Proposed solutions Exam 2023-05-31 Econometrics II

Problem 1

a) (ii.)

b) (iv.)

c) (iv.)

d) (ii.)

e) (iii.)

Problem 2.

M1: $y_t = 5 + \epsilon_t - 0.7\epsilon_{t-1} + 0.3\epsilon_{t-2}$ is an MA(2) model.

The autocovariance of an MA(2) model $y_t = \mu + \epsilon_t - \theta_1 \cdot \epsilon_{t-1} - \theta_2 \cdot \epsilon_{t-1}$ at lags 1 and 2 are:

$$\gamma(1) = (-\theta_1 + \theta_1 \cdot \theta_2) \cdot \sigma^2$$

$$\gamma(2) = -\theta_2 \cdot \sigma^2$$

The autocorrelation function ACF is

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}, \quad \gamma(0) = V[y_t] = \sigma^2(1 + \theta_1^2 + \theta_2^2)$$

i.e

$$\rho(1) = \frac{-\theta_1 + \theta_1 \cdot \theta_2}{1 + \theta_1^2 + \theta_2^2}, \quad \rho(2) = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

inserting $\theta_1 = 0.7$ and $\theta_2 = -0.3$ gives

$\rho(1) \approx -0.58$

$\rho(2) \approx 0.19$,

is consistent with ACF plot B1.

The ACF plot B1 indicates that observations 1 lag apart are negatively correlated. This is consistent with time series plot A2 where observations one lag apart tend to be (more often than not) on the opposite side of $E[y_t]$

M2: $y_t = 5 + \epsilon_t + 0.9\epsilon_{t-1} + 0.5\epsilon_{t-2}$ is

also an MA(2) model,

with $\theta_1 = -0.9$ and $\theta_2 = -0.5$.

using the formulas above,

$\rho(1) \approx 0.66$

$\rho(2) \approx 0.24$,

consistent with ACF plot B2.

This ACF plot indicates a positive correlation up to two lags, which is consistent with Time series plot A3, which shows a moderately positive autocorrelation.

M3: $y_t = 0.35 + 0.93 y_{t-1} + \epsilon_t$ is an AR(1), which has an exponentially decaying ACF, consistent with ACF plot B3. There is a significant autocorrelation, even for large k, consistent with time series plot A1, where for a long stretch the process is below the $E[y_t] = 5$.

[Recall $E[y_t] = \frac{\delta}{1-\phi_1} = \frac{0.35}{0.07} = 5$ in stationary AR(1) model].

Answer:

M1 + A2 + B1

M2 + A3 + B2

M3 + A1 + B3

Problem 3

a) ARIMA(1,2,1)

$$(1-\phi_1 B)(1-B)^2 y_t = \delta + (1-\theta_1 B)\epsilon_t$$

b) ARIMA(0,2,2)

$$(1-B)^2 y_t = \delta + (1-\theta_1 B-\theta_2 B^2)\epsilon_t$$

c) ARIMA(1,1,1) x (1,1,2) with s=4

$$(1-\phi_{1,4}B^4)(1-\phi_1 B)(1-B)(1-B^4)y_t = \delta + (1-\theta_{1,4}B^4-\theta_{1,8}B^8)(1-\theta_1 B)\epsilon_t$$

Problem 4

Suppose that

$$y_t = \alpha_0 + \delta_1 \cdot x_{t-1} + \delta_2 \cdot x_{t-2} + \delta_3 \cdot x_{t-3} + u_t ,$$

with y_t = "Marketing spending at time t"

x_t = "Sales at time t",

y_t and x_t are weakly dependent and stationary.

a) This is a finite distributed lag model of order 3. Note that $\delta_0 = 0$.

For consistency, TS1'-TS3' need to be fulfilled. TS1' is fulfilled by assumption.

TS2' trivially fulfilled since x_t varies.

We need to add TS3':

$E[u_t \mid x_{t-1}, x_{t-2}, x_{t-3}] = 0$, i.e

contemporaneous exogeneity.

If u_t is serially correlated does not change consistency results.

b) The immediate change in y after a unit temporary increase of x at time t is δ_0, hence $10 \cdot \delta_0$ million.

However, $\delta_0 = 0$, so no immediate change.

c) The change in y two periods after a unit temporary increase of x at time t is δ_2, hence $20 \cdot \delta_2$ million

d) The cumulative effect on y 2 time periods after a unit permanent increase of x at time t is $\delta_0 + \delta_1 + \delta_2$. So for 5 million units,

$5(\delta_0 + \delta_1 + \delta_2) = 5(\delta_1 + \delta_2)$ $(\delta_0 = 0)$ million.

e) If the permanent increase is 1 unit, then this is the $LRP = \delta_0 + \delta_1 + \delta_2 + \delta_3 = \delta_1 + \delta_2 + \delta_3$.

For 10 million units, $10(\delta_1 + \delta_2 + \delta_3)$ million.

Problem 5

a) Simple exponential smoothing:

$$\tilde{y}_T = \lambda \cdot y_T + (1-\lambda)\,\tilde{y}_{T-1}, \quad T = 1, \dots, 9.$$

The process in Figure 3 is

$$y_t = \beta_0 + \beta_1 \cdot t + \varepsilon_t, \quad \varepsilon_t \text{ iid } E[\varepsilon_t] = 0.$$

Since $E[\tilde{y}_T] = \beta_0 + \beta_1 \cdot T - \frac{(1-\lambda)}{\lambda}\beta_1$, by (4.17) in MJK.

From Figure 3, $\beta_1 > 0$ and since $0 < \lambda < 1$,

$E[\tilde{y}_T] < \beta_0 + \beta_1 \cdot T$, so simple exponential smoothing is biased downwards (underestimates the process).

b) Using the equations for double exponential smoothing (with $\lambda = 0.6$) for $T = 6, 7, 8, 9$

$$\tilde{y}_T^{(1)} = \lambda y_T + (1-\lambda)\,\tilde{y}_{T-1}^{(1)}$$

$$\tilde{y}_T^{(2)} = \lambda \tilde{y}_T^{(1)} + (1-\lambda)\,\tilde{y}_T^{(2)},$$

with $\tilde{y}_5^{(1)} = 21.47$ and $\tilde{y}_5^{(2)} = 20.67$

we get

	$T=6$	$T=7$	$T=8$	$T=9$
$\tilde{y}_T^{(1)}$	22.508	23.8232	25.48928	26.275 71
$\tilde{y}_T^{(2)}$	21.7728	23.00304	24.49478	25.56334

$\hat{y}_T = 2\tilde{y}_T^{(1)} - \tilde{y}_T^{(2)}$ is unbiased for the underlying linear process.

Thus (rounded to 2 decimals)

T	$T=1$	$T=2$	$T=3$	$T=4$	$T=5$	$T=6$	$T=7$	$T=8$	$T=9$
$\hat{y}_T$	17.00	18.76	19.66	20.67	22.27	23.24	24.64	26.48	26.99

c) In double exponential smoothing,

$$\hat{y}_{T+1}(T) = \hat{y}_T + \hat{\beta}_{1,T},$$

$$\hat{\beta}_{1,T} = \frac{\lambda}{1-\lambda}\left(\hat{y}_T^{(1)} - \hat{y}_T^{(2)}\right) = \frac{0.6}{1-0.6}\left(26.27571 - 25.56334\right)$$

$$= 1.068555.$$

$$\hat{y}_{T+1}(T) = 26.99 + 1.068555 = 28.05855.$$

Actual observation $y_{10} = 25.6$

Relative forecast error (one-step ahead forecast).

$$re_{10}(1) = \left(\frac{y_{10} - \hat{y}_{10}(9)}{y_{10}}\right) \times 100$$

$$\left(\frac{25.6 - 28.05855}{25.6}\right) \times 100 \approx -9.6\%$$

d) Holt's method predicts

$$y_{t+\tau}(t) = L_t + T_t \cdot \tau ,$$

where the smoothing equations are

$$L_t = \alpha \cdot y_t + (1-\alpha)(L_{t-1} + T_{t-1})$$

$$T_t = \gamma \cdot (L_t - L_{t-1}) + (1-\gamma) T_{t-1} .$$

Smooth the values for $t = 1, \dots, 4$ (first four years of data).

With $\alpha = 0.5$, $\gamma = 0.6$, $L_0 = 16$, $T_0 = 1.2$ we get

$$L_1 = 0.5 \cdot 17 + (1-0.5)(16+1.2) = 17.1$$

$$T_1 = 0.6 \cdot (17.1 - 16) + (1-0.6) \cdot 1.2 = 1.14$$

$$L_2 = 18.67$$

$$T_2 = 1.398$$

$$L_3 = 19.884$$

$$T_3 = 1.2876$$

and at $t=4$

$L_4 = 20.9358$

$T_4 = 1.4612$

The one-step ahead forecast for 2014

$\hat{y}_5(4) = L_4 + T_4 \cdot 1 = 22.08192$

and one-step ahead forecast error

$e_5(1) = y_5 - \hat{y}_5(4) = 22.4 - 22.08192$

$= 0.31808.$

two-step ahead forecast for 2015

$\hat{y}_6(4) = L_4 + T_4 \cdot 2 = 23.22804$

and two-step ahead forecast error

$e_6(2) = y_6 - \hat{y}_6(4) = 23.2 - 23.22804$

$= -0.02804.$

Problem 6

Consider

$$y_t = \mu - \theta_2 \cdot \epsilon_{t-2} - \theta_4 \cdot \epsilon_{t-4} - \theta_5 \cdot \epsilon_{t-5},$$

with ϵ_t iid, $E[\epsilon_t] = 0$ and $V[\epsilon_t] = \sigma^2$.

a) Compute $\gamma_y(k) = Cov(y_t, y_{t+k})$,
$k = 1, 2, 3, 4, 5$.

$k=1$: $Cov(y_t, y_{t+1})$

$$= Cov(\mu - \theta_2 \cdot \epsilon_{t-2} - \theta_4 \epsilon_{t-4} - \theta_5 \epsilon_{t-5},$$
$$\mu - \theta_2 \cdot \epsilon_{t-1} - \theta_4 \epsilon_{t-3} - \theta_5 \epsilon_{t-4})$$

$$= \{ \epsilon_t \text{ iid}, Cov(\epsilon_t, t_s) = 0 \quad s \neq t \} =$$

$$= -\theta_4 \cdot -\theta_5 \, Cov(\epsilon_{t-4}, \epsilon_{t-4}) = \underline{\underline{\theta_4 \cdot \theta_5 \sigma^2}}$$

$k=2$: $Cov(y_t, y_{t+2})$

$$= Cov(\mu - \theta_2 \cdot \epsilon_{t-2} - \theta_4 \epsilon_{t-4} - \theta_5 \epsilon_{t-5},$$
$$\mu - \theta_2 \cdot \epsilon_t - \theta_4 \epsilon_{t-2} - \theta_5 \epsilon_{t-3})$$

$$= \{ \epsilon_t \text{ iid}, Cov(\epsilon_t, t_s) = 0 \quad s \neq t \} =$$

$$= -\theta_2 \cdot -\theta_4 \, Cov(\epsilon_{t-2}, \epsilon_{t-2}) = \underline{\underline{\theta_2 \cdot \theta_4 \sigma^2}}$$

$$k=3: \ Cov(y_t, y_{t+3})$$

$$= Cov(\mu - \theta_2 \cdot \epsilon_{t-2} - \theta_4 \epsilon_{t-4} - \theta_5 \epsilon_{t-5},$$

$$\mu - \theta_2 \cdot \epsilon_{t+1} - \theta_4 \epsilon_{t-1} - \theta_5 \epsilon_{t-2})$$

$$= \{ \epsilon_t \text{ iid}, \ Cov(\epsilon_t, t_s) = 0 \ s \neq t \} =$$

$$= -\theta_2 \cdot -\theta_5 \ Cov(\epsilon_{t-2}, \epsilon_{t-2}) = \underline{\underline{\theta_2 \cdot \theta_5 \sigma^2}}$$

$$k=4: \ Cov(y_t, y_{t+4})$$

$$= Cov(\mu - \theta_2 \cdot \epsilon_{t-2} - \theta_4 \epsilon_{t-4} - \theta_5 \epsilon_{t-5},$$

$$\mu - \theta_2 \cdot \epsilon_{t+2} - \theta_4 \epsilon_{t} - \theta_5 \epsilon_{t-1})$$

$$= \{ \epsilon_t \text{ iid}, \ Cov(\epsilon_t, t_s) = 0 \ s \neq t \} =$$

$$= \underline{\underline{0}}$$

$$k=5: \ Cov(y_t, y_{t+5})$$

$$= Cov(\mu - \theta_2 \cdot \epsilon_{t-2} - \theta_4 \epsilon_{t-4} - \theta_5 \epsilon_{t-5},$$

$$\mu - \theta_2 \cdot \epsilon_{t+3} - \theta_4 \epsilon_{t+1} - \theta_5 \epsilon_{t})$$

$$= \{ \epsilon_t \text{ iid}, \ Cov(\epsilon_t, t_s) = 0 \ s \neq t \} =$$

$$= \underline{\underline{0}}$$

For the autocorrelation, note that $\rho_y(-k) = \rho_y(k) = \dfrac{Cov(y_t, y_{t+k})}{V(y_t)}$

since y_t is stationary (MA always stationary).

Moreover,

$$V(y_t) = V(\mu - \theta_2 \cdot \epsilon_{t-2} - \theta_4 \cdot \epsilon_{t-4} - \theta_5 \cdot \epsilon_{t-5})$$

$$\{\epsilon_t \text{ iid}\} = \theta_2^2 V[\epsilon_{t-2}] + \theta_4^2 V[\epsilon_{t-4}] + \theta_5^2 V[\epsilon_{t-5}]$$

$$= \sigma^2(\theta_2^2 + \theta_4^2 + \theta_3^2)$$

Thus

$$\rho_y(2) = \frac{\theta_2 \cdot \theta_4 \sigma^2}{\sigma^2(\theta_2^2 + \theta_4^2 + \theta_5^2)} = \underline{\underline{\frac{\theta_2 \cdot \theta_4}{(\theta_2^2 + \theta_4^2 + \theta_5^2)}}}$$

and

$$\rho_y(3) = \underline{\underline{\frac{\theta_2 \cdot \theta_5}{(\theta_2^2 + \theta_4^2 + \theta_5^2)}}}$$

b) The optimal $\hat{y}_{t+1}$ that minimises

$$E[(y_{t+1} - \hat{y}_{t+1})^2]$$

is $\hat{y}_{t+1} = E[y_{t+1} \mid I_t]$,

i.e the conditional expectation of y_{t+1} given the information set I_t. When $t=15$ we get:

$$y_{16} = \mu - \theta_2 \cdot \epsilon_{14} - \theta_4 \cdot \epsilon_{12} - \theta_5 \cdot \epsilon_{11} ,$$

hence

$$E[y_{16} \mid I_{15}]$$

$$E[\mu - \underbrace{\theta_2 \cdot \epsilon_{14} - \theta_4 \cdot \epsilon_{12} - \theta_5 \cdot \epsilon_{11}}_{\text{Not random given } I_{15}} \mid I_{15}]$$

$$= \mu - \theta_2 \cdot \epsilon_{14} - \theta_4 \cdot \epsilon_{12} - \theta_5 \cdot \epsilon_{11}$$

$= \{$ inserting the values given in the text $\}$

$= 2 - (-0.2 \cdot -0.5) - 0.4 \cdot 0.2 - 0.4 \cdot 1.05$

$= 1.4$,

i.e

$\hat{y}_{t+1} = 1.4$ (for $t = 15$).