CI $= \{\theta : \min_i(X_i) - a < \theta < \min_i(X_i)\}$

$= \{\theta : 9{,}3 - 0{,}19 < \theta < 9{,}3\}$

$= \{\theta : 9{,}1 < \theta < 9{,}3\}$

Rejecting 2 sided hypothesis $\to$ Symmetric CI

**Note**: if we have the hypothesis $\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta \neq \theta_0 \end{cases}$, $\theta_0 > 0$

The LRT would be: $\lambda = \frac{f(x|\theta_0)}{f(x|\min(x))}$, since $\hat{\theta} = \min(X)$

This test would be rejected when $\min(X) \leq k$, and inverting this test would give same CI, as stated above.

Answer: CI 90% $= (9{,}1 < \theta < 9{,}3)$

Poäng: (Points) 20

Lärarens kommentar: (*Teacher's note*)

Poäng: (*Points*)

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Uppg.nr.: (Task no.) 5

Lärarens kommentar: (Teacher's note)

$$\begin{cases} H_0: \lambda = 2 = \lambda_0 \\ H_1: \lambda = 5 = \lambda_1 \end{cases}$$

$X_i \overset{iid}{\sim} Pois(\lambda)$

This is a simple hypothesis, so Neyman-Pearson's Lemma can be used to construct UMP test.

NPL states: $x \in RR$ if $\frac{f(x|\theta_1)}{f(x|\theta_0)} > k > c$

and $\alpha = P(X \in RR|\theta_0)$

$$\frac{f(\vec{x}|\theta_1)}{f(\vec{x}|\theta_0)} = \frac{\frac{\lambda_1^{\Sigma x} e^{-\lambda_1 n}}{\prod_{i=1}^{n} x_i!}}{\frac{\lambda_0^{\Sigma x} e^{-\lambda_0 n}}{\prod_{i=1}^{n} x_i!}} = \left(\frac{\lambda_1}{\lambda_0}\right)^{\Sigma x} e^{-n(\lambda_1 - \lambda_0)}$$

- The likelihood ratio becomes large when $\Sigma x$ is large, since $\lambda_1 > \lambda_0$ $(5>2)$.
- This is expected since Poisson has MLR w.r.t the sufficient statistic $(\Sigma x)$, since it is exponential family.

$\Rightarrow$ Reject $H_0$ when $\Sigma X > k'$

we determine $k'$ by: $\alpha = P(\Sigma X > k' | \lambda_0 = \lambda)$

note that $\Sigma X \sim Pois(\Sigma \lambda_i)$, and under $H_0$ $\Sigma \lambda_i = \Sigma \lambda_0 = n\lambda_0$

$\alpha = P(\Sigma X > k' | \lambda_0 = \lambda) \Leftrightarrow \alpha = P(T > k')$

where $T \sim Pois(n \cdot \lambda_0)$

Poäng: (Points) 20

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

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Uppg.nr.: (Task no.) 5
Lärarens kommentar: (Teacher's note)

Answer:

Reject $H_0$ when $\sum X > k'$

we can decide $k'$, by specifying a significance level $\alpha$ using:

$$\alpha = P(\sum X > k' \mid \lambda = \lambda_0) = P(T > k')$$

where $T \sim Pois(n\lambda_0) = Pois(2n)$

Poäng: (Points)

Tentamensservice/SU/typ_a

Lärarens kommentar:
(*Teacher's note*)

Poäng:
(*Points*)

## Regler i skrivsalen

- Följ tentamensvärds anvisningar.
- Väskor och ytterkläder ska placeras på anvisad plats.
- Placera ID-handling väl synlig på bordet framför dig.
- Ingen student får lämna skrivsalen under de första 30 minuterna.
- Endast en student i taget får besöka toaletten. Vid toalettbesök skriv ditt namn och klockslag på avsedd lista. Efter toalettbesöket ska du åter ange klockslag på listan.
- Elektronisk utrustning som mobiltelefon eller Smartwatch ska vara avstängd och placerad på anvisad plats.
- Under tentamen gäller tystnad – det är förbjudet att prata, eller på annat sätt kommunicera, med andra studenter under pågående tentamen.
- Innan tentamenshandlingarna lämnas in; skriv sidnummer, anonymiseringskod och datum på alla inlämnade papper.

Om något är oklart – fråga gärna tentamensvärden. Lycka till!

## Rules in the examination hall

- Follow the invigilator's instructions.
- Bags and outerwear must be placed at the designated place.
- Place your ID document clearly visible on the table in front of you.
- No student may leave the examination hall for the first 30 minutes.
- Only one student at a time may visit the toilet. Before visiting the toilet, write your name and time on the intended list. After the toilet visit, enter the time on the list again.
- Electronic equipment such as a mobile phone or Smartwatch must be switched off and placed at the designated place.
- During the exam, silence applies – you are not allowed to talk, or otherwise communicate, with other students during the exam.
- Before submitting the examination documents; remember to write the page number, anonymization code, and date on all papers.

Please do not hesitate to ask the invigilator if anything is unclear. Good luck!